

Fast Wavelet Transform;

Lecture 18: Various Extensions

Note Title

★ Assumptions on an input signal

- In all practical applications, \exists a finest and a coarsest scale of interest.
- For simplicity, let's assume that an input signal is a vector $f = (f_0, \dots, f_{N-1})^T$, $N = 2^n$, and periodic.
- Let's assume the finest scale is $2^0 = 1$, and we view an input signal $\in V_0$, $\dim(V_0) = N$.
- Also assume the coarsest scale is 2^J with $1 \leq J \leq n$. This implies that $\dim(V_J) = 2^{n-J} = N/2^J$, and
$$V_0 = V_J \oplus \bigoplus_{j=1}^J W_j.$$
- Finally assume the given samples f_0, \dots, f_{N-1} are the finest scale coefficients, i.e.,
$$f_k = \langle f, \phi_{0,k} \rangle =: S_k^0$$

$\{f_k\}$ are given. So, we implicitly deal with "fictitious" $f = \sum_{k=0}^{N-1} f_k \phi_{0,k} = \sum_{k=0}^{N-1} S_k^0 \phi_{0,k}$.

Hence in this case, $f = P_{V_0} f$.

- If you know $f(x)$ over $[0, 1]$, and want to have $f_k \approx f(\frac{k}{N})$, then you need to design

ϕ with high vanishing moments \Rightarrow "coiflets".
 \perp normally ϕ does not have vanishing moments.

★ Fast Orthogonal Wavelet Transform

Let us write

$$P_{V_j} f = \sum_{k=0}^{2^{n-j}-1} s_k^j \phi_{j,k}, \quad P_{W_j} f = \sum_{k=0}^{2^{n-j}-1} d_k^j \psi_{j,k},$$

where $s_k^j := \langle f, \phi_{j,k} \rangle$, $d_k^j := \langle f, \psi_{j,k} \rangle$

Sum

difference

Forward transf: Given $P_{V_0} f$, compute

$$P_{W_1} f, P_{W_2} f, \dots, P_{W_J} f, P_{V_J} f.$$

\Leftrightarrow Given $\{s_k^0\}_{k=0}^{N-1}$, compute $\{d_k^j\}_{k=0}^{2^{n-j}-1}$, $j=1, \dots, J$
and $\{s_k^J\}_{k=0}^{2^{n-J}-1}$.

Inverse transf: Given $P_{W_1} f, \dots, P_{W_J} f, P_{V_J} f$,
reconstruct $P_{V_0} f$.

\Leftrightarrow Reconstruct $\{s_k^0\}_{k=0}^{N-1}$ from $\{d_k^j\}_{k=0}^{2^{n-j}-1}$, $j=1, \dots, J$
and $\{s_k^J\}_{k=0}^{2^{n-J}-1}$.

Thm (Mallat 1989)

discrete convolution

Forward transf.

$$\begin{cases} s_k^{j+1} = \sum_{l \in \mathbb{Z}} h_{l-2k} s_l^j = (s^j * \tilde{h})_{2k} \\ d_k^{j+1} = \sum_{l \in \mathbb{Z}} g_{l-2k} s_l^j = (s^j * \tilde{g})_{2k} \end{cases} \quad k=0, \dots, 2^{n-j-1}.$$

where $\tilde{h}_l := h_{-l}$

↳ subsampling

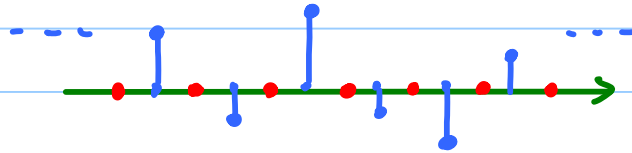
(every other samples)

Inverse transf.

$$s_k^j = \sum_{l \in \mathbb{Z}} h_{k-2l} s_l^{j+1} + \sum_{l \in \mathbb{Z}} g_{k-2l} d_l^{j+1} \quad k=0, \dots, 2^{n-j}.$$

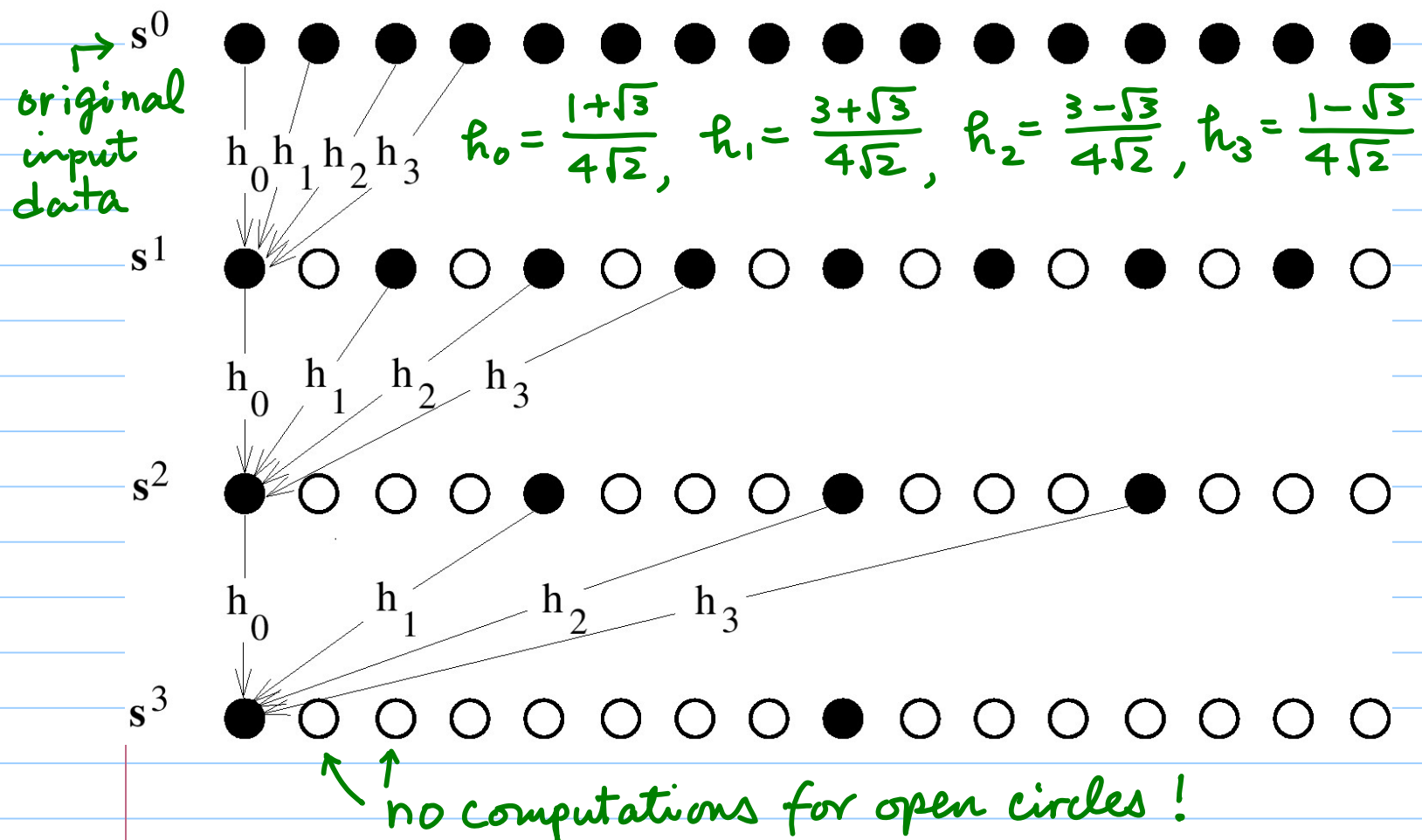
$$= \left(\overset{\vee}{S}_\cdot^{j+1} * h \right)_k + \left(\overset{\vee}{d}_\cdot^{j+1} * g \right)_k$$

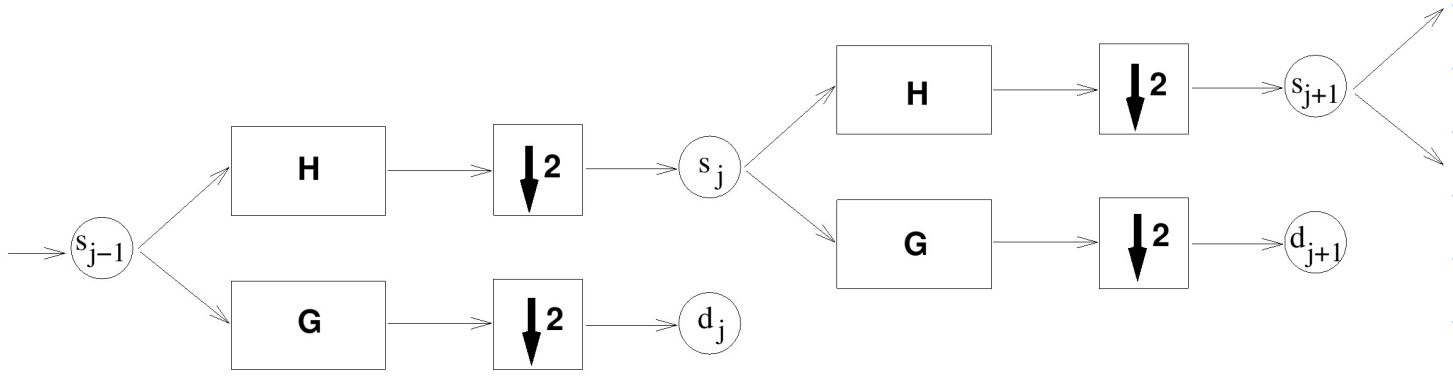
where $\overset{\vee}{\cdot}$ is an **up sampling** operation (with D_s):
 for $\{x_l\}_{l \in \mathbb{Z}}$, $\overset{\vee}{x}_l := \begin{cases} x_k & \text{if } l=2k \\ 0 & \text{if } l=2k+1 \end{cases}$.



Note that for compactly supported wavelets, only finite numbers of $\{h_k\}$, $\{g_k\}$ are nonzeros.

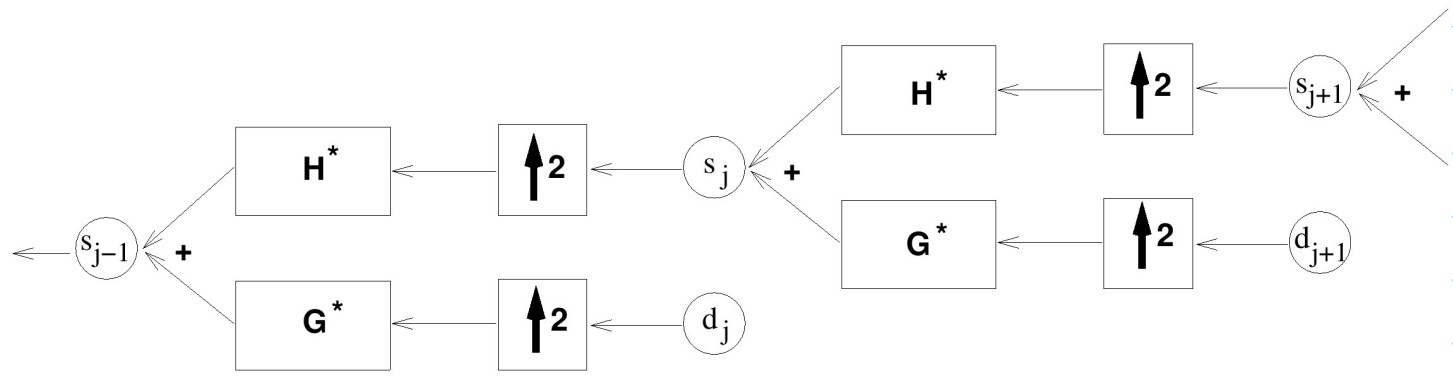
Ex. Daubechies's wavelet $p=2$





Decomposition by QMFs

Quadrature Mirror Filters
 \supset CMF



Reconstruction by QMFs

• Computational Complexity

Recall
 FFT's
 cost
 $O(N \log N)$

If $|\text{supp } h| = |\text{supp } g| = K$ (taps),
 then the cost for the forward/inverse transf.
 is at most $2KN$, i.e., $O(KN)$ or
 even you can say $O(N)$.

(Proof of the Thm)

Since $\phi_{j+1, l} \in V_{j+1} \subset V_j$,

$$(*) \quad \phi_{j+1, k} = \sum_{l \in \mathbb{Z}} \langle \phi_{j+1, k}, \phi_{j, l} \rangle \phi_{j, l}$$

$$\begin{aligned} \langle \phi_{j+1, k}, \phi_{j, l} \rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2^{j+1}}} \phi\left(\frac{x-2^{j+1}k}{2^{j+1}}\right) \overline{\frac{1}{\sqrt{2^j}} \phi\left(\frac{x-2^j l}{2^j}\right)} dx \\ &\stackrel{t=2^{-j}x-k}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right) \overline{\phi(t-l+2k)} dt \\ &= \langle \phi_{1,0}, \phi_{0, l-2k} \rangle = h_{l-2k} \quad (***) \end{aligned}$$

So, (*) is in fact

$$\phi_{j+1, k} = \sum_{l \in \mathbb{Z}} h_{l-2k} \phi_{j, l}$$

$$\begin{aligned} \Rightarrow S_k^{j+1} = \langle f, \phi_{j+1, k} \rangle &= \sum_{l \in \mathbb{Z}} h_{l-2k} \langle f, \phi_{j, l} \rangle \\ &= \sum_{l \in \mathbb{Z}} h_{l-2k} S_l^j \quad \checkmark \end{aligned}$$

Similarly, it's easy to derive

$$d_k^{j+1} = \sum_{l \in \mathbb{Z}} g_{l-2k} S_l^j \quad \checkmark$$

As for the inverse transf., note that

$$\begin{aligned} V_{j+1} \oplus W_{j+1} &= V_j \\ \text{Hence } \phi_{j, k} &= \sum_{l \in \mathbb{Z}} \langle \phi_{j, k}, \phi_{j+1, l} \rangle \phi_{j+1, l} \\ &\quad + \sum_{l \in \mathbb{Z}} \langle \phi_{j, k}, \psi_{j+1, l} \rangle \psi_{j+1, l}. \\ (***) \quad &\downarrow \\ h_k, g_k \in \mathbb{R} &\downarrow \\ &= \sum_l h_{k-2l} \phi_{j+1, l} + \sum_l \bar{g}_{k-2l} \psi_{j+1, l} \\ &= \sum_l h_{k-2l} \phi_{j+1, l} + \sum_l g_{k-2l} \psi_{j+1, l} \quad \equiv \equiv \equiv \end{aligned}$$

★ Other potential problems of fast discrete wavelet transforms with compactly supported wavelets

- Boundary treatment
- Lack of translation invariance
- Lack of symmetry/antisymmetry
- Lack of high frequency resolution
- Lack of orientation sensitivity in 2D & higher

(1) Boundary treatment

DWT requires information of the outside of the input signal $f = [f_0, \dots, f_{N-1}]^T$, i.e., needs f_j for some $j < 0$ and $j \geq N$, due to the convolution operations with $\{h_k\}$ & $\{g_k\}$.

Possible solutions:

- Periodize f
 - \Rightarrow creates artificial discontinuity because in general, the head and tail of f may be quite different.
 - \Rightarrow creates large wavelet coeff's, i.e., no good although it's easy to implement
- **Even-reflect** f at the boundary
 - \Rightarrow no artificial discontin., recommended!
- Design the "boundary" wavelets, i.e., use different ϕ & ψ toward the boundary (Cohen, Daubechies, Vial, 1993)
 - \Rightarrow Great, but cumbersome to implement.

(2) Lack of translation invariance

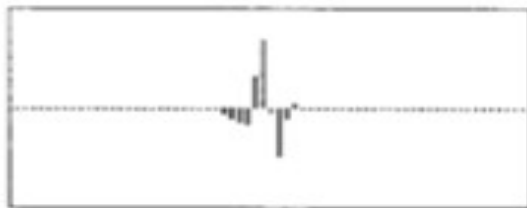
Due to the subsampling operations in DWT, the wavelet coef's of f and those of the shifted version of f are completely different, i.e., they are very sensitive to translations of an input signal.

It's quite a contrast to DFT where a translation amounts to a simple phase factor, i.e., $D_N[\tau_l f](k) = \omega_N^{-kl} D_N[f]$.

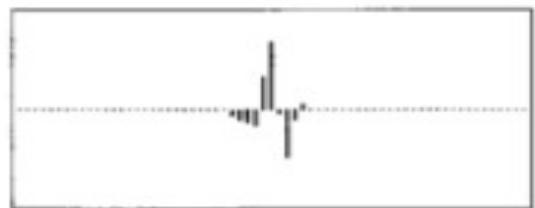
an input signal

a shifted input signal

S_k^0



(a)

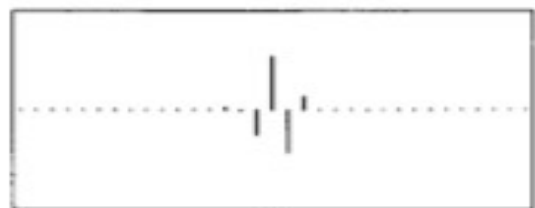


(e)

d_k^1



(b)



(f)

d_k^2



(c)

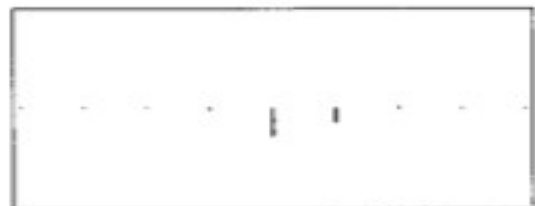


(g)

d_k^3



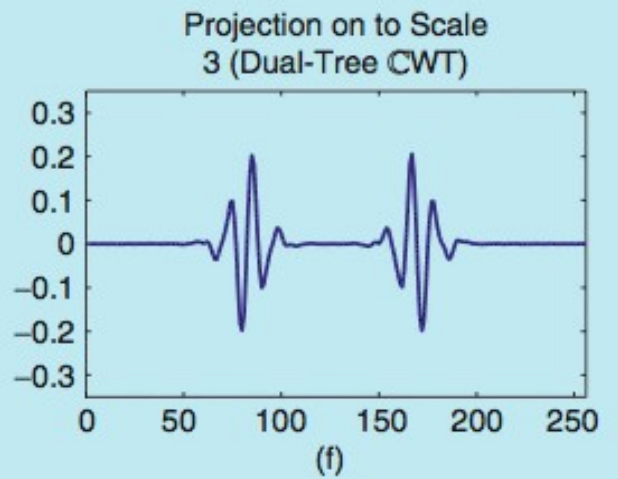
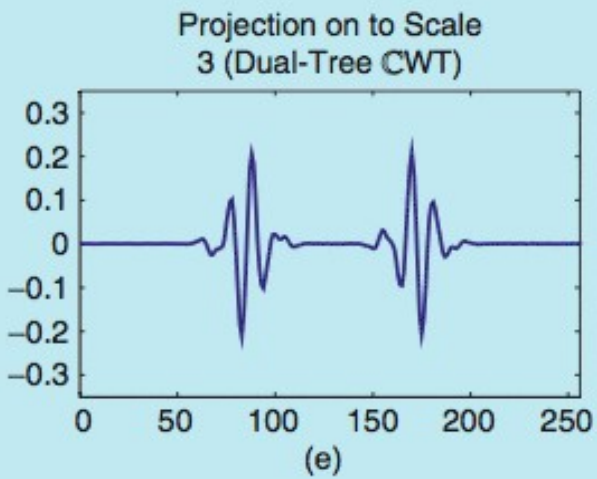
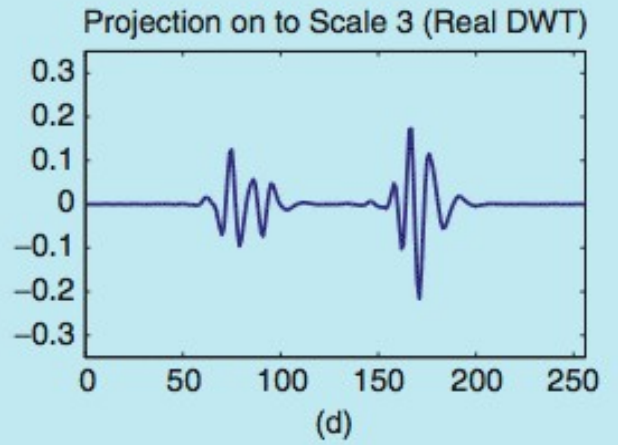
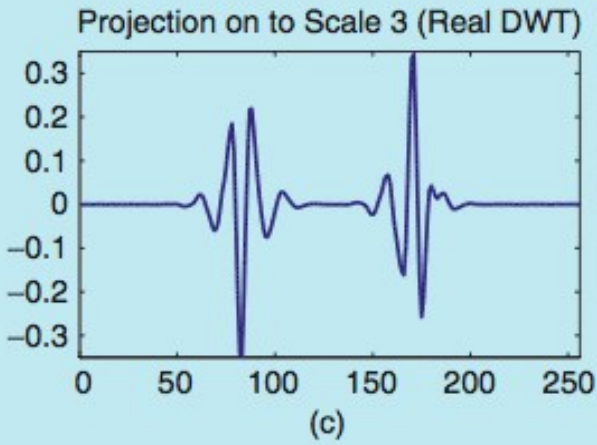
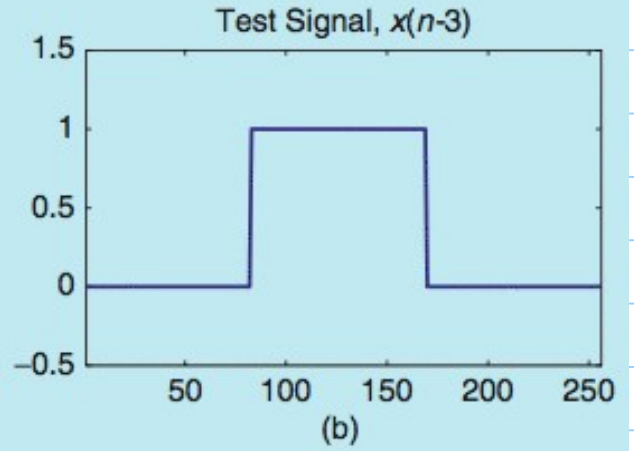
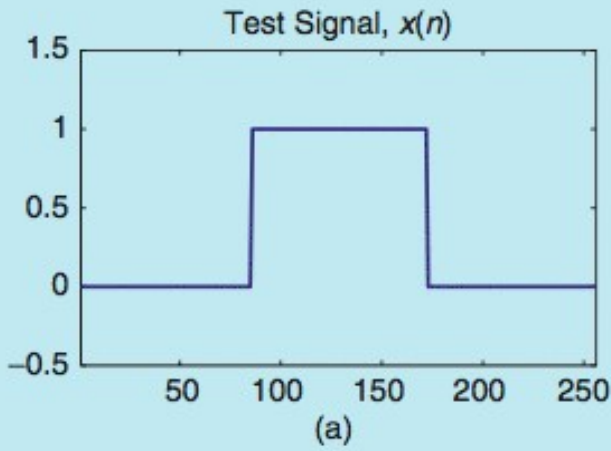
(d)



(h)

Possible solutions:

- Abandon the basis (non redundancy) and use the special frame (stationary wavelet transform) \Rightarrow no subsampling at each level.
Beylkin (1992), Nason & Silverman (1995)
Redundancy factor: $J+1$ where $J = \#$ levels scales
- Abandon the exact translation invariance but shoot for near trans. invariance in the magnitude of the wavelet coef's.
 \Rightarrow Shiftable multiscale transf.
Simoncelli, Freeman, Adelson, & Heeger (1992)
Here, the energy of each subspace is trans. inv.
They also developed such 'shiftability' in orientation & scale for 2D transf.
It's a tight frame with redundancy factor $\propto \#$ orientations $\times 4/3$
 \Rightarrow Dual-tree complex wavelet transf. (DWT)
Kingsbury (2001), Selesnick, Baramik & Kingsbury (2005). Can have some oriented basis fcn's and near translation invariance.
Redundancy factor: 2^d $d=1$ for 1D signal
 $= 2$ for 2D images.



Daubchies
wavelets
with $p=7$
→

(3) Lack of symmetry/antisymmetry

ϕ & ψ of Daubechies's cannot have symmetry/antisymmetry for $p > 1$.

$\left\{ \begin{array}{l} p=1 \Rightarrow \text{Haar, so } \phi: \text{symmetric, } \psi: \text{antisymmetric} \\ p \rightarrow \infty \Rightarrow \text{Shannon, so both } \phi \& \psi: \text{symmetric} \\ \text{but not compactly supported!} \end{array} \right.$

The source of the problem is the difficulty in finding symmetric/antisymmetric CMF coef's $\{h_k\}$ of finite taps.

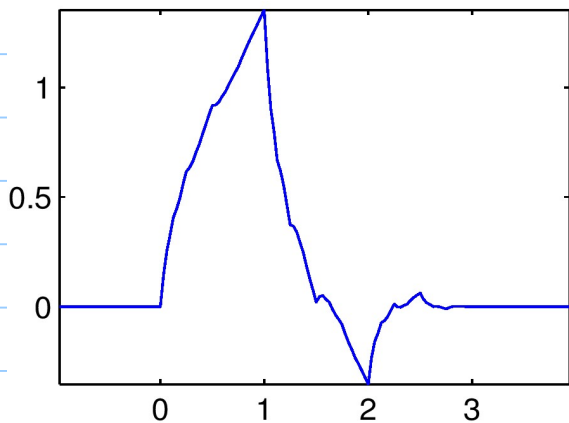
Possible solutions:

• Abandon true symmetry/antisymmetry and seek near **linear phase** CMF $\{h_k\}$.

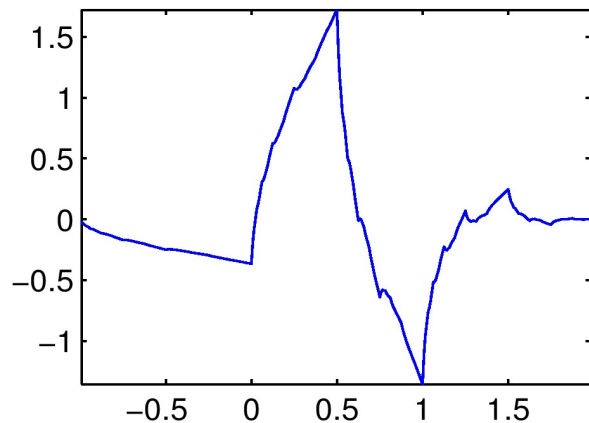
If $\{h_k\}$ is symmetric at $k=l$, say, $h_k = h_{2l-k}$, $k \in \mathbb{Z}$, then we can show $\hat{h}(\xi) = e^{-\underbrace{2\pi i l \xi}_{\text{linear phase}}} |\hat{h}(\xi)|$

To allow symmetry at half integers, we need to extend the definition of linear phase by including piecewise linear phase with constant slope whose discontinuities occur only at zeros of $|\hat{h}(\xi)|$ (e.g., the Haar case). Daubechies (1990) found a way to optimize the choice of $\{h_k\}$ to have almost linear phase with $\text{supp } h = [-p+1, p]$.
 \Rightarrow 'Symmlets'

Father

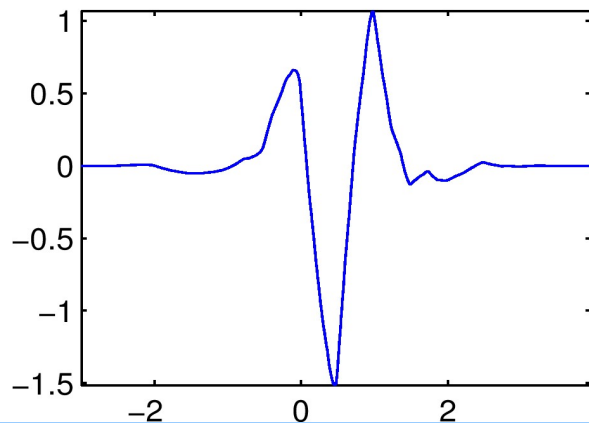
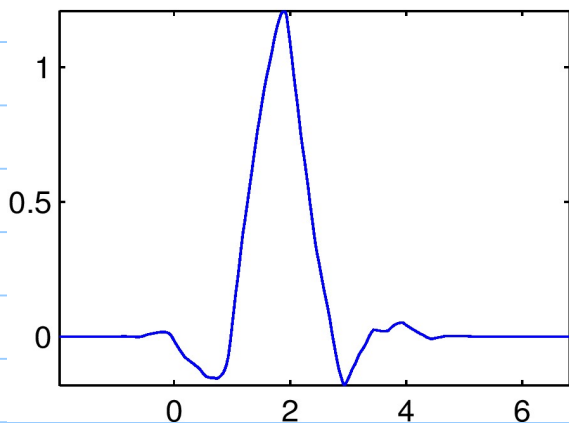


Mother

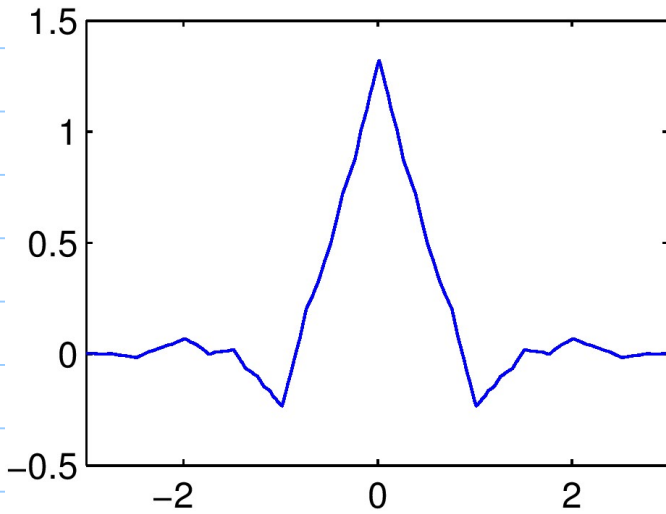
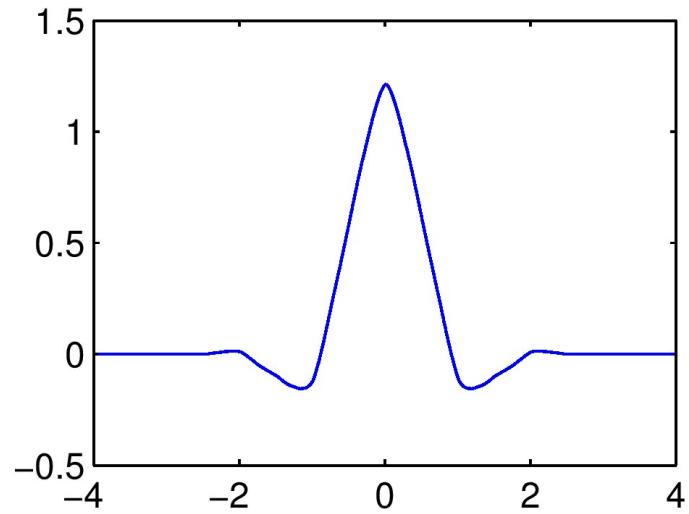
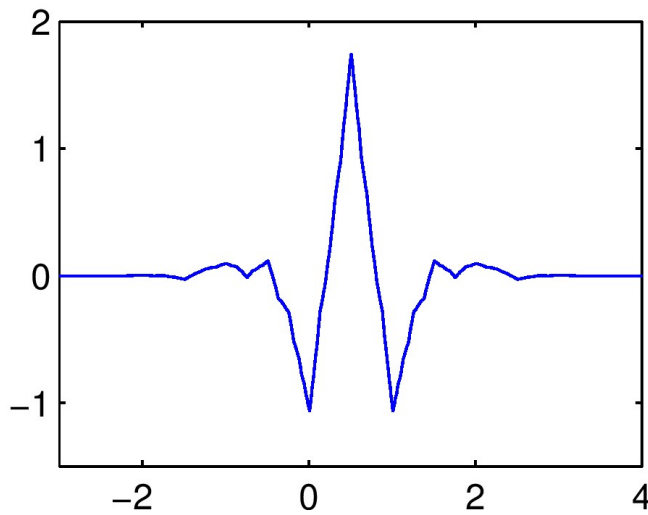
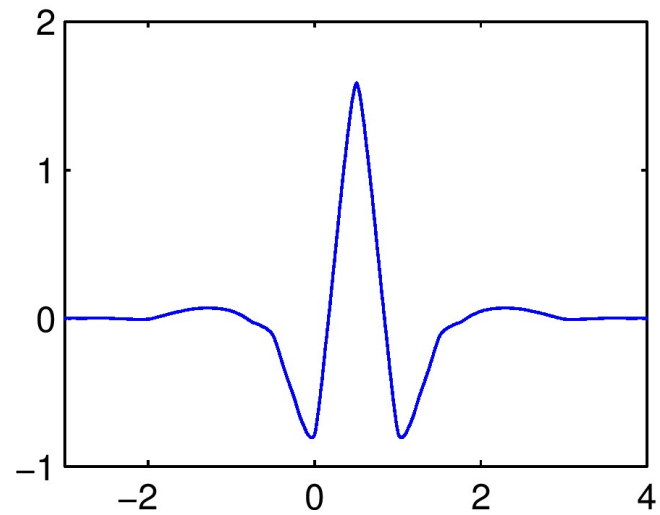


Default Daubechies's wavelets $p=2$

Symmlets $p=2$



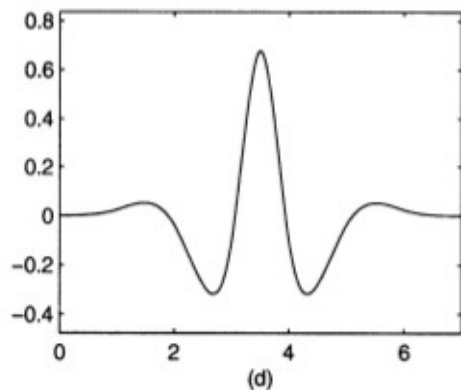
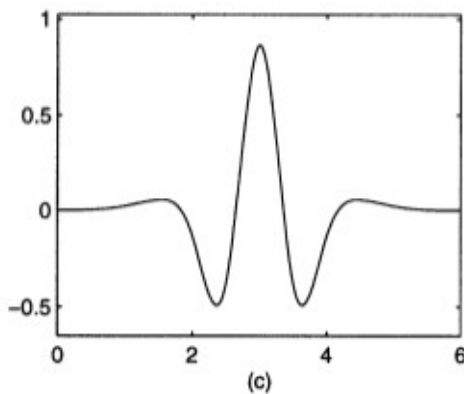
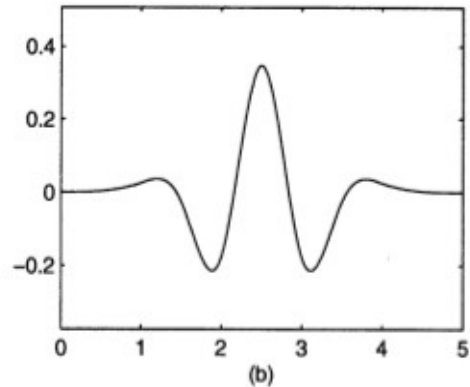
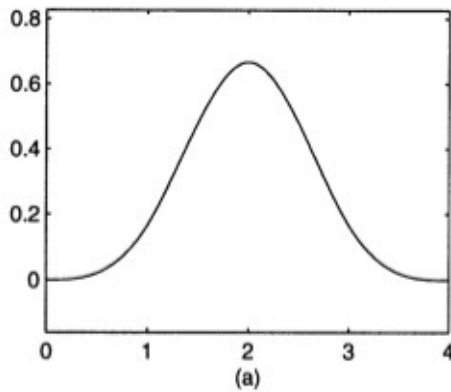
- Abandon the orthogonality for symmetry \Rightarrow Biorthogonal wavelet bases
 Cohen, Daubechies, & Feauveau (1992)
 Needs to use two sets of families
 $\{ \phi_{j,k}, \psi_{j,k} \}$ for analysis and
 $\{ \tilde{\phi}_{j,k}, \tilde{\psi}_{j,k} \}$ for synthesis (or vice versa)
 Quite flexible in terms of filter design,
 e.g., vanishing moments for ψ & $\tilde{\psi}$ can
 be different as well as their support.
 JPEG 2000 standard recommends
 the following biorthogonal wavelets:

$\phi(x)$  $\tilde{\phi}(x)$  $\psi(x)$  $\tilde{\psi}(x)$ 

They are referred to as "9/7" biorthogonal wavelets since $|\text{supp } h| = 9$, $|\text{supp } \tilde{h}| = 7$.

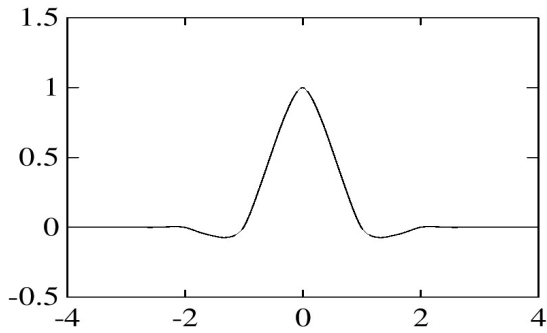
- Use frame, e.g., use more than one pair of father & mother wavelets \Rightarrow wavelet frames (framelets)
 Ron & Shen (1997),
 Benedetto & Li (1998),
 Daubechies, Han, Ron, & Shen (2003)
 and many others ...

Tight framelets system's father and 3 mother wavelets ($p=4$)

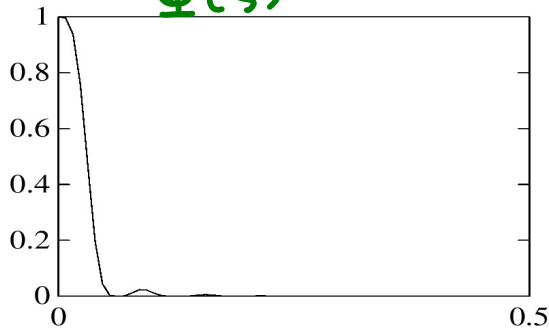


- Use the **autocorrelations** of father & mother wavelets of Daubechies
 - \Rightarrow **Auto correlation shell**
 - Saito & Beylkin (1993)
 - This is a special frame, i.e., redundant, nonorthogonal, but translation invariant & symmetric
- | | | | |
|---|------------------------|--------------------------------|---------------------|
| { | $p=1$ | : autocorr of boxcar = hat fn | → linear interp. |
| | $p=2$ | : " of Daub. 4 = Delcouriers - | Dubuc interpolation |
| | $p \rightarrow \infty$ | : " of sinc = sinc | → BL interp. |

$$\Phi(x) = \phi * \tilde{\phi}(x)$$

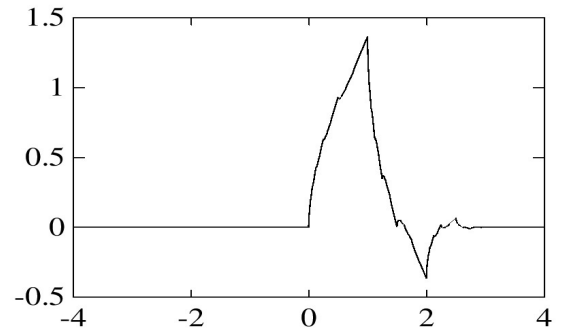


$$\hat{\Phi}(\xi)$$

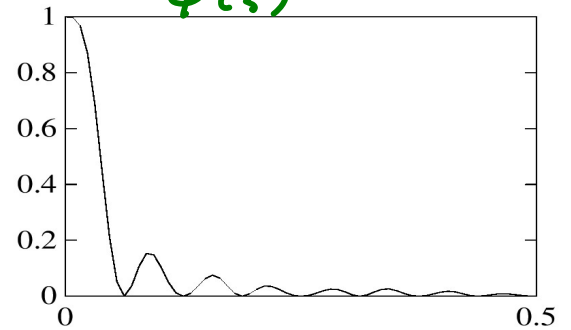


(c)

$$\phi(x)$$

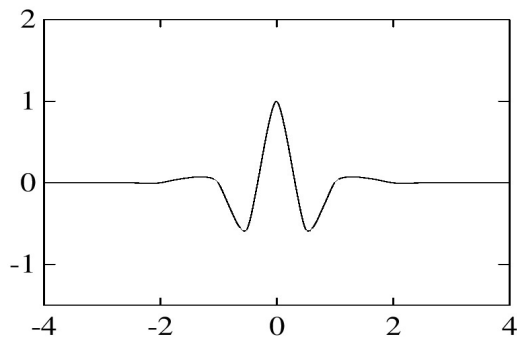


$$\hat{\phi}(\xi)$$

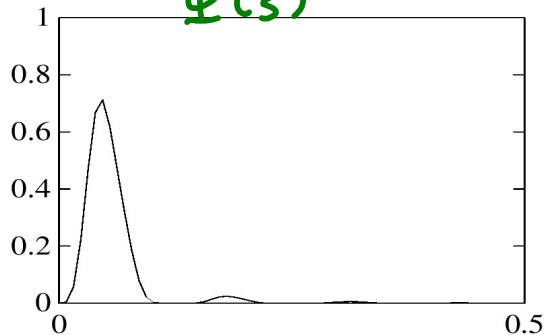


(d)

$$\Psi(x) = \psi * \tilde{\psi}(x)$$

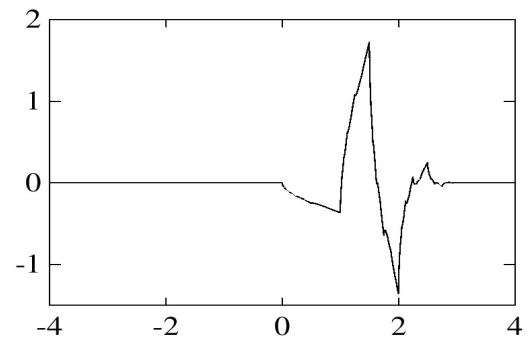


$$\hat{\Psi}(\xi)$$

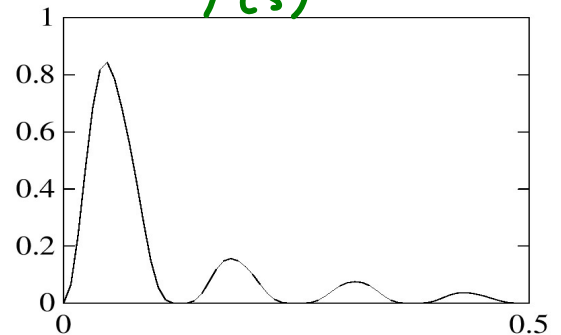


(c)

$$\psi(x)$$

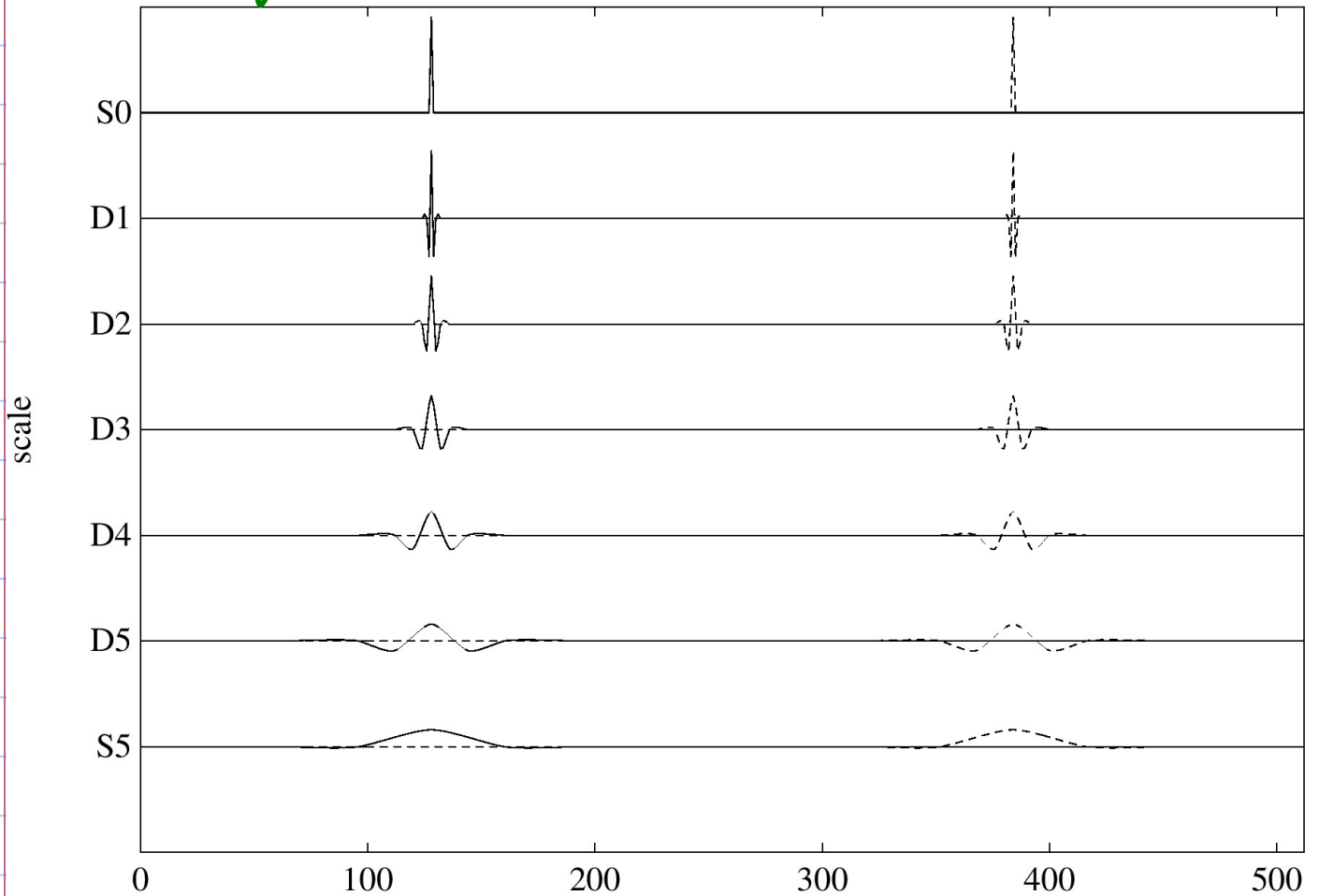


$$\hat{\psi}(\xi)$$

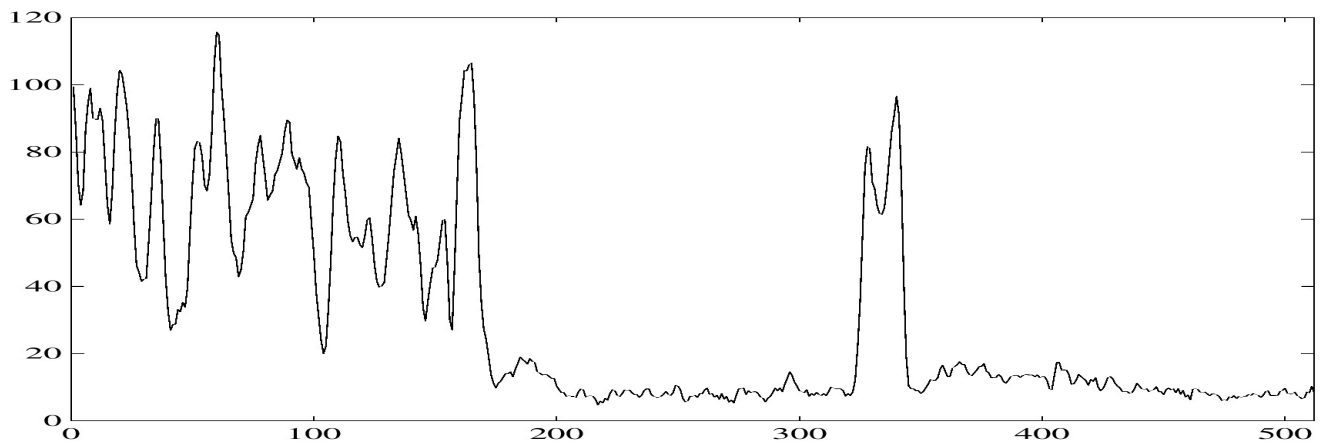


(d)

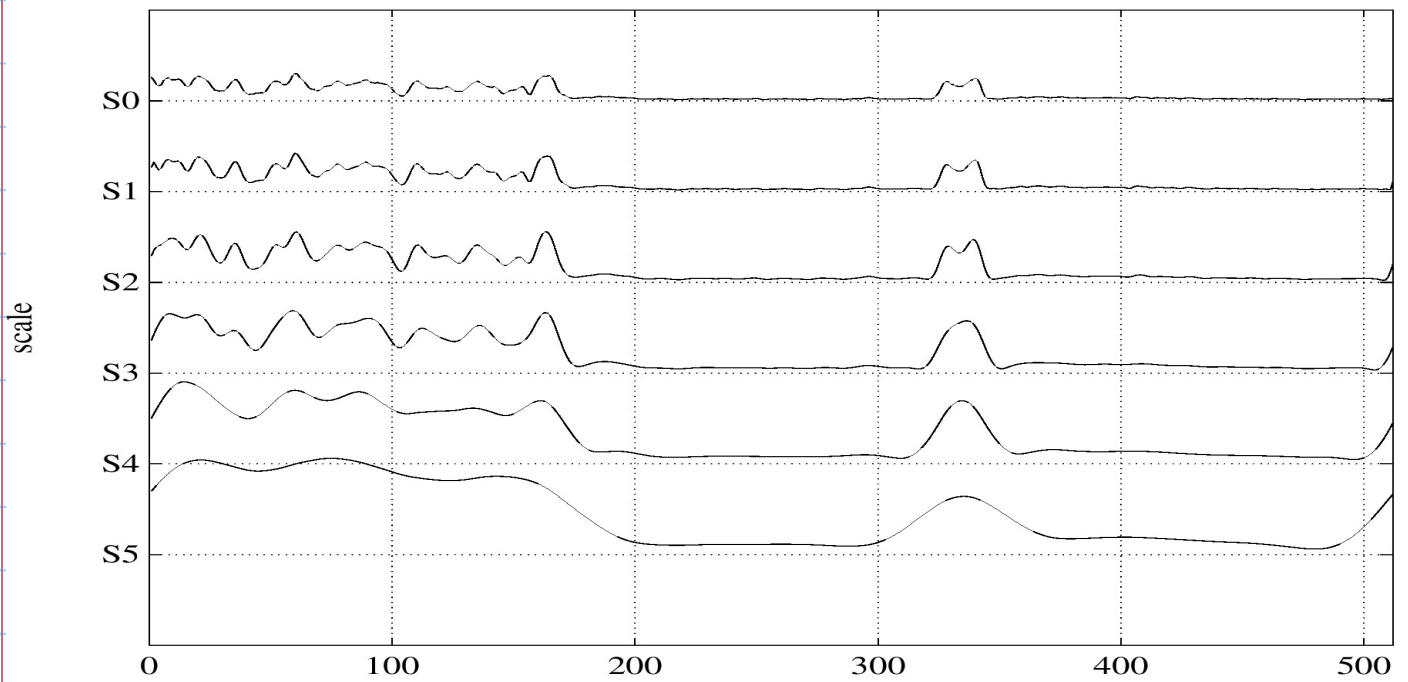
Demonstration of translation invariance of Auto correlation Shell



Original Signal to be decomposed:



multiscale Averages in Autocorr. Shell



multiscale Differences in Autocorr. Shell

