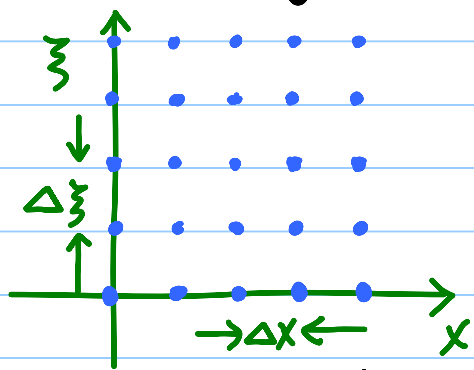
Lecture 12: { Intro to Frame Theory / The Balian–Low Theorem

Note Title

☆ Sampling & WTF



Consider sampling $f \in L^2(\mathbb{R})$ on the time-frequency plane via WF-atoms of the form:

$$g_{m,n}(x) := g(x - m\Delta x)\, e^{2\pi i n \Delta\xi x}, \quad m, n \in \mathbb{Z}$$

instead of $g_{x_0,\xi_0}(x) = g(x - x_0)\, e^{2\pi i \xi_0 x}$.

This is Gabor's proposal (1946).

$$\underbrace{f(x) = \sum \alpha_{m,n}\, \tilde{g}_{m,n}(x)}_{\text{Synthesis}}, \qquad \underbrace{\alpha_{m,n} = \langle f, g_{m,n} \rangle}_{\text{Analysis}}$$

$\tilde{g}_{m,n}$: dual basis (or dual frame)

$\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ may be quite redundant (i.e., more than a basis), and may not be orthogonal.

These ideas lead to:

☆ The Frame Theory

Def. Let $\mathcal{H}$ be a Hilbert space with its norm $\|\cdot\|$ and let $\{\phi_\gamma\}_{\gamma\in\Gamma} \subset \mathcal{H}$.
Then $\{\phi_\gamma\}$ is said to constitute a frame of $\mathcal{H}$ if $\forall f \in \mathcal{H}$, $\exists\, A, B \gneq 0$ s.t.

$$(*) \qquad A\|f\|^2 \le \sum_{\gamma\in\Gamma} |\langle f, \phi_\gamma \rangle|^2 \le B\|f\|^2$$

where the constants A, B are referred to as the frame bounds.

Remarks:

(1) Existence of the frame bounds $B \geq A > 0$ is a necessary & sufficient condition for the invertibility of the frame operator, i.e., you can reconstruct your original fcn f from the frame coefficents $\{\langle f, \phi_\gamma \rangle\}_{\gamma \in \Gamma}$.

(2) If $\|\phi_\gamma\| = 1$, $\forall \gamma \in \Gamma$, and $A > 1$, then the frame is redundant, and this A can be interpreted as a minimum redundancy factor.

(3) If $\|\phi_\gamma\| = 1$, $\forall \gamma \in \Gamma$, and $\{\phi_\gamma\}_{\gamma \in \Gamma}$ are linearly independent, then $A \leq 1 \leq B$. In this case, $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is not redundant and forms a basis of $\mathcal{H}$, which is called a Riesz basis.

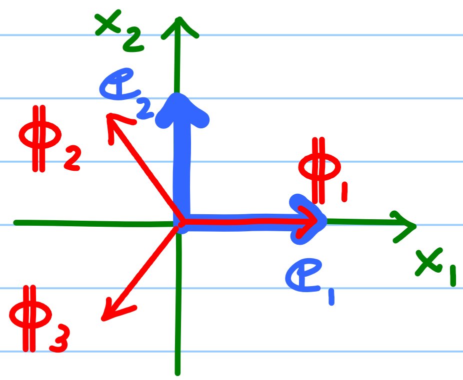
(4) $A = B = 1$ $\underset{\not\Rightarrow}{\Leftarrow}$ $\{\phi_\gamma\}_{\gamma \in \Gamma}$: an ONB of $\mathcal{H}$.

$\Rightarrow$ a Parseval frame

$\exists$ a Parseval frame that is not an ONB. If we assume $\{\phi_\gamma\}$: a Riesz basis, then $A = B = 1$ implies $\{\phi_\gamma\}$: an ONB.

(5) In the case of $A = B > 1$, $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is called a tight frame, and is redundant. Its redundancy is measured by A.

A simple example in $\mathbb{R}^2$



$$\begin{cases} \phi_1 = e_1 \\ \phi_2 = \frac{-1}{2} e_1 + \frac{\sqrt{3}}{2} e_2 \\ \phi_3 = \frac{-1}{2} e_1 - \frac{\sqrt{3}}{2} e_2 \end{cases}$$

$\{\phi_1, \phi_2, \phi_3\}$ form a frame of $\mathbb{R}^2$.

Let $\forall f \in \mathbb{R}^2$, $f = f_1 e_1 + f_2 e_2$, $f_1, f_2 \in \mathbb{R}$.

Then $\langle f, \phi_1 \rangle = f_1$, $\langle f, \phi_2 \rangle = -\frac{1}{2} f_1 + \frac{\sqrt{3}}{2} f_2$

$\langle f, \phi_3 \rangle = -\frac{1}{2} f_1 - \frac{\sqrt{3}}{2} f_2$

$$\Rightarrow \sum_{\gamma=1}^{3} |\langle f, \phi_\gamma \rangle|^2 = f_1^2 + \left(-\tfrac{1}{2} f_1 + \tfrac{\sqrt{3}}{2} f_2\right)^2 + \left(-\tfrac{1}{2} f_1 - \tfrac{\sqrt{3}}{2} f_2\right)^2$$

$$= \frac{3}{2}(f_1^2 + f_2^2) = \frac{3}{2} \| f \|^2$$

$\Rightarrow A = B = \frac{3}{2}$, i.e., it's a tight frame with 1.5 redundancy (agrees with our intuition!)

$\hookrightarrow \left\{ \sqrt{\tfrac{2}{3}} \phi_1, \sqrt{\tfrac{2}{3}} \phi_2, \sqrt{\tfrac{2}{3}} \phi_3 \right\}$ form a Parseval frame.

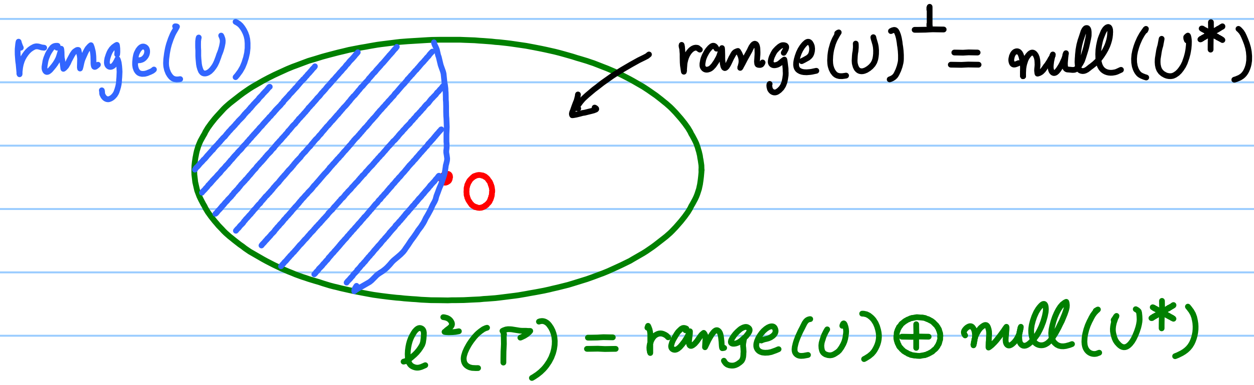
☆ Frame Operator and Reconstruction

Def. $U : \mathcal{H} \to \ell^2(\Gamma)$ is called a *frame operator* if $Uf[\gamma] = \langle f, \phi_\gamma \rangle$ for $f \in \mathcal{H}$.

Note that Uf is a sequence indexed by $\gamma \in \Gamma$.

Prop. If $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is a frame and linearly *dependent*, then

$$\text{range}(U) \subsetneq \ell^2(\Gamma).$$



(Pf) $\forall f \in \mathcal{H}$,

$$\|Uf\|^2 = \sum_{\gamma\in\Gamma} |\langle f, \phi_\gamma\rangle|^2 \le B\|f\|^2 < \infty$$

So clearly $Uf \in \ell^2(\Gamma)$ so $\text{range}(U) \subset \ell^2(\Gamma)$.

But $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is linearly dependent, so

$\exists\, \mathbb{c} \in \ell^2(\Gamma)$, $\mathbb{c} \neq \mathbb{0}$ s.t. $\sum_{\gamma\in\Gamma} c[\gamma]\phi_\gamma(x) = 0$.

Then for any $f \in \mathcal{H}$,

$$\sum_{\gamma\in\Gamma} \bar{c}[\gamma]\, Uf[\gamma] = \sum_{\gamma\in\Gamma} \bar{c}[\gamma] \langle f, \phi_\gamma\rangle$$

$$= \sum_{\gamma\in\Gamma} \langle f, c[\gamma]\phi_\gamma\rangle = \langle f, \underbrace{\sum_{\gamma\in\Gamma} c[\gamma]\phi_\gamma}_{\equiv 0}\rangle = 0$$

$\Rightarrow$ $Uf \perp \mathbb{c} \neq \mathbb{0}$, i.e., $\text{range}(U) \perp \mathbb{c} \neq \mathbb{0}$.

$\Rightarrow$ $\mathbb{c} \in \text{range}(U)^\perp = \text{null}(U^*)$

and $\text{range}(U) \subsetneqq \ell^2(\Gamma)$. ///

<u>Thm</u> The frame operator U has a pseudo inverse

$$U^+ = (U^* U)^{-1} U^*$$

s.t. $\underbrace{\|U^+\|}_{\text{operator norm}} \le \dfrac{1}{\sqrt{A}}$.

$$\|U^+\| = \sup_{\substack{\alpha\in\ell^2(\Gamma)\\ \alpha\neq 0}} \frac{\|U^+\alpha\|_{\mathcal{H}}}{\|\alpha\|_{\ell^2(\Gamma)}}$$

<u>Thm</u> Let $\{\phi_\gamma\}_{\gamma\in\Gamma}$ be a frame of $\mathcal{H}$ with its frame bounds A, B.

Define $\tilde{\phi}_\gamma := (U^* U)^{-1} \phi_\gamma \quad , \gamma \in \Gamma$

Then $\forall f \in \mathcal{H}$,

$$\frac{1}{B} \|f\|^2 \leq \sum_{\gamma\in\Gamma} |\langle f, \tilde{\phi}_\gamma \rangle|^2 \leq \frac{1}{A} \|f\|^2$$

and

$$f = U^+ U f = \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \tilde{\phi}_\gamma$$

$$= \sum_{\gamma\in\Gamma} \langle f, \tilde{\phi}_\gamma \rangle \phi_\gamma$$

<u>If</u> $A = B$ (i.e., tight), <u>then</u> $\tilde{\phi}_\gamma = \frac{1}{A}\phi_\gamma$

$A = B = 1, \|\phi_\gamma\| = 1, \forall \gamma \in \Gamma \Leftrightarrow \tilde{\phi}_\gamma = \phi_\gamma$, $\{\phi_\gamma\}$: an ONB of $\mathcal{H}$.

The system $\{\tilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ is called the **dual frame** of $\mathcal{H}$ relative to $\{\phi_\gamma\}_{\gamma\in\Gamma}$.

(Pf) Not difficult if we define the dual frame operator $\tilde{U} := U(U^*U)^{-1}$, and assume the following lemma:

<u>Lemma</u> Let $\mathcal{L}$ be a self-adjoint operator in $\mathcal{H}$ s.t. $\forall f \in \mathcal{H}$, $\exists B \geq A > 0$,

$$A\|f\|^2 \leq \langle \mathcal{L}f, f \rangle \leq B\|f\|^2.$$

Then $\mathcal{L}$ is invertible and

$$\frac{1}{B}\|f\|^2 \leq \langle \mathcal{L}^{-1} f, f \rangle \leq \frac{1}{A}\|f\|^2.$$

See Mallat's book (sec. 5.1.2 for the detail). ///

Let's go back to the WFT - Gabor proposal.

Thm (Daubechies, 1990) [The Necessary Cond.]

The WF family $\{g_{m,n}(x) = g(x - m\Delta x)\, e^{2\pi i n \Delta\xi x}\}_{(m,n) \in \mathbb{Z}^2}$

constitute a frame of $L^2(\mathbb{R})$

only if $\quad \Delta x \Delta\xi \le 1$.

$\Longrightarrow$

The frame bounds A, B necessarily satisfy

$$(*) \quad A \le \frac{1}{\Delta x \Delta\xi} \le B$$

$$\begin{cases} A \le \dfrac{1}{\Delta\xi} \displaystyle\sum_{m \in \mathbb{Z}} |g(x - m\Delta x)|^2 \le B & \forall x \in \mathbb{R} \\ A \le \dfrac{1}{\Delta x} \displaystyle\sum_{n \in \mathbb{Z}} |\hat{g}(\xi - n\Delta\xi)|^2 \le B & \forall \xi \in \mathbb{R} \end{cases}$$

$\to$ no gaps on the x and ξ axes.

Remarks:

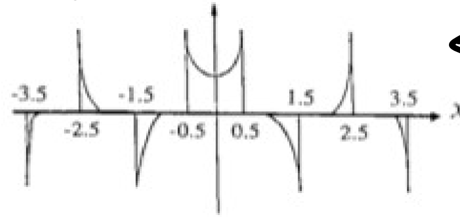
(1) If we want to make $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ an ONB of $L^2(\mathbb{R})$, then we must have $\Delta x \Delta\xi = 1$ (critical sampling) since $A = B = 1$ together with $(*)$ forces this.

necessary but not sufficient

(2) In 1980, M. Bastiaans tried to compute the dual of the Gabor frame with $\Delta x \Delta\xi = 1$. He got very singular $\{\tilde{g}_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$

g is Gaussian

e.g., $\sigma_\xi(\tilde{g}_{m,n}) = +\infty$



$\leftarrow \tilde{g}_{0,0}(x)$

(3) In early 1980's, Roger Balian tried to construct an ONB of the form $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ for a general window fcn g (not necessarily Gaussian) with $\|g\|=1$. Then he proved that if $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ form an ONB of $L^2(\mathbb{R})$, then $\Delta x \Delta \xi = 1$ without using the frame theory.

Examples of such $g = g_{0,0}$

- $g(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) \to$ generates block W.F. transform

  But, $\sigma_\xi(g_{m,n}) = +\infty$ as shown before.
- $g(x) = \operatorname{sinc}(x) \to$ generates the transf. by sharp partitioning of the frequency domain.

  But, $\sigma_x(g_{m,n}) = +\infty$

Is this an accident?

No! We have the following

Thm (Balian-Low, 1981, 1985)

Suppose $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ constitutes a windowed Fourier frame of $L^2(\mathbb{R})$ with $\Delta x \Delta \xi = 1$ (which includes the case of an ONB).

Then either $\sigma_x(g) = +\infty$ or $\sigma_\xi(g) = +\infty$.

(Pf) We only prove here the ONB case due to Guy Battle (1988). For the general non-orthogonal case including the Gabor frame case, see Daubechies & Janssen (1993).

Our strategy: assume $\sigma_x(g) < \infty$ and $\sigma_\xi(g) < \infty$. Then these lead to contradiction.

Consider $\langle xg, g' \rangle$, which also appeared in the proof of the Heisenberg inequality.

Note $xg, g' \in L^2(\mathbb{R})$ because

$$\|x g(x)\|^2 = \int x^2 |g(x)|^2 dx = \sigma_x^2(g) < +\infty$$

since $m_x(g) = 0$ and $\|g\| = 1$

Similarly, $\widehat{g'} = 2\pi i \xi \hat{g}$ & $\sigma_\xi(g) < +\infty$

lead to $\|g'\| < \infty$, i.e., $g' \in L^2(\mathbb{R})$.

Now, $\langle xg, g' \rangle \underset{\{g_{m,n}\}:\text{ONB}}{=} \sum_m \sum_n \langle xg, g_{m,n} \rangle \langle g_{m,n}, g' \rangle$

$$\overset{(a)}{=} \sum_m \sum_n \langle g_{-m,-n}, xg \rangle \langle -(g')_{m,n}, g \rangle$$

$$\overset{(b)}{=} \sum_m \sum_n \langle g_{-m,-n}, xg \rangle \langle -g', g_{-m,-n} \rangle$$

$$= \sum_m \sum_n \langle -g', g_{m,n} \rangle \langle g_{m,n}, xg \rangle$$

$$\underset{\{g_{m,n}\}:\text{ONB}}{=} -\langle g', xg \rangle \qquad (1)$$

We'll show the justification (a), (b) later.

Now, consider a fcn $f \in C_c^\infty(\mathbb{R})$, i.e., a space of C^∞ fcn's vanishing as $|x| \to \infty$.

Then, $\langle xf, f' \rangle = \int_{-\infty}^{\infty} x f(x) \overline{f'(x)} dx$

$$= \cancel{x f(x) \overline{f(x)} \Big|_{-\infty}^{\infty}} - \int \overline{f(x)} (x f'(x) + f(x)) dx$$

$$= -\int x \overline{f(x)} f'(x) dx - \|f\|^2$$

$$= -\langle f', xf \rangle - \|f\|^2$$

Since $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{H} = \{f \in L^2 \mid xf, f' \in L^2\}$, the window fcn g under consideration must satisfy $\langle xg, g' \rangle = -\langle g', xg \rangle - \|g\|^2$ (2)

Combining (1) & (2), we conclude $\|g\| = 0$, which contradicts with $\|g\| = 1$ #

Finally, the justification of (a):

$$\langle xg, g_{m,n} \rangle = \langle g_{-m,-n}, xg \rangle \ \& \ \langle g_{m,n}, g' \rangle = \langle -(g')_{m,n}, g \rangle$$

$\because$ $\langle xg, g_{m,n} \rangle = \int x g(x) \overline{g(x - m\Delta x)} e^{-2\pi i n \Delta\xi x} dx$

$y = x - m\Delta x$

$$= \int (y + m\Delta x) g(y + m\Delta x) \overline{g(y)} e^{-2\pi i n \Delta\xi y} \cdot \underbrace{e^{-2\pi i n \Delta\xi m \Delta x}}_{= e^{-2\pi i n m} = 1} dy$$

$$= \int (x + m\Delta x) g(x + m\Delta x) \overline{g(x)} e^{-2\pi i n \Delta\xi x} dx$$ since $\Delta x \Delta\xi = 1$.

$$= \langle g_{-m,-n}, xg \rangle + \underbrace{m\Delta x \langle g_{-m,-n}, g \rangle}_{=0}$$

$$= \langle g_{-m,-n}, xg \rangle \quad //$$

$=0$ since $g = g_{0,0}$. $\{g_{m,n}\}$: ONB. so, $(m,n) \neq (0,0)$. $g_{-m,-n} \perp g$ if $(m,n) = (0,0)$. $m\Delta x = 0$.

$$\langle g_{m,n}, g' \rangle = \int_{-\infty}^{\infty} g_{m,n}(x) \overline{g'(x)} dx$$

$$= g_{m,n}(x) \overline{g(x)} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (g_{m,n}(x))' \overline{g(x)} dx$$

$g \in L^2$

$$= \langle -(g')_{m,n}, g \rangle \underbrace{- 2\pi i n \Delta\xi \langle g_{m,n}, g \rangle}_{=0}$$

$$= \langle -(g')_{m,n}, g \rangle \quad //$$

the same logic

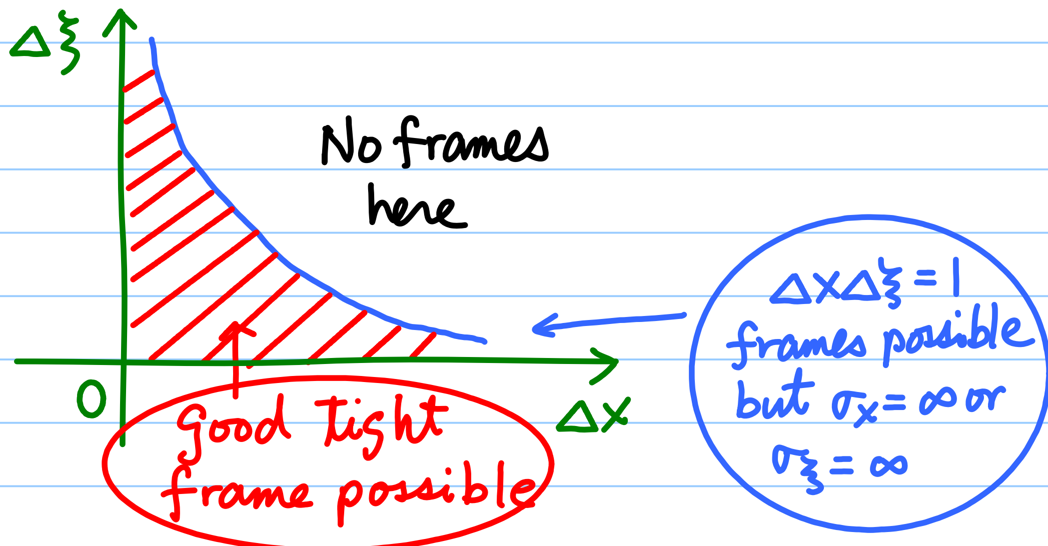
How about the justification (b)?

$\langle -(g')_{m,n}, g \rangle = \langle -g', g_{-m,-n} \rangle$

Use the same logic as the first part of (a). ///

Continuation of Remarks

(4) In 1990, Daubechies summarized the frame conditions for $\{g_{m,n}\}$ w.r.t. Δx & $\Delta\xi$ as follows:



$\Delta x \Delta\xi \le 1$ was the necessary cond. for $\{g_{m,n}\}$ to form a frame. How about the sufficient cond.?

Thm. (Daubechies, 1990)

See the original paper for the proof.

Let $\beta(u) := \sup_{0 \le x \le \Delta x} \sum_{m \in \mathbb{Z}} |g(x - m\Delta x)| \, |g(x - m\Delta x + u)|$

$$\Delta := \sum_{k \in \mathbb{Z}}{}' \left[\beta\left(\frac{k}{\Delta\xi}\right) \beta\left(\frac{-k}{\Delta\xi}\right) \right]^{\frac{1}{2}}$$

If Δx, $\Delta\xi$ satisfy

$$A_0 := \frac{1}{\Delta\xi} \left(\inf_{0 \le x \le \Delta x} \sum_{m \in \mathbb{Z}} |g(x - m\Delta x)|^2 - \Delta \right) > 0$$

$$B_0 := \frac{1}{\Delta\xi} \left(\sup_{0 \le x \le \Delta x} \sum_{m \in \mathbb{Z}} |g(x - m\Delta x)|^2 + \Delta \right) < \infty,$$

then $\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ is a frame with $\begin{cases} A_0 = \inf A \\ B_0 = \sup B. \end{cases}$