Lecture 6: { Functions of Bounded Variation / Fourier series on intervals II

Note Title

☆ Functions of Bounded Variations

why are we interested in fcns of BVs?
- often chosen as a model for piecewise smooth signals & images
- useful in data compression & statistical estimation
- Provide sharp info on the decay rate of the Fourier coeff's.

Let $g(x)$ be a fcn on a closed interval $I = [a, b]$. (I could be $\mathbb{R}$).
Let $D :=$ a subdivision of I, i.e.,

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Now, let's form the sum:

$$T_D(g) := \sum_{k=1}^{n} | g(x_k) - g(x_{k-1}) |$$

__Def.__ __If__ $T_D(g) < \infty$ for all possible subdivision D, __then__ g is said to be of **bdd. var.** in I, and the **total variation** of g in I is defined as

$$V_I[g] = V_a^b[g] := \sup_D T_D(g) .$$

$BV(I) :=$ a set of all fcns of bdd. var. in I.

__Fact:__ • $|g(b) - g(a)| \le V_a^b[g] < \infty$. ← Take $x_0 = a$, $x_1 = b$.

• __If__ $I \subset J$, __then__ $V_I[g] \le V_J[g]$

Thm 1. $g \in BV(I) \Rightarrow g$ is bdd. in I.

(Pf) $g(x) = g(a) + g(x) - g(a)$

$$\Rightarrow |g(x)| \le |g(a)| + |g(x) - g(a)|$$
$$\le |g(a)| + V_a^x[g]$$
$$\le |g(a)| + V_a^b[g] < \infty. \quad ///$$

One can also show that $BV(I)$ is a Banach space.

Thm 2. $g, h \in BV(I) \Rightarrow gh \in BV(I)$.

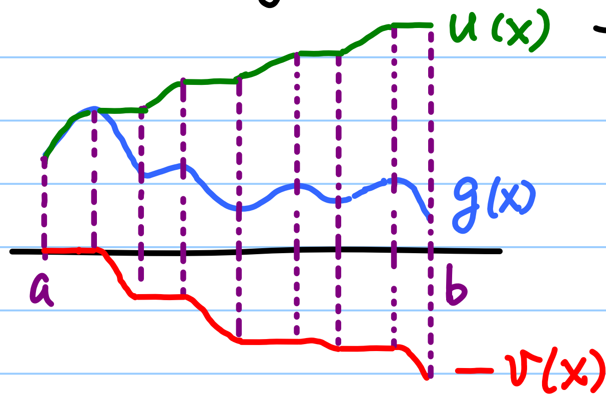
$g, h \in BV(I)$, $|h(x)| \ge {}^{\exists} m > 0$
$\Rightarrow g/h \in BV(I)$.

Thm 3. ${}^{\forall} c \in (a,b)$, $g \in BV[a,b]$
$\Leftrightarrow g \in BV[a,c]$ & $g \in BV[c,b]$.

Moreover, $V_a^b[g] = V_a^c[g] + V_c^b[g]$.

Remark: This can be generalized to
$a < c_1 < c_2 < \cdots < c_n < b$.

Thm. 4 $g \in BV(I) \Leftrightarrow g$ can be written as the difference of two nondecreasing fcns.



say, $u(x) - v(x)$

M. Taibleson's Thm (1967) 1 page paper!

If $f \in BV[0,1]$, $f(x) \sim \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x}$,

then $\alpha_k = O(1/k)$ as $k \to \infty$.

(Pf) Use the fact: $k \neq 0$

$$\int_{j/|k|}^{\frac{j+1}{|k|}} e^{-2\pi i k x}\, dx = 0, \quad j = 0, 1, \cdots, |k|,$$

$$\odot \quad \hookrightarrow = \frac{e^{-2\pi i (j+1)\frac{k}{|k|}} - e^{-2\pi i j \frac{k}{|k|}}}{-2\pi i k}$$

$$= \frac{1}{-2\pi i k}\left(e^{\mp 2\pi i (j+1)} - e^{\mp 2\pi i j}\right) = 0 \quad //$$

Now, fix k, and let $a_j := \frac{j}{|k|}$, $j = 0, 1, \cdots, |k|$.

Then define

$$g(x) := \sum_{j=0}^{|k|-1} f(a_j)\chi_{[a_j, a_{j+1}]}(x)$$ a step fcn approx. of f.

Then, $\underbrace{\alpha_k[g]}_{\text{the } k\text{th Fourier coeff. of } g} = \int_0^1 g(x) e^{-2\pi i k x} dx$

$$= \int_0^1 \sum_{j=0}^{|k|-1} f(a_j)\chi_{[a_j, a_{j+1}]}(x) e^{-2\pi i k x} dx$$

$$= \sum_{j=0}^{|k|-1} f(a_j) \underbrace{\int_{a_j}^{a_{j+1}} e^{-2\pi i k x} dx}_{=0} = 0.$$

$$\alpha_k[f] = \int_0^1 f(x) e^{-2\pi i k x} dx$$

$$|\alpha_k[f]| = |\alpha_k[f] - \alpha_k[g]| = |\alpha_k[f-g]|$$

$$= \left|\int_0^1 (f(x) - g(x)) e^{-2\pi i k x} dx\right|$$

$$\le \sum_{j=0}^{|k|-1} \int_{a_j}^{a_{j+1}} |f(x) - f(a_j)|\, dx$$

$$\le \sum_{j=0}^{|k|-1} V_{a_j}^{a_{j+1}}[f]\, \underbrace{(a_{j+1} - a_j)}_{= 1/|k|}$$

$$\overset{\text{Thm 3}}{=} \frac{1}{|k|} \underbrace{V_0^1[f]}_{\text{"const.}} = O(1/|k|). \quad ///$$

Thm (NS - J.F. Remy, 2006)

Let f be a 1-periodic fcn and $f \in C^m(\mathbb{R})$. Furthermore, let us assume that $f^{(m+1)}$ exists and in $BV[0,1]$. Then its Fourier coeff. $\alpha_k[f] = \hat{f}(k)$ decays as $O(|k|^{-m-2})$, where $m = 0, 1, \cdots$.

(Pf) Use $\begin{cases} \text{the periodicity, i.e., } f^{(\ell)}(0) = f^{(\ell)}(1),\ \ell = 0, \cdots, m. \\ \text{integration by parts!} \end{cases}$

$$\hat{f}(k) = \int_0^1 f(x)\, e^{-2\pi i k x}\, dx$$

$$= \cancel{\frac{e^{-2\pi i k x}}{-2\pi i k} f(x) \Big|_0^1} + \frac{1}{2\pi i k} \int_0^1 f'(x)\, e^{-2\pi i k x}\, dx$$

$$= \cancel{\frac{e^{-2\pi i k x}}{-(2\pi i k)^2} f'(x) \Big|_0^1} + \frac{1}{(2\pi i k)^2} \int_0^1 f''(x)\, e^{-2\pi i k x}\, dx$$

$$= \cdots = \cancel{\frac{e^{-2\pi i k x}}{-(2\pi i k)^{m+1}} f^{(m)}(x) \Big|_0^1} + \frac{1}{(2\pi i k)^{m+1}} \underbrace{\int_0^1 f^{(m+1)}(x)\, e^{-2\pi i k x}\, dx}$$

By assumption, $f^{(m+1)} \in BV[0,1]$. So, can use the Taibleson Thm to get:

$$|\hat{f}(k)| \le V_0^1[f^{(m+1)}](2\pi)^{-m-1} \cdot |k|^{-m-2}. \quad ///$$

☆ Fourier Series on Intervals II

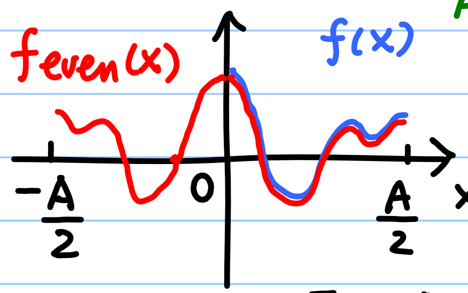
Suppose your fcn is defined on $[0, \frac{A}{2}]$ instead of $[-\frac{A}{2}, \frac{A}{2}]$.

$\Rightarrow \exists$ two ways to make it an A-periodic fcn:

(1) Even extension

$$f_{even}(x) = \begin{cases} f(x) & \text{if } x \in [0, \frac{A}{2}] \\ f(-x) & \text{if } x \in [-\frac{A}{2}, 0] \end{cases}$$

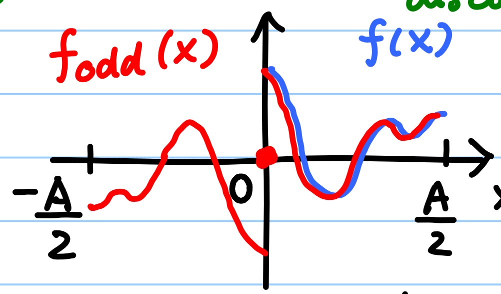
$f \in C[0, \frac{A}{2}] \Rightarrow f_{even} \in C(\mathbb{R})$ A-periodic



(2) Odd extension

$$f_{odd}(x) = \begin{cases} f(x) & \text{if } x \in (0, \frac{A}{2}] \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x \in [-\frac{A}{2}, 0) \end{cases}$$

$f \in C[0, A/2] \Rightarrow f_{odd}$: discontinuous



Then their Fourier series expansions get simpler. But before computing them, let's review the relationship between the Fourier coefficients $\{\alpha_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}} e^{2\pi i k x/A}\}$ and $\{a_k, b_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}}\} \cup \{\sqrt{\frac{2}{A}} \cos(\frac{2\pi k}{A} x)\} \cup \{\sqrt{\frac{2}{A}} \sin(\frac{2\pi k}{A} x)\}$.

Let $g(x)$ be an A-periodic L^2 fcn.

$$g(x) \sim \frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x/A}, \quad \alpha_k = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} g(x) e^{-2\pi i k x/A} dx$$

Then $\frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x/A}$

$$= \frac{1}{\sqrt{A}} \left[\alpha_0 + \sum_{1}^{\infty} (\alpha_k + \alpha_{-k}) \cos\left(\frac{2\pi k x}{A}\right) + i(\alpha_k - \alpha_{-k}) \sin\left(\frac{2\pi k x}{A}\right) \right]$$

$$= \frac{\alpha_0}{\sqrt{A}} + \sum_{1}^{\infty}\left[\frac{\alpha_k+\alpha_{-k}}{\sqrt{2}}\sqrt{\frac{2}{A}}\cos\left(\frac{2\pi k x}{A}\right) + \frac{\alpha_k-\alpha_{-k}}{\sqrt{2}}i\sqrt{\frac{2}{A}}\sin\left(\frac{2\pi k x}{A}\right)\right]$$

a_k $\qquad$ b_k

If we want, we can write a_0 instead of α_0.

Remark: In many books, the Fourier series is often defined on $[-\pi, \pi)$ and written as

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where $a_0 = 2c_0$, $\underset{k\geq 1}{a_k = c_k + c_{-k}}$, $b_k = i(c_k - c_{-k})$.

But $\{e^{ikx}\}$ is an orthogonal basis of $L^2[-\pi,\pi]$, but not normalized. To make it an ONB, one needs the factor $\frac{1}{\sqrt{2\pi}}$.

The same can be said for the orthogonal basis $\{1\} \cup \{\cos kx\} \cup \{\sin kx\}$.
The ONB is $\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos kx\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\sin kx\right\}$

i.e., $A = 2\pi$.

Compare this notation with mine in this lecture, which is the orthonormalized version:

$$\sum_{k=-\infty}^{\infty} \alpha_k \frac{1}{\sqrt{A}} e^{2\pi i k x/A} = \frac{1}{\sqrt{A}} a_0 + \sqrt{\frac{2}{A}} \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{2\pi k}{A}x\right) + b_k \sin\left(\frac{2\pi k}{A}x\right)\right]$$

where $a_0 = \alpha_0$, $\underset{k\geq 1}{a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}}$, $b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i$.

Now, let's go back to the Fourier series expansion of f_{even} & f_{odd}.

$$\alpha_k[f_{even}] = \frac{1}{\sqrt{A}}\int_{-A/2}^{A/2} f_{even}(x)\, e^{-2\pi i k x/A}\,dx$$
$$= \frac{2}{\sqrt{A}}\int_0^{A/2} f(x) \cos\left(\frac{2\pi k x}{A}\right) dx$$
$$= \alpha_{-k}[f_{even}]$$ thanks to the evenness of $\cos\theta$.

Recall the relationship:

$$a_0 = \alpha_0, \quad a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}, \quad b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}};$$

Hence, in this case of f_{even}, $a_0 = \alpha_0[f_{even}]$,

$a_k = \sqrt{2}\,\alpha_k[f_{even}]$, $b_k \equiv 0$, $k \geq 1$.

In other words, f_{even} can be written as the Fourier cosine series:

$$f_{even}(x) \sim \frac{1}{\sqrt{A}} a_0 + \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{A}} \cos\left(\frac{2\pi k x}{A}\right)$$

where $a_0 = \frac{2}{\sqrt{A}}\int_0^{A/2} f(x)\,dx = \alpha_0[f_{even}]$,

$$a_k = 2\sqrt{\frac{2}{A}}\int_0^{A/2} f(x)\cos\left(\frac{2\pi k x}{A}\right)dx$$
$$= \sqrt{2}\,\alpha_k[f_{even}].$$

Similarly, for $f_{odd}(x)$,

$$\alpha_k[f_{odd}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{odd}(x) e^{-2\pi i k x/A} dx$$

$$= \frac{-2i}{\sqrt{A}} \int_0^{A/2} f(x) \sin\left(\frac{2k\pi x}{A}\right) dx$$

$$= -\alpha_{-k}[f_{odd}]$$ due to the oddness of $\sin\theta$.

$$\left.\begin{aligned} \Rightarrow \quad a_k &= \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}} \equiv 0, \quad k \in \mathbb{N}. \\ a_0 &= \alpha_0 = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{odd}(x)\, dx = 0. \end{aligned}\right\} \text{No cosines}$$

$$b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i = \sqrt{2}\, i\, \alpha_k[f_{odd}].$$

So, $$f_{odd}(x) \sim \sum_{k=1}^{\infty} b_k \sqrt{\frac{2}{A}} \sin\left(\frac{2\pi k x}{A}\right)$$

$$b_k = 2\sqrt{\frac{2}{A}} \int_0^{A/2} f(x) \sin\left(\frac{2\pi k x}{A}\right) dx.$$

So, if a fcn is given on $[0, \frac{A}{2}]$, say $f \in C[0, A/2]$
$\exists$ three ways to extend it to a periodic fcn

$O(1/k)$ (1) Brute force periodization with period $A/2$.

$O(1/k^2)$ (2) Even extension followed by A-periodization.

$O(1/k)$ (3) Odd " " "

(2) is the best among these 3 in terms of the decay of the Fourier coeff's.

However, $\exists$ an even better way!

☆ The Lanczos Method (1938)

Suppose $f \in C^{2m}[0,1]$, but $f(0) \neq f(1)$ no match at $x=0,1$.
also assume $f^{(2m+1)} \in BV[0,1]$. $m = 1, 2, \cdots$.

Lanczos's idea:
decompose $f(x) = u(x) + v(x)$

where $u(x)$ = a polynomial of degree $2m-1$.
s.t. $\begin{cases} u^{(2k)}(0) = f^{(2k)}(0) \\ u^{(2k)}(1) = f^{(2k)}(1) \end{cases}$, $k = 0, 1, \cdots, m-1$.

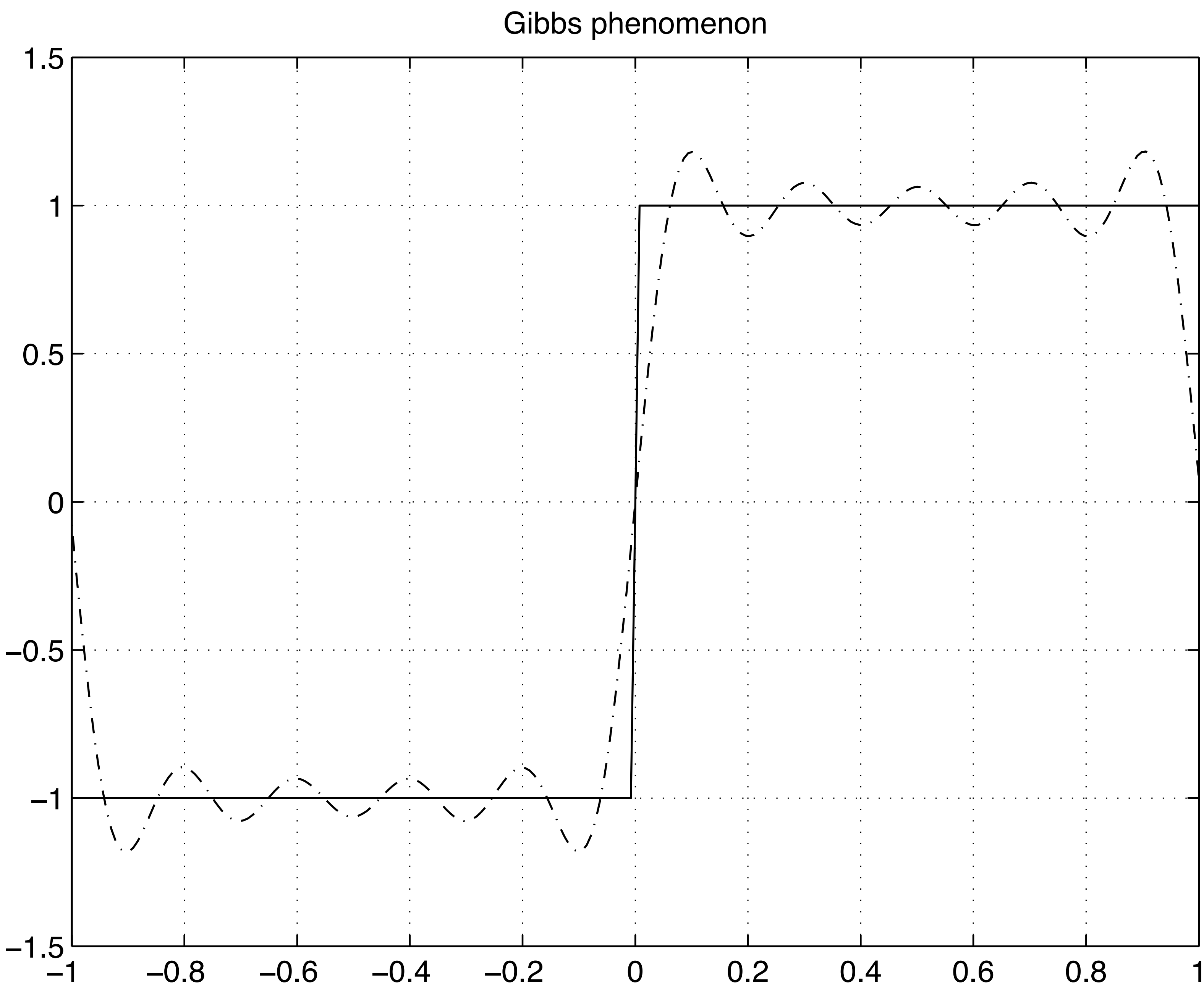
Then, consider the odd extension of v.
$\Rightarrow$ $v \in C^{2m-1}(\mathbb{R})$, $v^{(k)}(0) = v^{(k)}(1) = 0$
$k = 0, 1, 2, \cdots, 2m-1$.

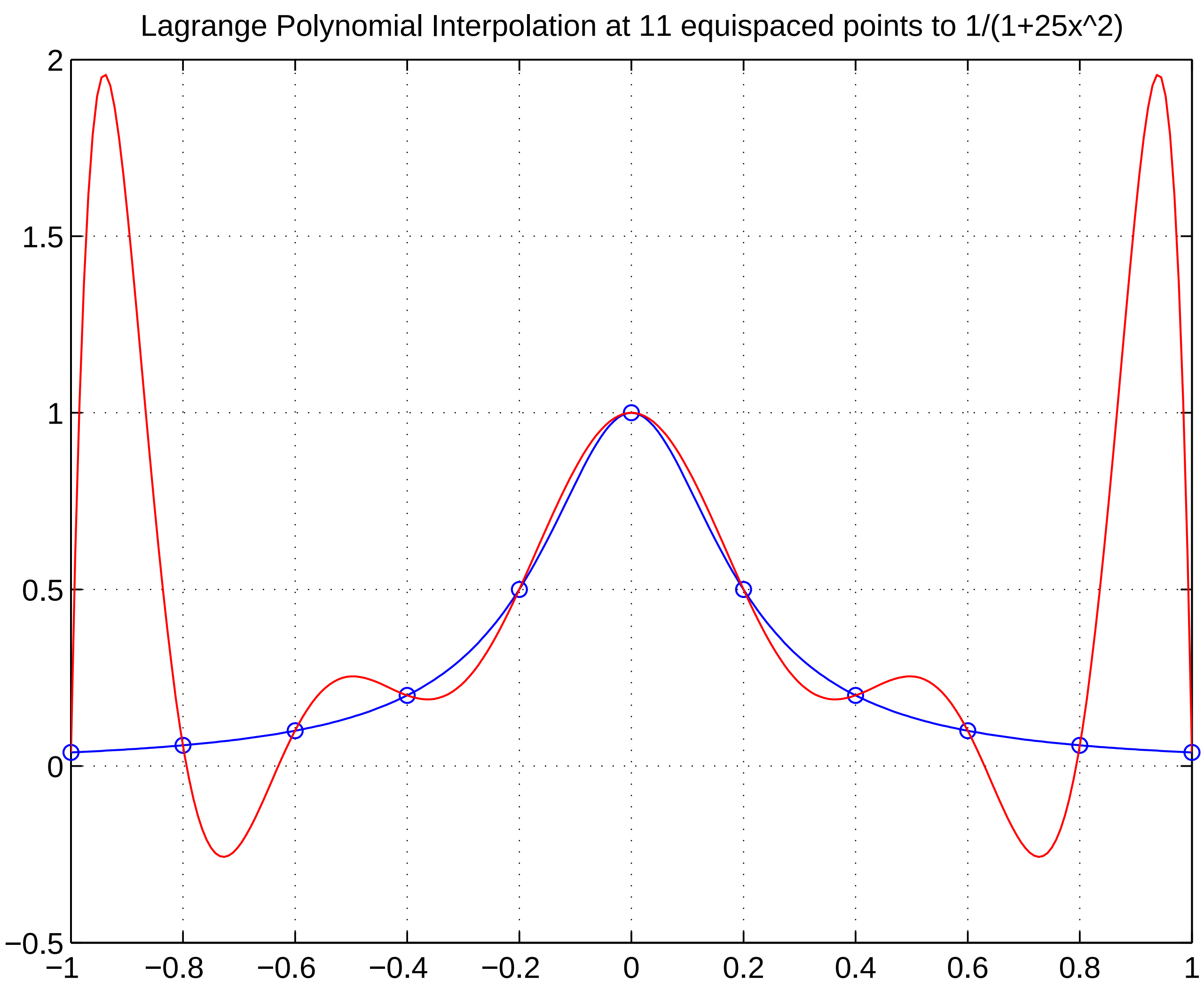
and the Fourier sine coefficients of $v(x)$ (with period 2) decay as
$b_k = O(|k|^{-2m-1})$

e.g., $m=1$ gives us $b_k = O(1/k^3)$.
and $u(x)$ is a straight line connecting $(0, f(0))$ & $(1, f(1))$.

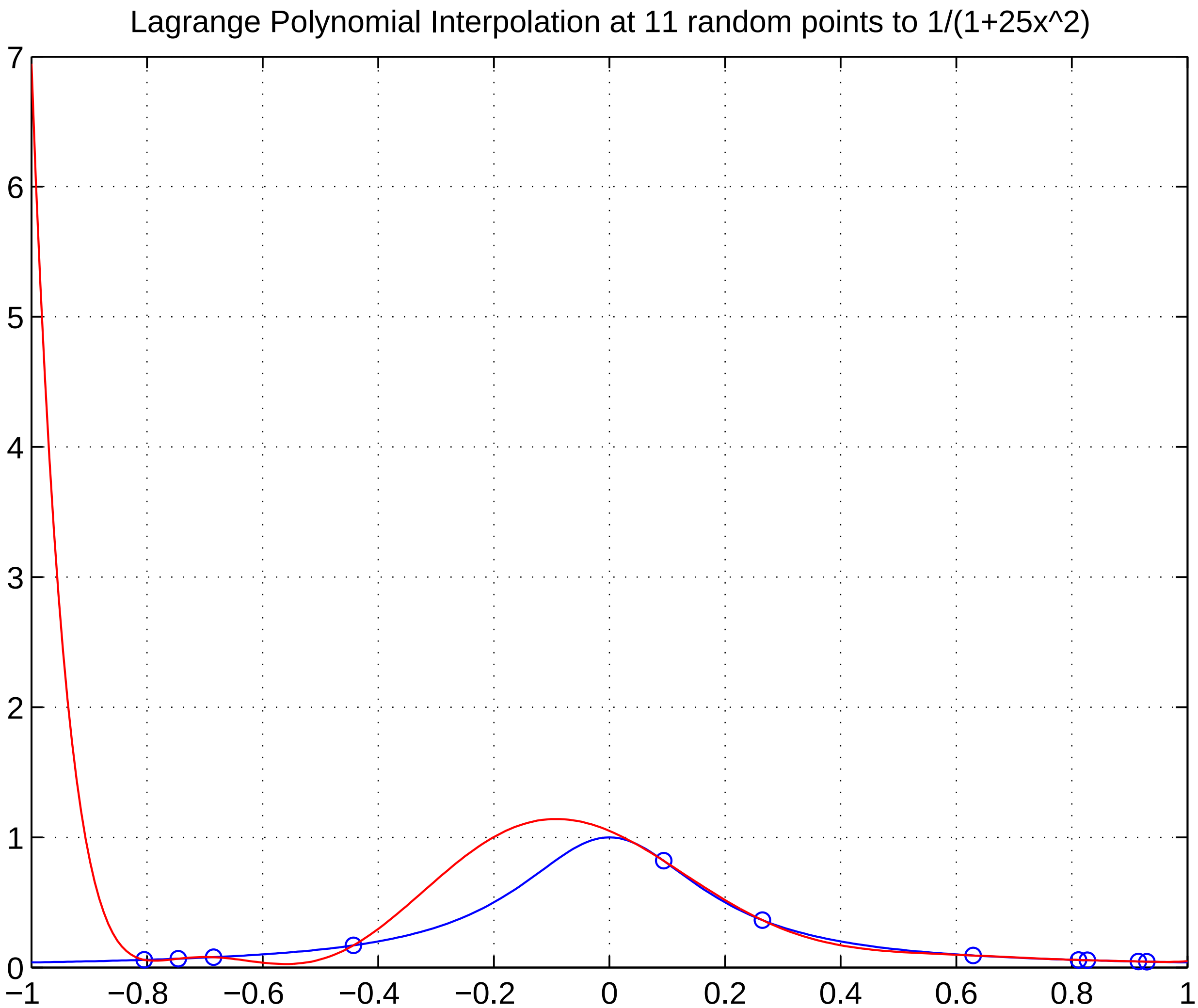
Remarks:
(1) $f = u + v$ = (an algebraic poly) + (a trig. poly).
can avoid both the Runge & Gibbs phenomena!

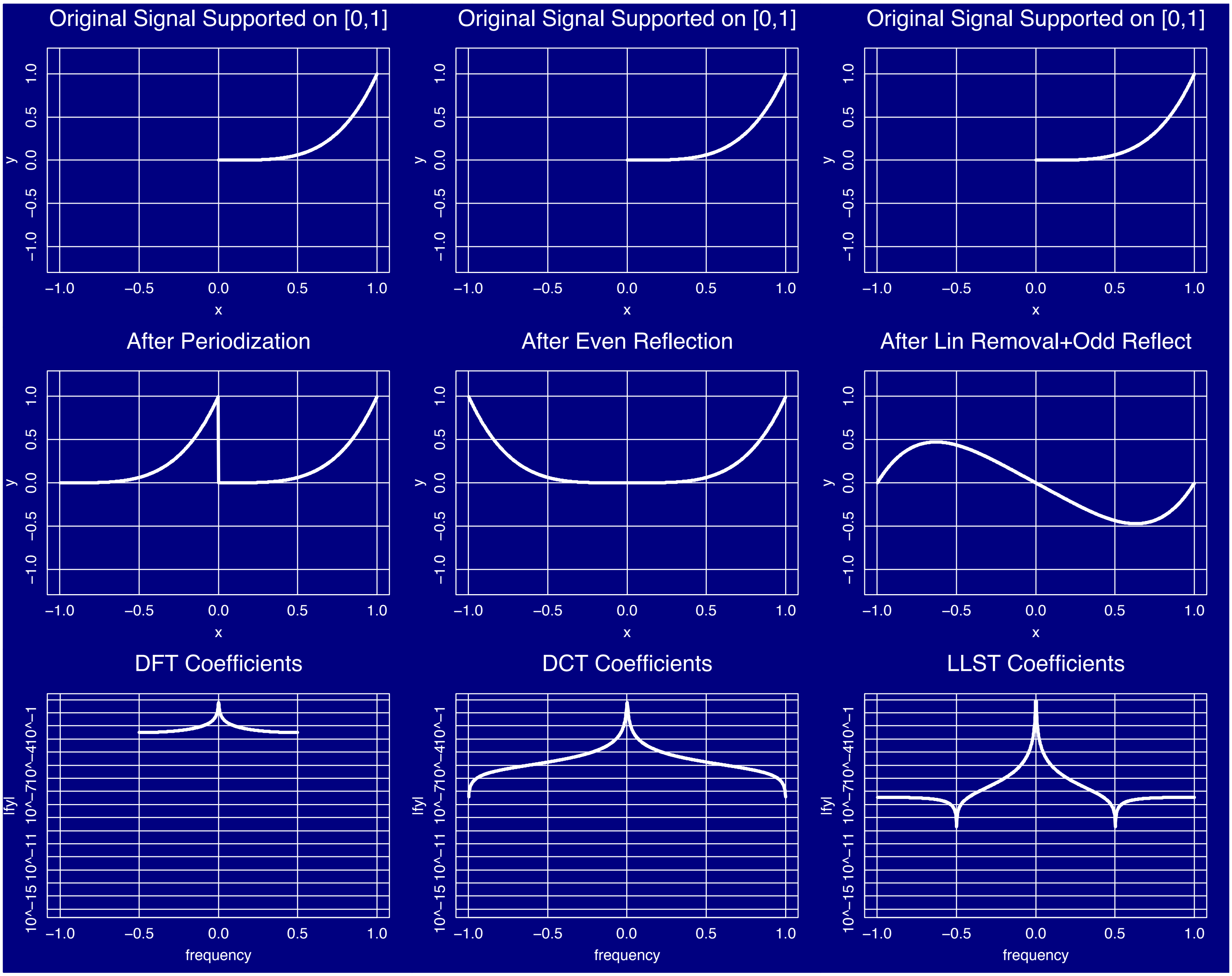
(2) NS–J.F. Remy (2006) generalized this to $\mathbb{R}^d$.


Gibbs phenomenon
1.5
1
0.5
0
−0.5
−1
−1.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



Lagrange Polynomial Interpolation at 11 equispaced points to 1/(1+25x^2)
2
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Lagrange Polynomial Interpolation at 11 random points to 1/(1+25x^2)
7
6
5
4
3
2
1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Original Signal Supported on [0,1]
Original Signal Supported on [0,1]
Original Signal Supported on [0,1]
After Periodization
After Even Reflection
After Lin Removal+Odd Reflect
DFT Coefficients
DCT Coefficients
LLST Coefficients
x
y
frequency
|fy|
−1.0
−0.5
0.0
0.5
1.0
10^−15
10^−11
10^−7
10^−4
10^−1