Lecture 11 : Time-Frequency Analysis/Synthesis

Note Title

* Now we are in a good position to discuss how to beat the Heisenberg uncertainty principle.
* Shortcomings of the Fourier basis:
  - Too global in space (or time)
  - Not good for edges, singularities in signals
  - Not good for nonstationary signals, i.e., signals changing their characteristics in time.

Our Roadmap

* (General) time - frequency atoms
* Windowed (or short-time) Fourier transf.
  - Boxcar window
  - Gaussian window → Gabor fcns
  - Local cosine transf. (LCT)
* Wavelet transforms
  - Continuous wavelet transf. (redundant)
  - discrete wavelet transf.
    - orthonormal
    - biorthogonal
    - redundant → Frame Theory
  - wavelet packets

☆ Time-Frequency Atoms

$\{\phi_\gamma\}_{\gamma\in\Gamma} \subset L^2(\mathbb{R})$, Γ: some multi index set

$\|\phi_\gamma\|_2 = 1.$ e.g., $\{(m,n)\}_{(m,n)\in\mathbb{Z}^2}$

m: time index, n: freq. index

The correlation of a given fcn $f \in L^2(\mathbb{R})$ with $\{\phi_\gamma\}_{\gamma\in\Gamma}$ can be measured by

$$Tf(\gamma) := \int_{-\infty}^{\infty} f(x)\,\overline{\phi_\gamma(x)}\,dx = \langle f, \phi_\gamma\rangle$$

By Plancherel's equality, $\langle f, \phi_\gamma\rangle = \langle \hat{f}, \hat{\phi}_\gamma\rangle$

Recall the Heisenberg uncertainty principle:

$$\Delta^2_{x_0} f \;\Delta^2_{\xi_0}\hat{f} \ge \frac{1}{16\pi^2} \quad \forall x_0, \xi_0 \in \mathbb{R}.$$

where $\Delta^2_{x_0} f := \int (x-x_0)^2 |f(x)|^2 dx \,/\, \|f\|_2^2$

Suppose $m_x(\phi_\gamma) := \int x\,|\phi_\gamma(x)|^2\,dx.$

the center of gravity of $|\phi_\gamma|^2$

Similarly $m_\xi(\hat{\phi}_\gamma) := \int \xi\,|\hat{\phi}_\gamma(\xi)|^2\,d\xi$

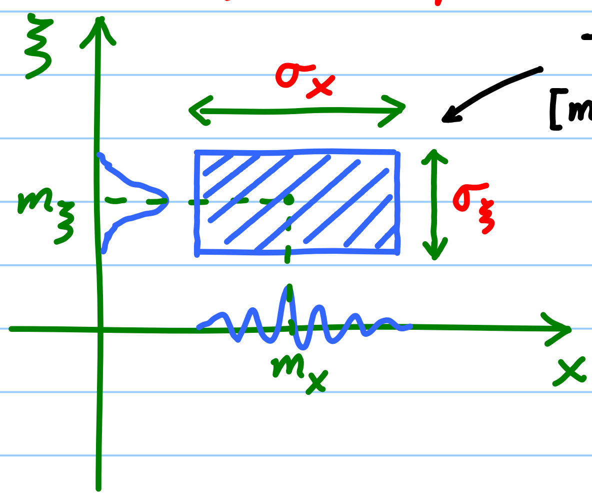
Since $\|\phi_\gamma\|_2 = \|\hat{\phi}_\gamma\|_2 = 1$, we have

$$\Delta^2_{m_x}\phi_\gamma \cdot \Delta^2_{m_\xi}\hat{\phi}_\gamma \ge \frac{1}{16\pi^2}$$

By defining $\begin{cases} \sigma_x := \sqrt{\Delta^2_{m_x}\phi_\gamma} \\ \sigma_\xi := \sqrt{\Delta^2_{m_\xi}\hat{\phi}_\gamma} \end{cases}$

the Heisenberg inequality can be written as

$$\sigma_x \, \sigma_\xi \geq \frac{1}{4\pi}$$



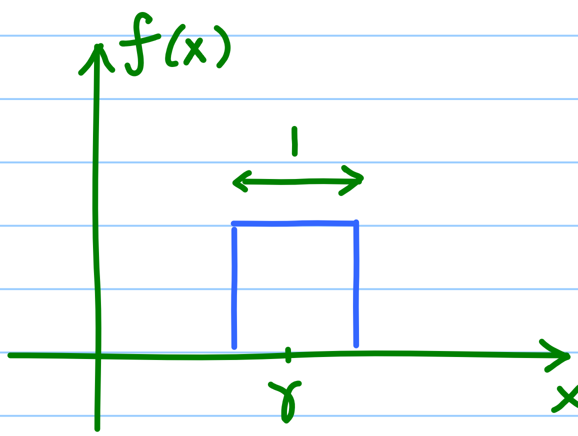
This box

$$\left[m_x - \frac{\sigma_x}{2}, m_x + \frac{\sigma_x}{2}\right] \times \left[m_\xi - \frac{\sigma_\xi}{2}, m_\xi + \frac{\sigma_\xi}{2}\right]$$

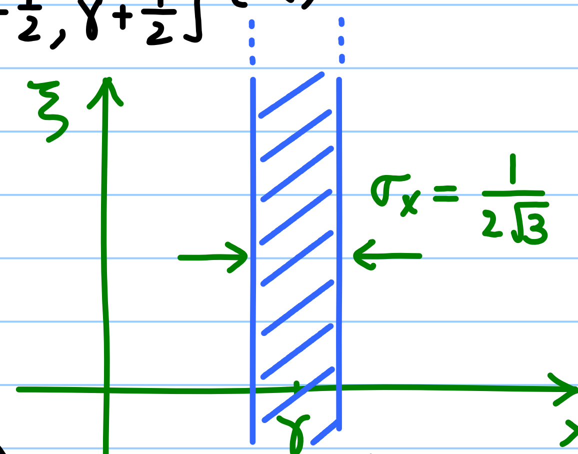
is called the Heisenberg box of ϕ_γ

Note that the main energy of ϕ_γ is in this box but not all the energy in it.

<u>Example</u> $\quad \phi_\gamma(x) = \chi_{[\gamma-\frac{1}{2}, \gamma+\frac{1}{2}]}(x)$



$\Rightarrow$



$$\phi_\gamma(x) = \tau_\gamma \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) \xrightarrow{\mathcal{F}} \hat{\phi}_\gamma(\xi) = e^{-2\pi i \gamma \xi} \operatorname{sinc}(\xi)$$

$$m_x = \int_{\gamma-\frac{1}{2}}^{\gamma+\frac{1}{2}} x \cdot 1 \, dx = \gamma \qquad m_\xi = \int_{-\infty}^{\infty} \underbrace{\xi}_{\text{odd}} \cdot \underbrace{\operatorname{sinc}^2(\xi)}_{\text{even}} \, d\xi = 0$$

$$\Delta^2_{m_x} \phi_\gamma = \int_{\gamma-\frac{1}{2}}^{\gamma+\frac{1}{2}} (x-\gamma)^2 \, dx = \frac{1}{12} \qquad \Delta^2_{m_\xi} \hat{\phi}_\gamma = \int_{-\infty}^{\infty} \xi^2 \cdot \frac{\sin^2 \pi \xi}{\pi^2 \xi^2} \, d\xi = +\infty$$

$\sigma_x = 1/2\sqrt{3}$ but $\sigma_\xi = \infty \Rightarrow$ No frequency resolution!

Use different families of atoms

(1) Windowed Fourier Atoms → Windowed (short-time) Fourier transf.
(aka. Time-Frequency Atoms)

$$\phi_\gamma(x) = g_{x_0,\xi_0}(x) := \underbrace{e^{2\pi i \xi_0 x}}_{\text{modulation}}\, g(\underbrace{x-x_0}_{\text{translation}})$$

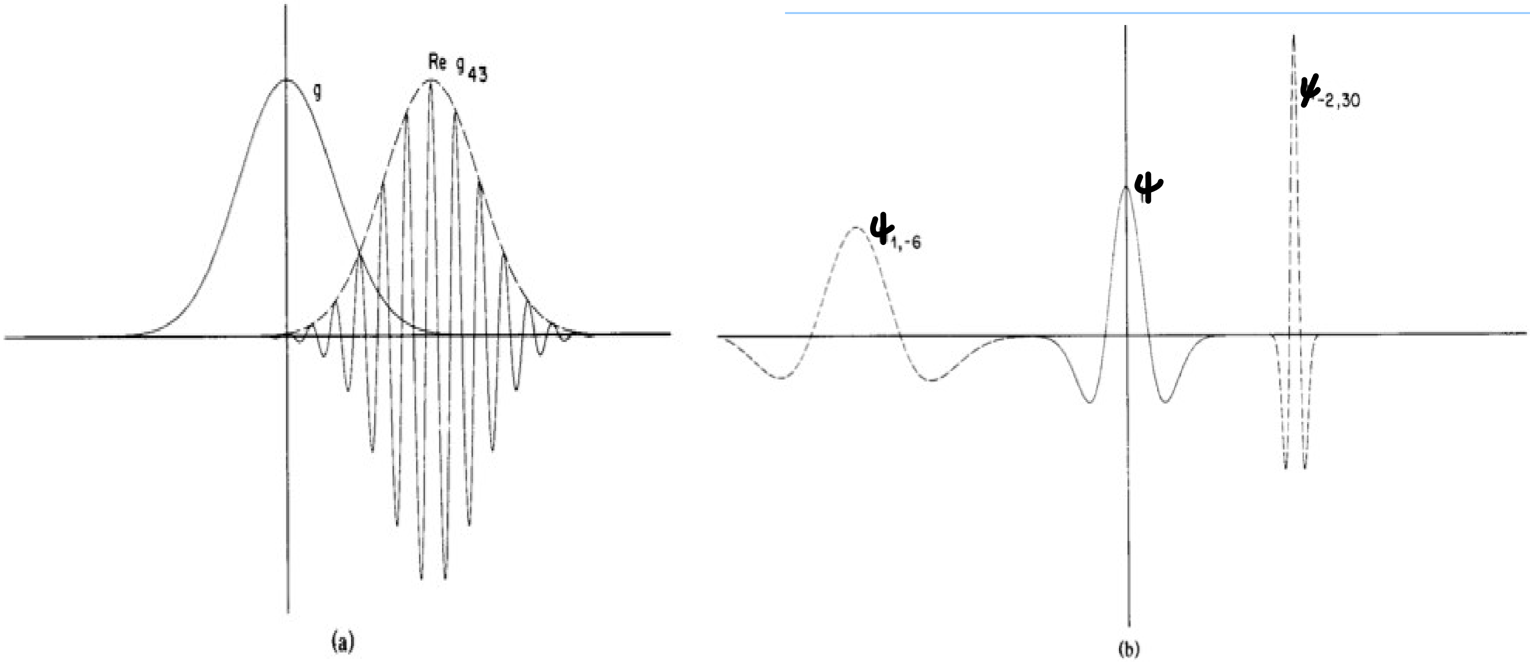
g: some window fcn. e.g., Gaussian.

modulation = translation in the freq. dom.

(2) Wavelet Atoms → Wavelet transf.
(aka. Time-Scale Atoms)

$$\phi_\gamma(x) = \psi_{a,b}(x) := \underbrace{\tau_b}_{\text{translation}}\, \underbrace{\delta_a}_{\text{dilation}} \psi(x) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{x-b}{a}\right)$$

ψ : a "mother" wavelet ⇒ must satisfy some conditions.



Windowed Fourier Atoms   Wavelet Atoms

(1) Windowed Fourier Transforms

$$g_{x_0,\xi_0}(x) = e^{2\pi i \xi_0 x} g(x - x_0)$$

assume $\begin{cases} \|g\|_2 = 1 \text{ so that } \|g_{x_0,\xi_0}\|_2 = 1. \\ g : \text{real-valued \& even} \end{cases}$

For $f \in L^2(\mathbb{R})$, define

$$Sf(x_0,\xi_0) := \langle f, g_{x_0,\xi_0} \rangle = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi_0 x} g(x - x_0)\, dx$$

$$= \int_{-\infty}^{\infty} f(x) g(x - x_0) e^{-2\pi i \xi_0 x} dx$$

$$= \mathcal{F}[f \cdot \tau_{x_0} g](\xi_0)$$

checking f around (x_0, ξ_0) on the t-f plane.

How good g_{x_0,ξ_0} is in terms of the Heisenberg box?

$$m_x(g_{x_0,\xi_0}) = \int_{-\infty}^{\infty} x\,|g_{x_0,\xi_0}(x)|^2 dx = \int_{-\infty}^{\infty} x\, g^2(x - x_0)\, dx$$

$$= \int_{-\infty}^{\infty} (y + x_0)\, g^2(y)\, dy \quad \leftarrow y = x - x_0$$

$$= \underbrace{\int_{-\infty}^{\infty} y\, g^2(y)\, dy}_{\text{odd}} + x_0 \underbrace{\int_{-\infty}^{\infty} g^2(y)\, dy}_{= \|g\|_2^2 = 1} = x_0$$

$$\hat{g}_{x_0,\xi_0}(\xi) = \mathcal{F}[e^{2\pi i \xi_0 x} \cdot \tau_{x_0} g](\xi)$$

$$= \tau_{\xi_0}\left(e^{-2\pi i \xi x_0} \hat{g}(\xi)\right)$$

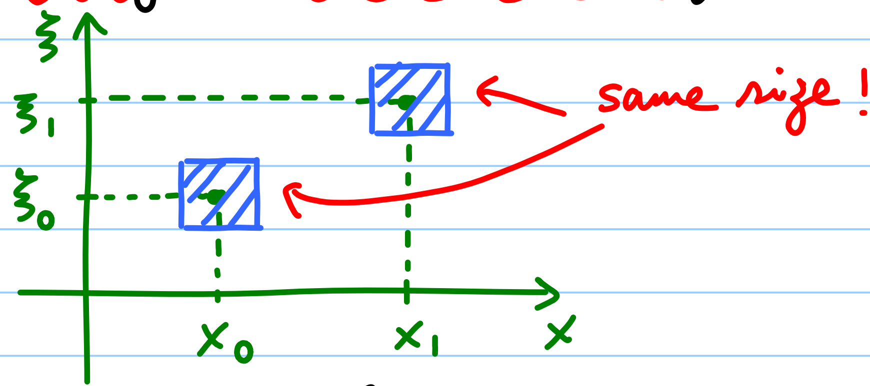
$$= e^{-2\pi i (\xi - \xi_0) x_0} \hat{g}(\xi - \xi_0)$$

$$\Rightarrow m_{\xi_0}(\hat{g}_{x_0,\xi_0}) = \int_{-\infty}^{\infty} \xi\, |\hat{g}_{x_0,\xi_0}(\xi)|^2 d\xi$$

$$= \int_{-\infty}^{\infty} \xi\, |\hat{g}(\xi - \xi_0)|^2 d\xi = \xi_0$$

$$\begin{cases}\Delta^2_{m_x} g_{x_0,\xi_0} = \int_{-\infty}^{\infty}(x-x_0)^2\,|g(x-x_0)|^2\,dx = \int_{-\infty}^{\infty} x^2 g^2(x)\,dx \\ \Delta^2_{m_\xi} \hat{g}_{x_0,\xi_0} = \int_{-\infty}^{\infty}(\xi-\xi_0)^2\,|\hat{g}(\xi-\xi_0)|^2\,d\xi = \int_{-\infty}^{\infty} \xi^2\,|\hat{g}(\xi)|^2\,d\xi\end{cases}$$

These quantities are completely specified by g only, independent from (x_0, ξ_0)



If $g(x) = \frac{1}{\sqrt[4]{\pi}} e^{-x^2/2}$, then $\|g\|_2 = 1$ and $\hat{g}(\xi) = \sqrt{2}\,\sqrt[4]{\pi}\,e^{-2\pi^2\xi^2}$. Hence, we have

$$\Delta^2_{m_x} g_{x_0,\xi_0} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx \overset{\text{Integration by parts}}{=} \frac{1}{2}$$

$$\Delta^2_{m_\xi} \hat{g}_{x_0,\xi_0} = 2\sqrt{\pi}\int_{-\infty}^{\infty} \xi^2 e^{-4\pi^2\xi^2}\,d\xi = 2\sqrt{\pi}\int_{-\infty}^{\infty}\left(\frac{y}{2\pi}\right)^2 e^{-y^2}\frac{dy}{2\pi}$$

$$= 2\sqrt{\pi}\,\frac{1}{8\pi^3}\cdot\frac{\sqrt{\pi}}{2} = \frac{1}{8\pi^2}$$

$$\Rightarrow \Delta^2_{m_x} g_{x_0,\xi_0}\cdot\Delta^2_{m_\xi}\hat{g}_{x_0,\xi_0} = \frac{1}{16\pi^2}$$

achieving the lower bound in the Heisenberg inequality!

How about using $g_\sigma(x)$ instead of $g(x)$?
Note that our previous definition of g_σ was

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, \quad \|g_\sigma\|_1 = 1$$

But we want here $\|g_\sigma\|_2 = 1$.
So, we redefine it by $g_\sigma(x) := \frac{1}{\sqrt[4]{\pi}\sqrt{\sigma}} e^{-x^2/2\sigma^2}$

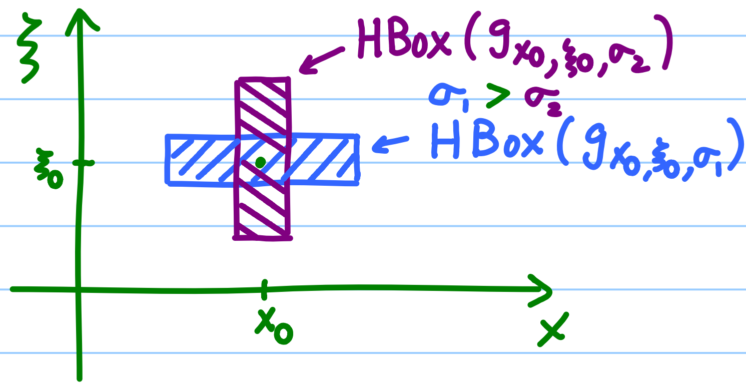
We can show:

$$m_x(g_{x_0,\xi_0,\sigma}) = x_0, \quad m_\xi(\hat{g}_{x_0,\xi_0,\sigma}) = \xi_0$$

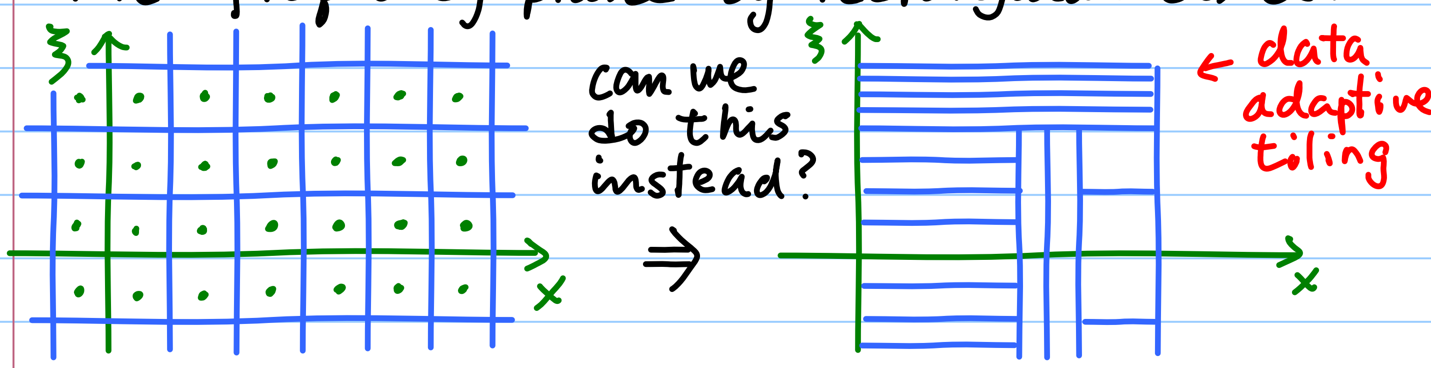
$$\Delta^2_{m_x} g_{x_0,\xi_0,\sigma} = \frac{\sigma^2}{2}, \quad \Delta^2_{m_\xi} \hat{g}_{x_0,\xi_0,\sigma} = \frac{1}{8\pi^2\sigma^2}$$

$$\Rightarrow \Delta^2_{m_x} g_{x_0,\xi_0,\sigma} \cdot \Delta^2_{m_\xi} \hat{g}_{x_0,\xi_0,\sigma} = \frac{1}{16\pi^2}$$

Still achieves the lower bd.



With a constant σ, we can "tile" the time-frequency plane by rectangular boxes.



· Show the Wavelab demo here!

So far, we have talked only "analysis". How about "synthesis" or "representation"?

<u>Thm</u> If $f \in L^2(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Sf(y,\eta)\, g_{y,\eta}(x)\, dy\, d\eta,$$

and $\|f\|_2^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Sf(x,\xi)|^2\, dx\, d\xi$

i.e., $Sf \in L^2(\mathbb{R}^2)$.

Note that WFT is very redundant!

$$S : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$$

(Pf) First of all, let's compute

$$\mathcal{F}_x[Sf(x,\eta)] = \widehat{Sf}(\xi,\eta). \quad \text{To do so,}$$

$$Sf(x,\eta) = \int_{-\infty}^{\infty} f(y)\, g(y-x)\, e^{-2\pi i \eta y}\, dy$$

$$= e^{-2\pi i \eta x}\int_{-\infty}^{\infty} f(y)\, g(y-x)\, e^{-2\pi i \eta (y-x)}\, dy$$

$g(y-x) = g(x-y)$ since g: even

$$= e^{-2\pi i \eta x}\int_{-\infty}^{\infty} f(y)\, g(x-y)\, e^{2\pi i \eta (x-y)}\, dy$$

$$= e^{-2\pi i \eta x} \cdot f * g_{0,\eta}(x)$$

$$\downarrow \mathcal{F}_x$$

(*) $$\widehat{Sf}(\xi,\eta) = \hat{f}(\xi+\eta)\cdot \hat{g}_{0,\eta}(\xi+\eta) = \hat{f}(\xi+\eta)\,\hat{g}(\xi)$$

because $g_{0,\eta}(x) = g(x)e^{2\pi i \eta x} \longrightarrow \hat{g}_{0,\eta}(\xi) = \hat{g}(\xi-\eta)$

Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Sf(y,\eta)\, g_{y,\eta}(x)\, dy\, d\eta$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} Sf(y,\eta)\, \underbrace{g(x-y)}_{= g(y-x) = \tau_x g(y)}\, dy \right\} e^{2\pi i \eta x}\, d\eta$$

$$= \langle Sf(\cdot,\eta), \tau_x g(\cdot) \rangle$$

Plancherel

$$= \langle \widehat{Sf}(\cdot,\eta), e^{-2\pi i x \cdot}\hat{g}(\cdot) \rangle$$

via (*)

$$= \int_{-\infty}^{\infty} \hat{f}(\xi+\eta)\, \hat{g}(\xi)\, e^{2\pi i \xi x}\, \overline{\hat{g}(\xi)}\, d\xi$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\xi+\eta)\, |\hat{g}(\xi)|^2\, e^{2\pi i (\xi+\eta) x}\, d\xi\, d\eta$$

Fubini

$$= \int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 \left\{ \underbrace{\int_{-\infty}^{\infty} \hat{f}(\xi+\eta)\, e^{2\pi i(\xi+\eta)x}\, d\eta}_{= f(x) \quad \text{inverse FT!}} \right\} d\xi$$

$$= f(x) \int_{-\infty}^{\infty} |\hat{g}(\xi)|^2\, d\xi = f(x)\, \underbrace{\|\hat{g}\|_2^2}_{= \|g\|_2^2 = 1} = f(x). \quad ///$$

Because the WFT is redundant, it is not true that any $\Phi \in L^2(\mathbb{R}^2)$ is a WFT of some $f \in L^2(\mathbb{R})$, i.e., not "onto".

<u>Prop</u>. Let $\Phi \in L^2(\mathbb{R}^2)$. Then,

$\exists f \in L^2(\mathbb{R})$ s.t. $\Phi(x,\xi) = Sf(x,\xi)$

$$\Longleftrightarrow \Phi(x_0,\xi_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x,\xi)\, K(x_0,x,\xi_0,\xi)\, dx\, d\xi$$

where $K(x_0,x,\xi_0,\xi) := \langle g_{x,\xi}, g_{x_0,\xi_0} \rangle$ a reproducing kernel

(Pf) Exercise!