Lecture 2: Basics of Fourier Transforms

Note Title

We will mainly focus on 1D signals.

1. Fourier Transform on $L^1(\mathbb{R})$

Let $f \in L^1(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \mid \|f\|_1 < \infty \}$

where $\|f\|_1 := \int_{-\infty}^{\infty} |f(x)|\,dx$ is called the L^1-norm of f. This integral is that of Lebesgue

Def. The Fourier transform of $f \in L^1(\mathbb{R})$ is the function $\hat{f}$ on $\mathbb{R}$ defined by

$$(*) \qquad \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

We also write $\mathcal{F}[f](\xi) = \hat{f}(\xi)$

(Note that in $\mathbb{R}^d$, the definition becomes

$$\hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} f(\mathbf{x})\, e^{-2\pi i \langle \boldsymbol{\xi}, \mathbf{x} \rangle}\, d\mathbf{x}.$$

the standard inner product in $\mathbb{R}^d$)

Now, $(*)$ is well defined, i.e., abs. conv.:

$$|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)|\,dx = \|f\|_1 < \infty$$

since $\left| e^{-2\pi i \xi x} \right| = 1$. //

In fact, $\mathcal{F} : L^1(\mathbb{R}) \longrightarrow BC(\mathbb{R})$

$$= C(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

* Basic Properties of the Fourier transf

Def. For any $a \in \mathbb{R}$, the translation operator τ_a is defined by

$$\tau_a f(x) := f(x-a)$$

Def. For any $s > 0$, the dilation operator δ_s is defined by

$$\delta_s f(x) := \frac{1}{\sqrt{s}} f\left(\frac{x}{s}\right)$$

This factor guarantees that δ_s is an isometry in $L^2(\mathbb{R})$

i.e., $\|f\|_2 = \|\delta_s f\|_2$

We'll do more of L^2 shortly.

$$\|f\|_2 := \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}$$

Def. The convolution of $f, g \in L^1$, is defined as

$$f * g(x) := \int f(x-y) g(y)\, dy$$

Note: $f * g = g * f$

$(f*g)*h = f*(g*h)$, etc.

Thm 1. Let $f, g \in L^1$. Then,

(a) $\mathcal{F}[\tau_a f] = \underbrace{e^{-2\pi i \xi a}}\hat{f}(\xi)$

$\mathcal{F}[\underbrace{e^{2\pi i a x}}_{\to \text{modulation}} f(x)] = \tau_a \hat{f}(\xi) = \hat{f}(\xi - a)$

(b) $\mathcal{F}[\delta_s f] = \sqrt{s}\,\hat{f}(s\xi) = \delta_{1/s}\hat{f}(\xi)$

clearly, $\mathcal{F}[\delta_{1/s} f] = \delta_s \hat{f}(\xi)$

(c) $\mathcal{F}[f * g] = \hat{f}\,\hat{g}$

$\mathcal{F}[f g] = \hat{f} * \hat{g}$

(d) Let $\partial_x := \frac{\partial}{\partial x}$

If $f \in C^k$ and $\partial_x^j f \in L^1$, $j = 1, \cdots, k$

and $\partial_x^j f \in C_0$, $j = 1, \cdots, k-1$,

C_0 ↳ continuous & vanishes at $\pm\infty$ ⇒ used in integration by parts!

then

$\mathcal{F}[\partial_x^k f] = (2\pi i \xi)^k \hat{f}(\xi)$

(e) On the other hand,

if $x^j f \in L^1$, $j = 1, \cdots, k$,

then $\hat{f} \in C^k$ and $\mathcal{F}[x^k f(x)] = \left(\frac{i}{2\pi}\right)^k \partial_\xi^k \hat{f}(\xi)$

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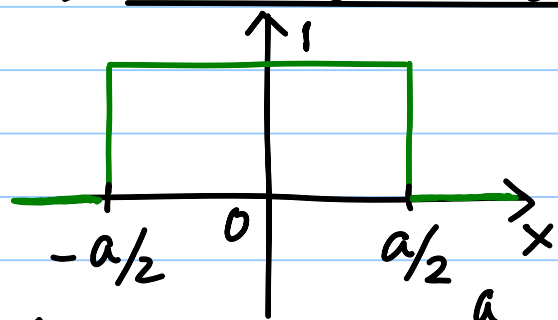
* Important Examples

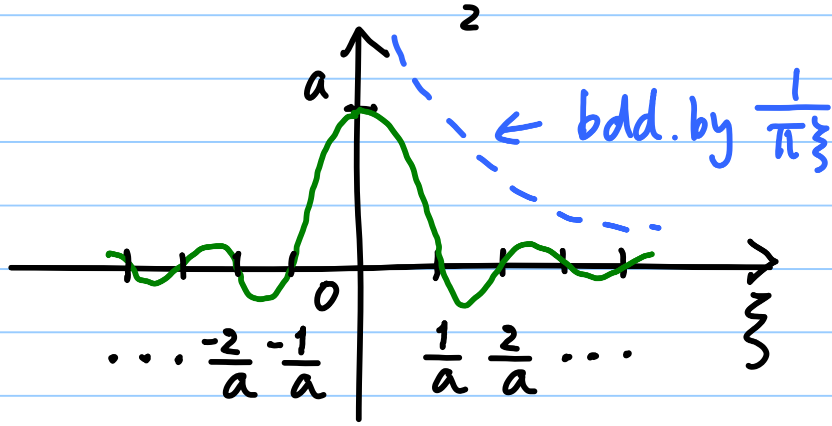
(1) The characteristic (indicator) fcn



$$\chi_{(-\frac{a}{2},\frac{a}{2})}(x) := \begin{cases} 1 & \text{if } |x| < \frac{a}{2} \\ 0 & \text{o.w.} \end{cases}$$

$$\hat{\chi}_{(-\frac{a}{2},\frac{a}{2})}(\xi) = \int_{-\frac{a}{2}}^{\frac{a}{2}} 1 \cdot e^{-2\pi i \xi x}\, dx = \frac{e^{-\pi i a \xi} - e^{\pi i a \xi}}{-2\pi i \xi}$$

$$= \underbrace{\frac{\sin \pi a \xi}{\pi \xi}}$$



zeros are at $\xi = n/a$, $n \in \mathbb{Z} - \{0\}$.

If we define $\text{sinc}(x) := \dfrac{\sin \pi x}{\pi x}$,

then we can write

$$\hat{\chi}_{(-\frac{a}{2},\frac{a}{2})}(\xi) = \underbrace{a \,\text{sinc}(a\xi)}$$

Note that this fcn is not in L^1 but in L^2!

$\because \displaystyle\int_{-\infty}^{\infty} \left| \frac{\sin \pi a \xi}{\pi \xi} \right| d\xi = +\infty$ but $\displaystyle\int_{-\infty}^{\infty} \left| \frac{\sin \pi a \xi}{\pi \xi} \right|^2 d\xi = a < \infty$ //

The details are left as an exercise.

(2) The Gaussian fcn

$$g(x;\sigma) = g_\sigma(x) := \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad \sigma > 0, \; x \in \mathbb{R}$$

How to compute $\hat{g}_\sigma(\xi)$?

$\Rightarrow$ HW problem! $\exists$ a neat way.

$$\hat{g}_\sigma(\xi) = e^{-2\pi^2\sigma^2\xi^2}$$

Incidentally, in general, we have

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i 0 \cdot x} dx = \int_{-\infty}^{\infty} f(x) dx$$

$=$ The DC component of $f(x)$

EE terminology

Hence, $\hat{g}_\sigma(0) = 1$, i.e., $g_\sigma(x)$ is a probability density fcn (pdf)

Also note that if $\sigma = 1/\sqrt{2\pi}$, then

$$e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi \xi^2}$$

invariant w.r.t. $\mathcal{F}$!

* The Riemann-Lebesgue Lemma

If $f \in L^1$, then $\hat{f}(\xi) \to 0$ as $\xi \to \pm\infty$

(i.e., $\mathcal{F}[L^1(\mathbb{R})] \subset C_0(\mathbb{R})$)

" If $f \in L^1$, very high frequency components vanish. " (non rigorous statement !)

* The Fourier Inversion Thm

states a procedure to recover f from $\hat{f}$.

Want to have

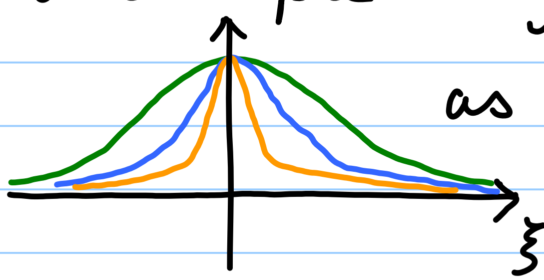
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

But the problem is :

$\hat{f}$ may not be in L^1. e.g., sinc $\notin L^1$

Hence, we use the " cut off " fcn

a good example is $\hat{g}_a(\xi) = e^{-2\pi^2 a^2 \xi^2}$



as $a \uparrow$, more localized

Consider

$$(*) \quad \int \hat{f}(\xi)\, \hat{g}_\sigma(\xi)\, e^{2\pi i \xi x}\, d\xi$$

$$= \iint f(y)\, e^{-2\pi i \xi y}\, dy\, e^{-2\pi^2\sigma^2\xi^2} e^{2\pi i \xi x}\, d\xi$$

OK by Fubini

$$\stackrel{\downarrow}{=} \iint f(y)\, e^{2\pi i \xi (x-y)}\, e^{-2\pi^2\sigma^2\xi^2}\, d\xi\, dy$$

Now $\displaystyle\int e^{2\pi i \xi (x-y)}\, e^{-2\pi^2\sigma^2\xi^2}\, d\xi$

$$= \mathcal{F}_\xi\left[e^{-2\pi^2\sigma^2\xi^2}\right](y-x)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(y-x)^2}{2\sigma^2}} = g_\sigma(y-x)$$

$$= g_\sigma(x-y)$$

So $(*) = \displaystyle\int f(y)\, g_\sigma(x-y)\, dy = f * g_\sigma(x)$

We can show that

$$\| f * g_\sigma - f \|_1 \to 0 \text{ as } \sigma \to 0$$

and $\displaystyle\lim_{\sigma\to 0} f * g_\sigma = f$ a.e.

almost everywhere

So, if $\hat{f} \in L^1$, then things become easy!

Def. Define $\check{f}(x) := \mathcal{F}^{-1}[f(\xi)](x)$

$$= \int_{-\infty}^{\infty} f(\xi)\, e^{2\pi i \xi x}\, d\xi$$

This is the inverse Fourier transf. of $f \in L^1$

Thm 2. If f & $\hat{f}$ are both in L^1,
then $(\hat{f})^{\vee} = (\check{f})^{\wedge} = f$ a.e.

Cor. If $\hat{f} = \hat{g}$, then $f = g$ a.e.
$\hat{f} = 0$, $f = 0$ a.e.

- many fcns in L^1 has their F.T. in L^1, but not all of the L^1 fcns have $\hat{f} \in L^1$.

For $f \in L^1$ to have $\hat{f} \in L^1$, f must be a little smooth.

$\Rightarrow$ $\hat{f}(\xi)$ must decay as $\xi \uparrow$.

e.g., if $|\hat{f}(\xi)| \le \dfrac{C}{1+\xi^2}$ for $\exists C > 0$.
then $\hat{f} \in L^1$ (easy to show.)

If $f \in C^2$ & $f', f'' \in L^1$, then $\hat{f}$ decays as above.

"Smoothness of a fcn
$\Longleftrightarrow$ Decay of the Fourier transf. at high freq."

In particular, if $\hat{f}$ has a compact support, then it's great!
Such f is called a band-limited fcn.

2. The Fourier Transform on L^2

So far, we have dealt with FT on L^1.
Simply assuming $f \in L^2(\mathbb{R})$,

$\int f(x) e^{-2\pi i \xi x} dx$ may not converge.

e.g., $f(x) = \operatorname{sinc}(x) \in L^2$

but it is not in L^1.

- We will overcome this problem as follows.

Define a subspace $\mathcal{X} \subset L^1$ s.t.

$$\mathcal{X} := \{ f \in L^1 \mid \hat{f} \in L^1 \}$$

For any $f \in \mathcal{X}$, both f & $\hat{f}$ are in $BC(\mathbb{R})$
So, $\mathcal{X} \subset L^2$ (because $f \in L^1 \cap BC$
$\Rightarrow f \in L^2$, which comes from the general

Thm: $L^p \cap L^r \subset L^q$ for

$0 < p < q < r \leq \infty$.

set $p=1, q=2, r=\infty$.

can be proved via Hölder's ineq.)

This general thm also implies

$$\|f\|_q \leq \|f\|_p \vee \|f\|_r$$
$$= \max(\|f\|_p, \|f\|_r).$$

Also, you can show that $\mathcal{X}$ is dense in L^2.

- Hence, for any $f \in L^2$, we can find a sequence $\{f_n\} \subset \mathcal{X}$ s.t. $\|f_n - f\|_2 \xrightarrow[n\to\infty]{} 0$

$\{f_n\} \subset \mathcal{X}$ implies $\{\hat{f}_n\} \subset \mathcal{X}$.

Now, we can show $\|\hat{f}_n - \hat{f}_m\|_2 \overset{\text{the Plancherel eq.}}{=} \|f_n - f_m\|_2 \to 0$ as $n, m \to \infty$

That is, $\{\hat{f}_n\}$ is a Cauchy sequence in L^2. Since L^2 is complete, $\hat{f}_n$ has a limit in L^2, which we define this limit as $\hat{f}$, the Fourier transf. of f.

Parseval is for Fourier series.

* **The Plancherel Thm** ($\mathcal{F}$ is an isometry in L^2)

For any $f, g \in L^2$,

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \text{ hence } \|f\|_2 = \|\hat{f}\|_2$$

- Finally, we can do the following:

Suppose $\phi(x) \in L^2$. Then $\exists\ \hat{f} \in L^2$ s.t. $\phi(x) = \hat{f}(x)$.

Then, $\hat{\phi}(\xi) = f(-\xi)$. Ex: $\phi(x) = \text{sinc}(x) \Rightarrow \hat{\phi}(\xi) = \chi_{(-\frac{1}{2},\frac{1}{2})}(\xi)$.