MAT 271: Applied & Computational Harmonic Analysis
Lecture 7: *Discrete Fourier Transform* (DFT)

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Outline

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Definitions

- The DFT can be viewed as either an *approximation to the Fourier transform* or an *approximation to the Fourier series coefficients*.
- Suppose $f \in L^2[-A/2, A/2]$, and $f(x) = 0$ for $|x| > A/2$. That is, f is a *space-limited*, square integrable function, which is a reasonable assumption in practice.
- Then, we can invoke the *dual* version of *the Sampling Theorem* in the *frequency* domain, which can also be stated in terms of Fourier coefficients of the Fourier series (Recall that *the Fourier transform of the periodic functions gives the line spectrum in the frequency domain*).
- In fact, we have the following relationship:

$$\hat{f}(k/A) = \int_{-A/2}^{A/2} f(x) \mathrm{e}^{-2\pi \mathrm{i} kx/A} \mathrm{d}x = \left\langle f, \mathrm{e}^{2\pi \mathrm{i} k\cdot/A} \right\rangle = \sqrt{A}\alpha_k, \quad k \in \mathbb{Z}. \quad (1)$$

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- In general, $f \in L^2[-A/2, A/2]$ is *not* a *band-limited* function; Recall the *Uncertainty Principle*!
- Therefore, to have a finite length vector representing the frequency samples (or the Fourier coefficients), we need to truncate the frequency information for $|\xi| > \Omega/2$ for some $\Omega > 0$.
- This is the *first* source of error of DFT approximation to FT/FS.
- This truncation allows us to consider only k with $|k| \leq A\Omega/2$.

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- We now need to approximate the integration in (1) numerically. We use the *trapezoid rule*. Here is the *second* source of the error of DFT.
- Let's divide the interval $[-A/2, A/2]$ into N (*positive even integer*[1]) subintervals of equal length of $\Delta x = A/N$. Let $x_\ell = \ell \Delta x$, $\ell = (-N/2):(N/2)$ be the points used in the trapezoid rule. Let $g(x) = f(x)\mathrm{e}^{-2\pi \mathrm{i} kx/A}$. Then we have

$$\hat{f}(k/A) \approx \frac{\Delta x}{2}\left\{ g(-A/2) + 2\sum_{\ell=-N/2+1}^{N/2-1} g(x_\ell) + g(A/2)\right\}.$$

- If we assume $f(-A/2) = f(A/2)$ (which we can do by extending f by reflection, windowing, or zero-padding followed by redefining A), then the above approximation is simplified:

$$\hat{f}(k/A) \approx \Delta x \sum_{\ell=-N/2}^{N/2-1} g(x_\ell) = \frac{A}{N}\sum_{\ell=-N/2}^{N/2-1} f(\ell A/N)\,\mathrm{e}^{-2\pi \mathrm{i} k\ell/N},$$

[1] All the subsequent matrix representations assume this. See [2, Sec. 3.1] for N being positive odd integer as well as the other cases, e.g., different starting and ending indices.

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- Now, let $f_\ell := f(\ell A/N)$ Then, *the N-point DFT* is defined as follows:

$$F_k := \frac{1}{\sqrt{N}} \sum_{\ell=-N/2}^{N/2-1} f_\ell \, \mathrm{e}^{-2\pi \mathrm{i} k \ell / N}, \quad k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1. \tag{2}$$

- The factor $1/\sqrt{N}$ is to make DFT a *unitary* transformation (i.e., ℓ^2-norm (energy) preserving transformation, so that the Parseval & Plancherel equalities holds.)[2]
- We now have the following relationship.

$$\hat{f}(k/A) = \sqrt{A}\alpha_k \approx \frac{A}{\sqrt{N}} F_k.$$

- *The N-point inverse DFT* is defined, as you can imagine, as follows.

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The proof of this formula gets easier when we use the vector-matrix notation later in this lecture.

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The Reciprocity Relations

- Let $\Delta\xi$ be a sampling rate in the frequency domain, i.e., $\Delta\xi = 1/A$. Since we know $\Delta x = A/N$, and $k/A = \Omega/2$ at $k = N/2$ (highest frequency in consideration), we have the following fundamental relations:

$$\Delta x \Delta\xi = \frac{1}{N}, \quad A\Omega = N.$$

- Interpretation of these relations is very important! For example:
  - *For fixed N: $A \uparrow \Longrightarrow \Delta x \uparrow \; \Omega \downarrow \; \Delta\xi \downarrow$ (coarser space sampling leads to finer frequency sampling, but the frequency bandwidth also decreases).*
  - *For fixed A: $N \uparrow \Longrightarrow \Delta x \downarrow \; \Omega \uparrow \; \Delta\xi = \text{const.} = 1/A$ (finer space sampling leads to the increase of the frequency bandwidth).*

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The vector-matrix notation of DFT

- We can gain great insights by expressing DFT using the *vector-matrix notation*. To do this, we need to define a couple of things.
- Let $\omega_N := \mathrm{e}^{2\pi\mathrm{i}/N}$, i.e., the *Nth root of unity*.
- Note that $\overline{\omega}_N = \omega_N^{-1}$; $\omega_N^0 = \omega_N^N = 1$; $\omega_N^{N/2} = -1$; and $\omega_N^{k+N} = \omega_N^k$ for any $k \in \mathbb{Z}$.
- Then, define a column vector:

$$\boldsymbol{w}_N^k := \frac{1}{\sqrt{N}}\left(\omega_N^{k\cdot 0}, \omega_N^{k\cdot 1}, \ldots, \omega_N^{k\cdot\frac{N}{2}}, \ldots, \omega_N^{k\cdot(N-1)}\right)^{\mathsf{T}}, \quad k = 0, \ldots, N-1.$$

- We also define another column vector:

$$\widetilde{\boldsymbol{w}}_N^k := \frac{1}{\sqrt{N}}\left(\omega_N^{k\cdot(-\frac{N}{2})}, \omega_N^{k\cdot(-\frac{N}{2}+1)}, \ldots, \omega_N^{k\cdot 0}, \ldots, \omega_N^{k\cdot(\frac{N}{2}-1)}\right)^{\mathsf{T}}, \quad k = -\frac{N}{2}, \ldots, \frac{N}{2}-1.$$

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- Using the properties of ω_N listed above, one can easily show that

$$\widetilde{\boldsymbol{w}}_N^k = S_N \boldsymbol{w}_N^k,$$

where S_N is equivalent to `fftshift` in MATLAB:

$$S_N := \begin{bmatrix} O_{\frac{N}{2}} & I_{\frac{N}{2}} \\ I_{\frac{N}{2}} & O_{\frac{N}{2}} \end{bmatrix},$$

i.e., $S_N(a_1, \ldots, a_N)^\mathsf{T} = \left(a_{\frac{N}{2}+1}, \ldots, a_N, a_1, \ldots, a_{\frac{N}{2}}\right)^\mathsf{T}$.

- Note that $S_N^\mathsf{T} = S_N^{-1} = S_N$ if N is an *even* integer, which is our assumption here. But you have to be careful for odd integer cases. Hence, in general, it is safer to use `ifftshift` in MATLAB to undo `fftshift`.

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$$S_N := \begin{bmatrix} O_{\frac{N}{2}} & I_{\frac{N}{2}} \\ I_{\frac{N}{2}} & O_{\frac{N}{2}} \end{bmatrix},$$

i.e., $S_N(a_1, \ldots, a_N)^\mathsf{T} = \left(a_{\frac{N}{2}+1}, \ldots, a_N, a_1, \ldots, a_{\frac{N}{2}}\right)^\mathsf{T}$.

- Note that $S_N^\mathsf{T} = S_N^{-1} = S_N$ if N is an *even* integer, which is our assumption here. But you have to be careful for odd integer cases. Hence, in general, it is safer to use `ifftshift` in MATLAB to undo `fftshift`.

- Let $\boldsymbol{f} = \left(f_{-\frac{N}{2}}, \dots, f_{\frac{N}{2}-1}\right)^{\mathsf{T}}$ be a vector of sampled points $f_\ell = f(\ell \Delta x)$.
- Now DFT can be written as follows:

$$F_k = \left\langle \boldsymbol{f}, \widetilde{\boldsymbol{w}}_N^k \right\rangle, \quad k = -\frac{N}{2}, \dots, \frac{N}{2} - 1.$$

- Finally, define an *N-point DFT matrix* commonly used in the literature:

$$W_N := \left[\boldsymbol{w}_N^0 \;\middle|\; \boldsymbol{w}_N^1 \;\middle|\; \cdots \;\middle|\; \boldsymbol{w}_N^{N-1} \right]$$

- On the other hand, we define the following matrix compliant with our definition of DFT in Eq. (2):

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- In fact, $\widetilde{W}_N^* = \left(S_N W_N S_N^{\mathsf{T}}\right)^* = S_N W_N^* S_N^{\mathsf{T}}$.
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All the eigenvalues of W_N and $\widetilde{W}_N$ are $1, -1, \mathrm{i}, -\mathrm{i}$.

(Proof) See e.g., [1, 2, 3]. Note that from this theorem we have $W_N^4 = \widetilde{W}_N^4 = I_N$.

Theorem (McClellan-Parks [3])

The multiplicities of the eigenvalues of W_N are summarized as:

N	mult(1)	mult(−1)	mult(i)	mult(−i)
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
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Research Opportunity: W_N and $\widetilde{W}_N$ are already the ONBs of $\mathbb{C}^N$. *What is the use of their eigenvectors?*

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Outline

Using the properties of ω_N, in particular the periodicity with period N, we have:

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The following figures show the matrix W_N^* with $N = 16$ as waveforms.

Note that the first row vector of a matrix is displayed in the bottom while the last row in the top in each figure.

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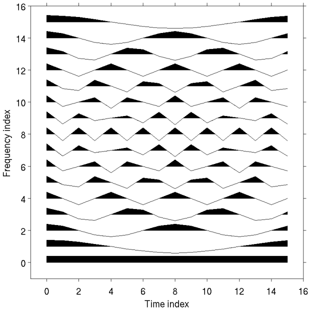
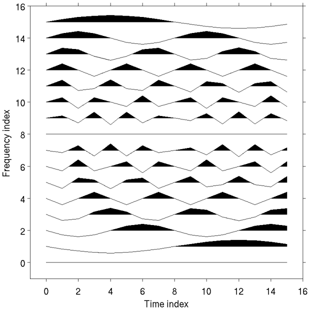


(a) $\mathrm{Re}(W^*)$



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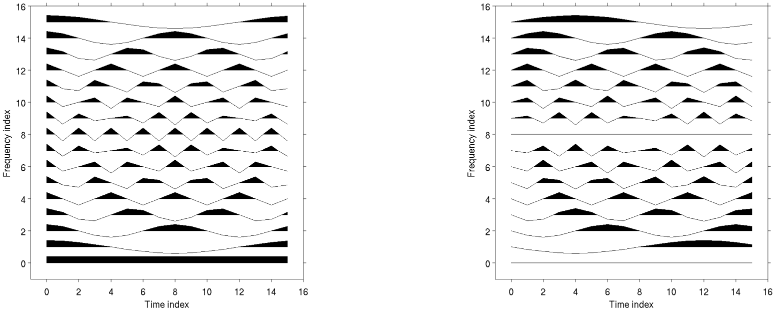

(a) $\mathrm{Re}(W^*)$ (b) $\mathrm{Im}(W^*)$

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Now, how about $\widetilde{W}_N^*$?

Note the change of the locations of the basis vectors as well as symmetry $(W_N^*)^\top = W_N^*$, $(\widetilde{W}_N^*)^\top = \widetilde{W}_N^*$.

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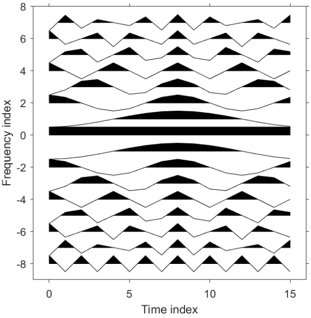
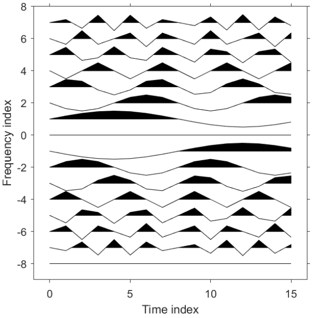


(a) $\mathrm{Re}(\widetilde{W}^*)$



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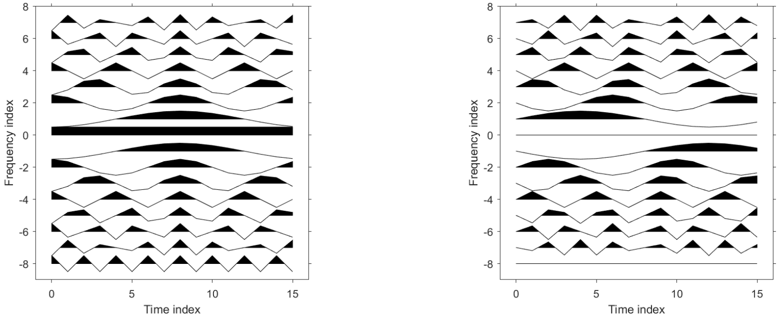

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Outline

1. Definitions
2. The Reciprocity Relations
3. The Vector-Matrix Notation of DFT
4. Pictorial View of W_N^*
5. Different Definitions of DFT
6. References

- It is amazing to know that the definition of DFT varies with the software systems one uses. We should always be careful what is the exact default definition of the DFT for each software system.

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Further caution:

- If an input argument to the DFT/FFT function is a *matrix* (or *multidimensional array*), then MATLAB applies DFT on *each column vector for a matrix* (or *the first non-singleton dimension for a 3D or higher dimensional array*.
- On the other hand, the DFT functions in the other packages perform the *multidimensional DFT* on the input.

Outline

References

For more information about the DFT including higher-dimensional versions, see [2].
Also, the DFT matrix has more *profound* properties. See the challenging and deep paper by [1].

[1] L. Auslander and R. Tolimieri, *Is computing with the finite Fourier transform pure or applied mathematics?*, Bull. Amer. Math. Soc., 1 (1979), pp. 847–897.

[2] W. L. Briggs and V. E. Henson, *The DFT: An Owner's Manual for the Discrete Fourier Transform*, SIAM, Philadelphia, PA, 1995.

[3] J. H. McClellan and T. W. Parks, *Eigenvalue and eigenvector decomposition of the discrete Fourier transform*, IEEE Trans. Audio Electacoust., AU-20 (1972), pp. 66–74.
See also comments appeared in AU-21, pp. 65, 1973.