Basics of Analytic Signals

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Motivation

- Many natural and man-made signals exhibit *time-varying frequencies* (e.g., chirps, FM radio waves).
- Characterization and analysis of such a signal, $u(t)$, based on *instantaneous amplitude* $a(t)$, *instantaneous phase* $\phi(t)$, and *instantaneous frequency* $\omega(t) := \phi'(t)$, are very important:

$$u(t) = a(t)\cos\phi(t).$$

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Analytic Signal

- It is convenient to use a complexified version of the signal whose real part is a given real-valued signal $u(t)$.
- Given $u(t)$, however, there are infinitely many ways to define the instantaneous amplitude and phase (IAP) pairs so that

$$u(t) = a(t)\cos\phi(t).$$

- This is due to the arbitrariness of the complexified version of u, i.e.,

$$f(t) = u(t) + \mathrm{i}v(t)$$

where $v(t)$ is an arbitrary real-valued signal; yet this yields the IAP representation of $u(t)$ via

$$a(t) = \sqrt{u^2(t) + v^2(t)}, \quad \phi(t) = \arctan\frac{v(t)}{u(t)}.$$

- The *instantaneous frequency* is defined as

$$\omega(t) := \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{u(t)v'(t) - u'(t)v(t)}{u^2(t) + v^2(t)}.$$

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Analytic Signal

- Gabor (1946) proposed to use the *the Hilbert transform* of $u(t)$ as $v(t)$, and called the complex-valued $f(t)$ an *analytic signal*.
- Vakman (1972) proved that $v(t)$ must be of the Hilbert transform of $u(t)$ if we impose some a priori physical assumptions:
  1. $v(t)$ must be derived from $u(t)$.
  2. Amplitude continuity: a small change in $u \Longrightarrow$ a small change in $a(t)$.
  3. Phase independence of scale: if $cu(t)$, $c \in \mathbb{R}$ arbitrary scalar, then the phase does not change from that of $u(t)$ and its amplitude becomes c times that of $u(t)$.
  4. Harmonic correspondence: if $u(t) = a_0 \cos(\omega_0 t + \phi_0)$, then $a(t) \equiv a_0$, $\phi(t) \equiv \omega_0 t + \phi_0$.

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Analytic Signal . . .

- For simplicity, we assume that our signals are 2π-periodic in $\theta \in [-\pi, \pi)$.
- Hence, we work on the unit circle and unit disk $\mathbb{D}$ in $\mathbb{C} = \mathbb{R}^2$.
- Note that the signals over $\mathbb{R} = (-\infty, \infty)$ can be treated similarly by considering the real axis and the upper half plane of $\mathbb{C}$.
- The analytic signal of a given signal $u(\theta) \in \mathbb{R}$ is often and simply obtained via the *Hilbert transform*:

$$f(\theta) = u(\theta) + \mathrm{i}\mathscr{H}u(\theta), \quad \mathscr{H}u(\theta) := \frac{1}{2\pi}\,\mathrm{pv}\int_{-\pi}^{\pi} u(\tau)\cot\frac{\theta-\tau}{2}\,\mathrm{d}\tau.$$

- Note that

$$u(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k \cos k\theta + b_k \sin k\theta) \Rightarrow \mathscr{H}u(\theta) = \sum_{k\geq 1}(a_k \sin k\theta - b_k \cos k\theta).$$

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Analytic Signal . . .

We can gain a deeper insight by viewing this as *the boundary value* of an *analytic function* $F(z)$ where

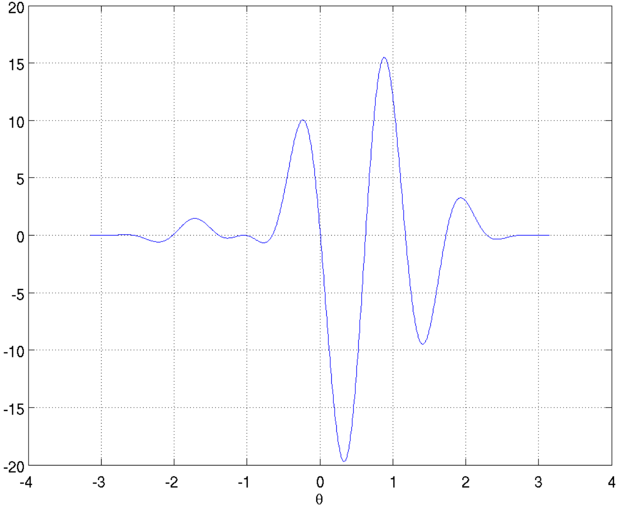
$$F(z) := U(z) + \mathrm{i}\widetilde{U}(z), \quad z \in \mathbb{D},$$

where

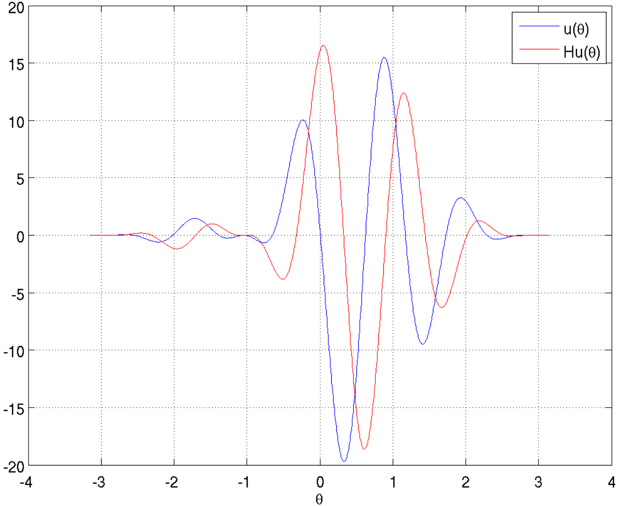
$$U(z) = U\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = P_r * u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\tau)+r^2}\, u(\tau)\,\mathrm{d}\tau,$$
$$\widetilde{U}(z) = \widetilde{U}\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = Q_r * u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r\sin(\theta-\tau)}{1-2r\cos(\theta-\tau)+r^2}\, u(\tau)\,\mathrm{d}\tau.$$

In other words, the original signal $u(\theta) = U\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ is *the boundary value of the harmonic function U on $\partial\mathbb{D}$*, which is constructed by *the Poisson integral*. $\widetilde{U}$ and $Q_r(\theta)$ are referred to as the conjugate harmonic function and the conjugate Poisson kernel, respectively.

Analytic Signal . . . An Example: $u(\theta)$



Analytic Signal . . . An Example: $u(\theta)$ and $\mathscr{H}u(\theta)$



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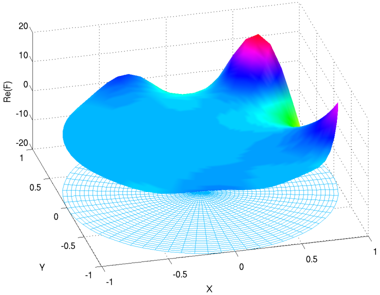
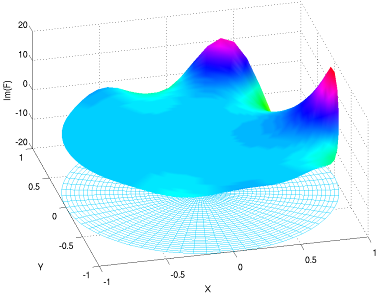


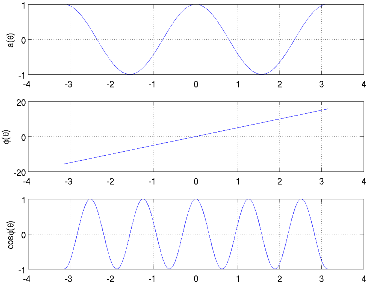
(a) $U(z)$



(b) $\widetilde{U}(z)$

Even if we use the analytic signal, its IAP representation is not unique as shown by Cohen, Loughlin, and Vakman (1999):

- $f(\theta) = a(\theta)e^{i\phi(\theta)}$, where $a(\theta) = u(\theta)\cos\phi(\theta) + v(\theta)\sin\phi(\theta)$ may be negative though $\phi(\theta)$ is continuous;
- $f(\theta) = |a(\theta)|e^{i(\phi(\theta)+\pi\alpha(\theta))}$, where $\alpha(\theta)$ is an appropriate phase function, which may be discontinuous.



(a) Continuous phase

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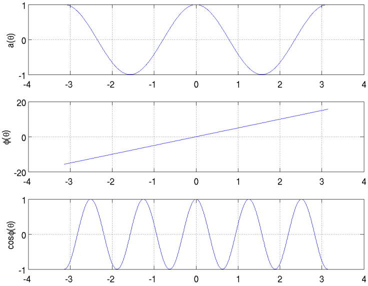
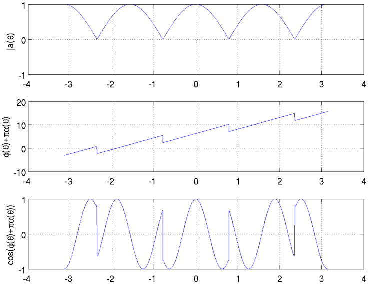


(a) Continuous phase



(b) Nonnegative amplitude