Lecture 14: Continuous Wavelet Transf. II

Note Title

In order to discuss the so-called analytic wavelets, we need to know a bit about the concept of analytic signals.

better than real-valued wavelets in
1) capturing phase info; 2) time-freq. tiling.

☆ Analytic Signal

Def. $f_a \in L^2(\mathbb{R})$ is said to be analytic if $\hat{f}_a(\xi) = 0 \quad \forall \xi < 0$.

$f_a(x) \in \mathbb{C}$, but $\exists$ a special relationship between $\mathrm{Re}(f_a)$ & $\mathrm{Im}(f_a)$:

$$\mathrm{Im}(f_a)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Re}(f_a)(y)}{x-y}\,dy$$

$$= \frac{1}{\pi x} * \mathrm{Re}(f_a) = \mathcal{H}[\mathrm{Re}(f_a)](x)$$

The Hilbert transform on $\mathbb{R}$

Let $f(x) = \mathrm{Re}(f_a)(x)$. Then,

$$f_a(x) = f(x) + i\,\mathcal{H}f(x)$$

$$\downarrow \mathcal{F}$$

$$\hat{f}_a(\xi) = \hat{f}(\xi) + i\left(\frac{1}{\pi x}\right)^{\wedge} \cdot \hat{f}(\xi) = \hat{f}(\xi) + i(-i\,\mathrm{sgn}\,\xi)\hat{f}(\xi)$$

$$= \hat{f}(\xi)(1+\mathrm{sgn}(\xi)) = \begin{cases} 2\hat{f}(\xi) & \text{if } \xi \ge 0 \\ 0 & \text{if } \xi < 0. \end{cases}$$

Ex. $f(x) = a\cos(2\pi\xi_0 x + \theta)$, $a, \theta \in \mathbb{R}$, $\xi_0 > 0$.

$\Rightarrow f_a(x) = a e^{i(2\pi\xi_0 x + \theta)}$ an easy exercise!

$f, f_a \notin L^2(\mathbb{R})$, but the above result still holds thanks to the theory of distributions.

Def. an *analytic wavelet fcn* $\psi \in L^2(\mathbb{R})$ is a wavelet that is also an analytic signal, i.e., it's $\mathbb{C}$-valued and satisfies

basic prop.
- $\int_{-\infty}^{\infty} \psi(x)\, dx = 0$ (i.e., $\hat{\psi}(0) = 0$)
- $\| \psi \|_2 = 1$
- $\psi(x)$ is centered around $x = 0$

admissibility cond. $\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi < +\infty$

analyticity • $\hat{\psi}(\xi) = 0$ for $\xi < 0$ (i.e., $\xi \leq 0$)

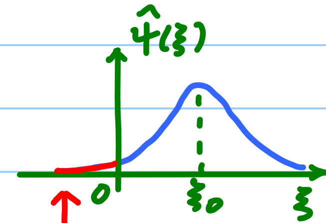
Examples

- **Morlet wavelet**

$$\psi(x) = \pi^{-1/4} e^{2\pi i \xi_0 x} e^{-x^2/2}$$

$\rightarrow$ $\hat{\psi}(\xi) = \sqrt{2}\, \pi^{-1/4} e^{-2\pi^2(\xi - \xi_0)^2}$

$\rightarrow$ Not exactly analytic, but close.



$< 5.037 \times 10^{-9}$ if $\xi_0 > 1$.

- **Generalized Morse wavelet (family)**

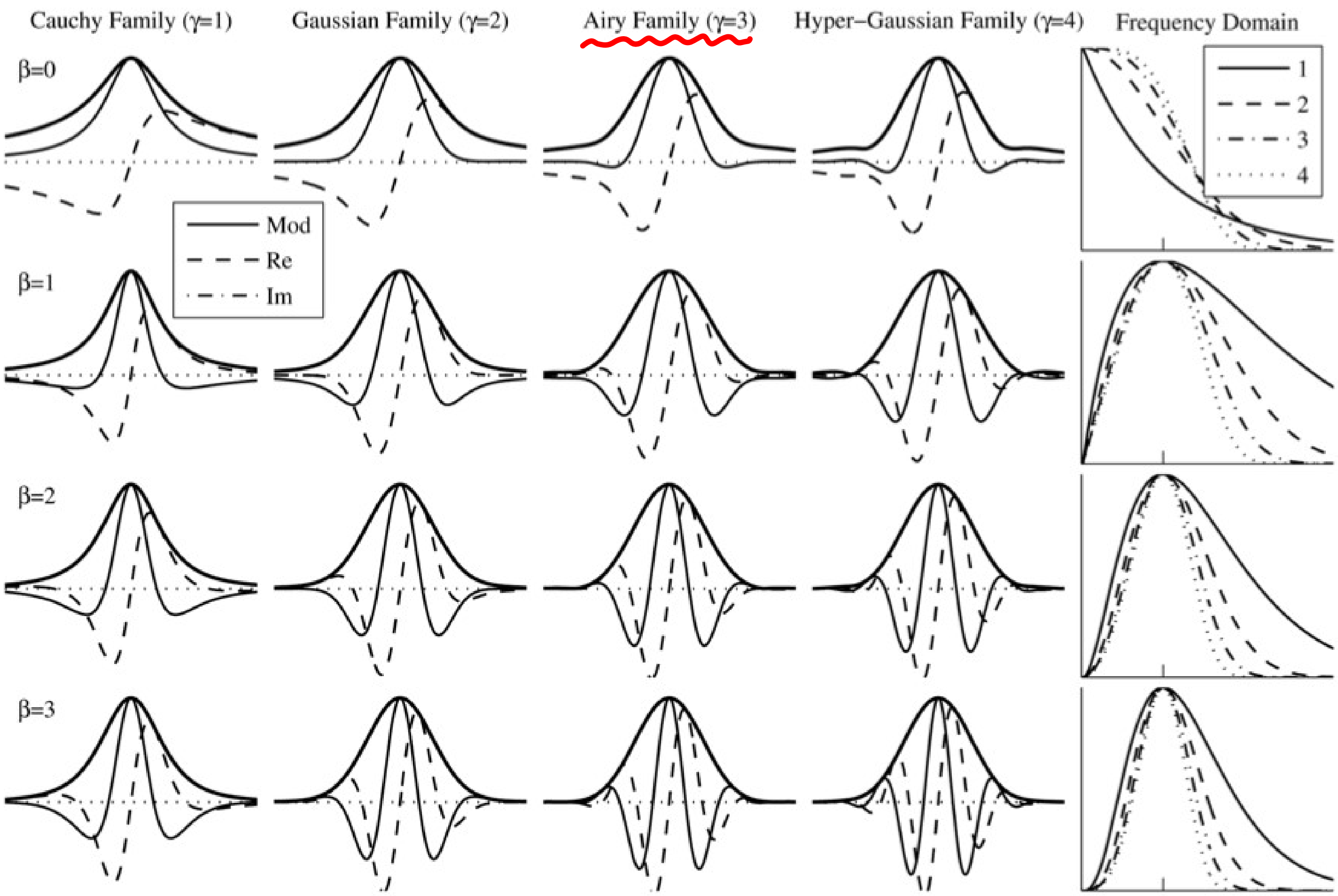
$$\hat{\psi}(\xi) = \chi_{[0,\infty)}(\xi)\, c_{\beta,\gamma}\, \xi^{\beta} e^{-(2\pi\xi)^{\gamma}}, \quad \beta, \gamma > 0$$

$c_{\beta,\gamma}$ is a normalization const.

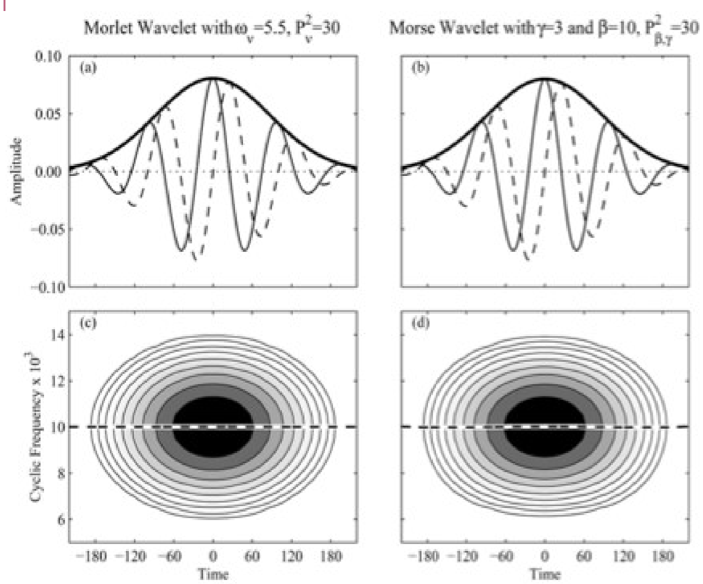
$\rightarrow$ Exactly analytic!

$\rightarrow$ This family includes many of the previously proposed analytic wavelets e.g., Bessel, Cauchy, analytic version of Mexican hat, Shannon.

$\rightarrow$ $\gamma = 3$ case closely approximates Morlet.



Morlet vs Morse



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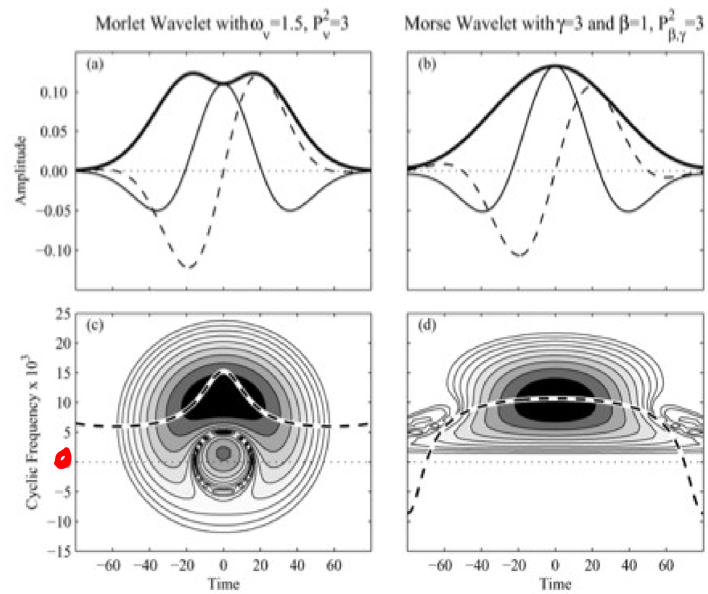
Morlet vs Morse



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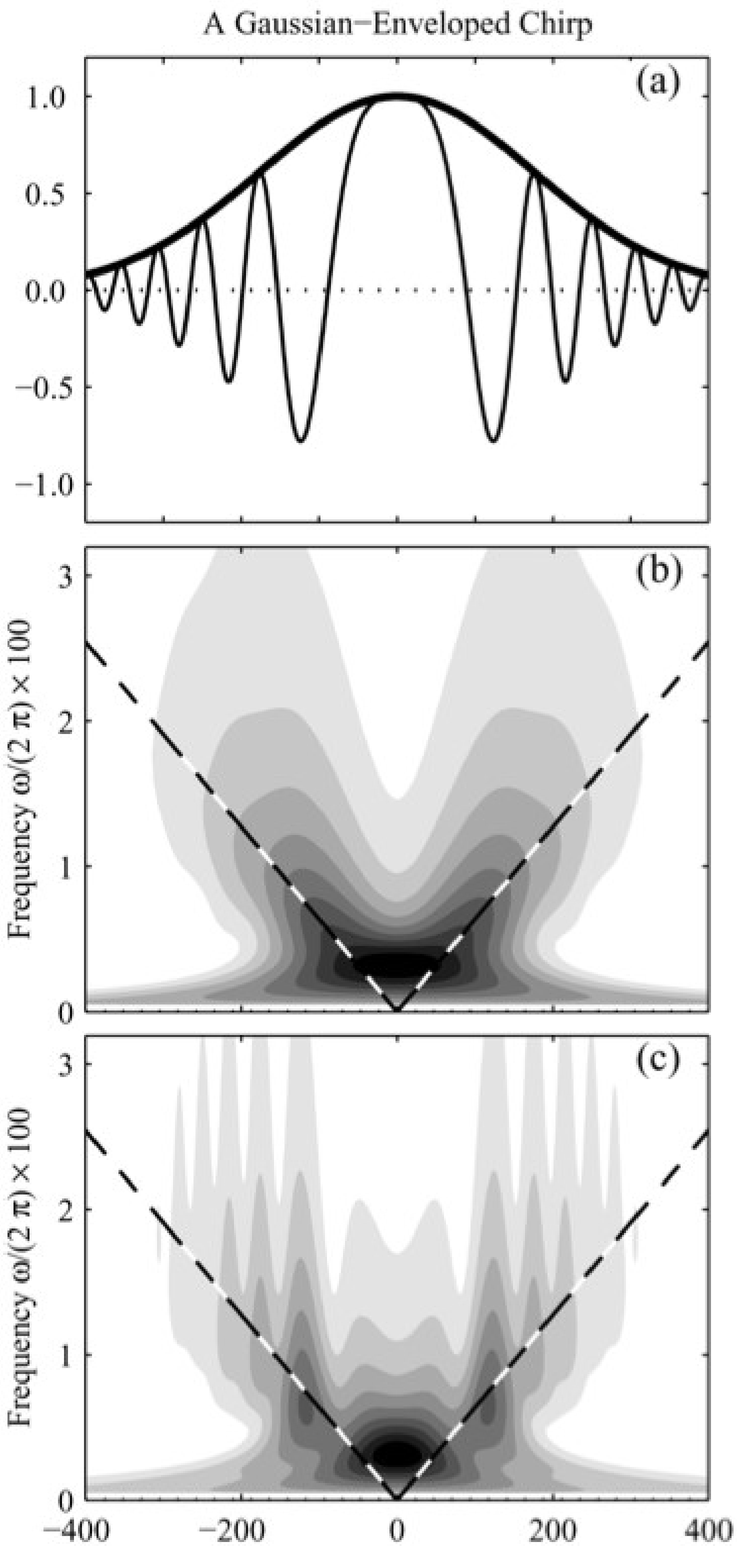


Exact analyticity is important for signal analysis;

non-analyticity leads to interference and artifacts in the time-freq. plane, and consequently to errorneous amplitude & phase estimates.

Morse

Morlet

☆ Heisenberg Box of analytic wavelets

$$Wf(a,b) = \langle f, \psi_{a,b} \rangle = \int f(x) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx$$

Suppose $m_x(\psi) = \int_{-\infty}^{\infty} x\, |\psi(x)|^2 dx = 0$.

Then $m_x(\psi_{a,b}) = \int_{-\infty}^{\infty} x \frac{1}{a} \left|\psi\left(\frac{x-b}{a}\right)\right|^2 dx$

$\frac{x-b}{a} = y$

$$= \int_{-\infty}^{\infty} (ay+b)\, |\psi(y)|^2 dy$$

$$= b \int_{-\infty}^{\infty} |\psi(y)|^2 dy = b \quad ✓$$

$$\sigma_x^2(\psi_{a,b}) = \int_{-\infty}^{\infty} (x-b)^2 \frac{1}{a} \left|\psi\left(\frac{x-b}{a}\right)\right|^2 dy$$

$$= \int_{-\infty}^{\infty} a^2 y^2 |\psi(y)|^2 dy = a^2 \sigma_x^2(\psi)$$

Now, how about these in the freq. domain?

$$m_\xi(\psi) = \int_{-\infty}^{\infty} \xi\, |\hat{\psi}(\xi)|^2 d\xi = \underbrace{\int_{0}^{\infty} \xi\, |\hat{\psi}(\xi)|^2 d\xi}_{\text{the center freq. of } \psi}$$

$$m_\xi(\psi_{a,b}) = \int_{-\infty}^{\infty} \xi\, |\hat{\psi}_{a,b}(\xi)|^2 d\xi$$

$$= \int_{-\infty}^{\infty} \xi\, a\, |\hat{\psi}(a\xi)|^2 d\xi$$

$$= \frac{1}{a} \int_{0}^{\infty} \eta\, |\hat{\psi}(\eta)|^2 d\eta = \frac{m_\xi(\psi)}{a}$$

$$\sigma_\xi^2(\psi_{a,b}) = \int_{-\infty}^{\infty} \left(\xi - \frac{m_\xi}{a}\right)^2 |\hat{\psi}_{a,b}(\xi)|^2 d\xi$$

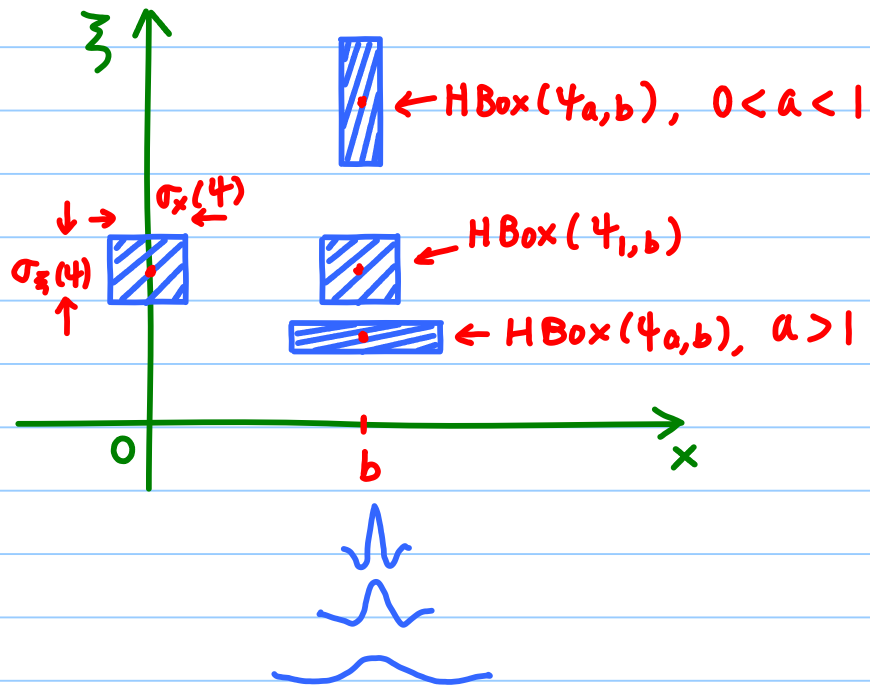
$$= \int_{-\infty}^{\infty} \left(\xi - \frac{m_\xi}{a}\right)^2 a \cdot |\hat{\psi}(a\xi)|^2 d\xi$$

$$= \frac{1}{a^2}\int_0^\infty (\eta - m_\xi(\psi))^2 |\hat{\psi}(\eta)|^2 d\eta$$

$$= \frac{\sigma_\xi^2(\psi)}{a^2}$$

<u>Summary</u>

$$\begin{cases} m_x(\psi_{a,b}) = b, & \sigma_x(\psi_{a,b}) = a\,\sigma_x(\psi) \\ m_\xi(\psi_{a,b}) = m_\xi(\psi)/a, & \sigma_\xi(\psi_{a,b}) = \sigma_\xi(\psi)/a \end{cases}$$



small a
~ high freq.

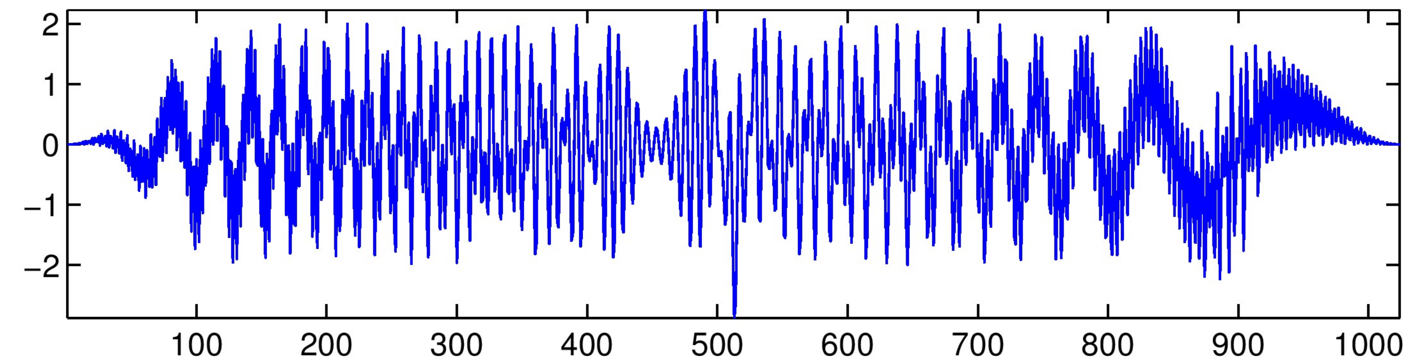
We can now compute the local time-freq. energy density of $f \in L^2(\mathbb{R})$ as

$$P_W f(x,\xi) := |Wf(a,x)|^2 = \left|Wf\left(\frac{m_\xi}{\xi}, x\right)\right|^2$$

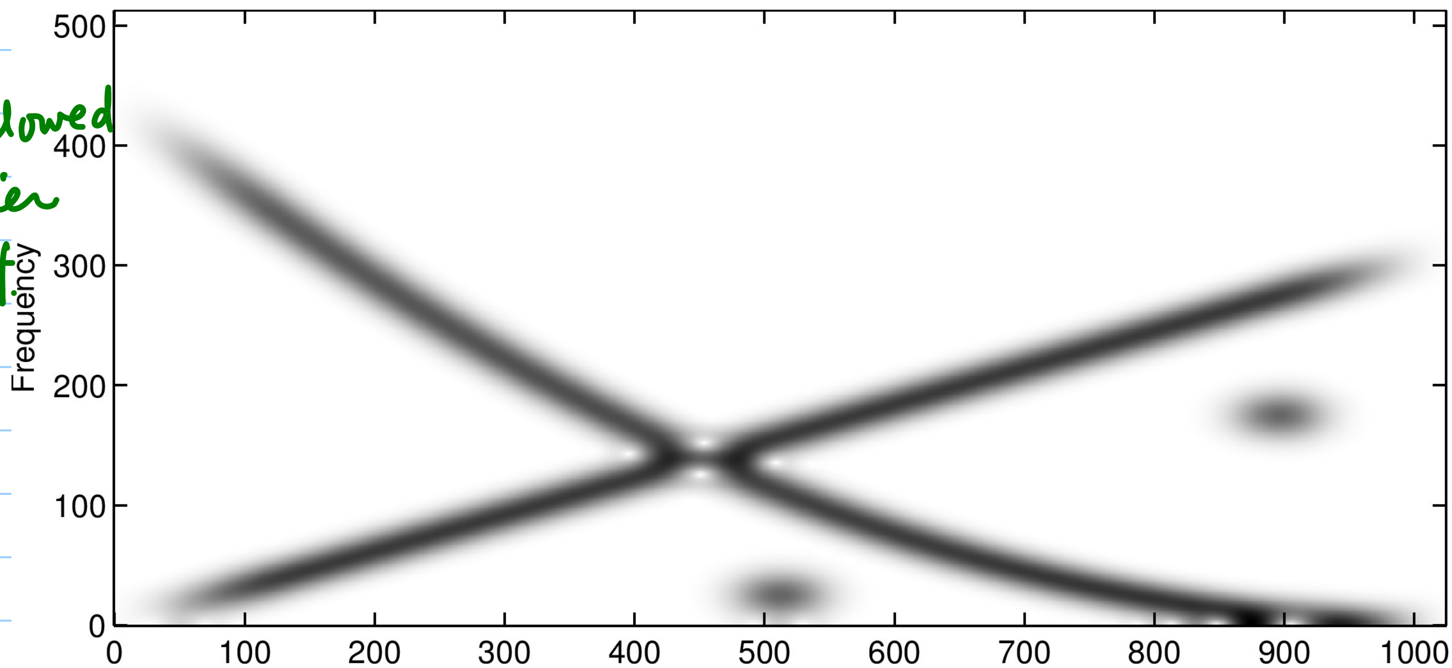
$$\xi = \frac{m_\xi(\psi)}{a}$$

This 2D plot is called the scalogram of f.

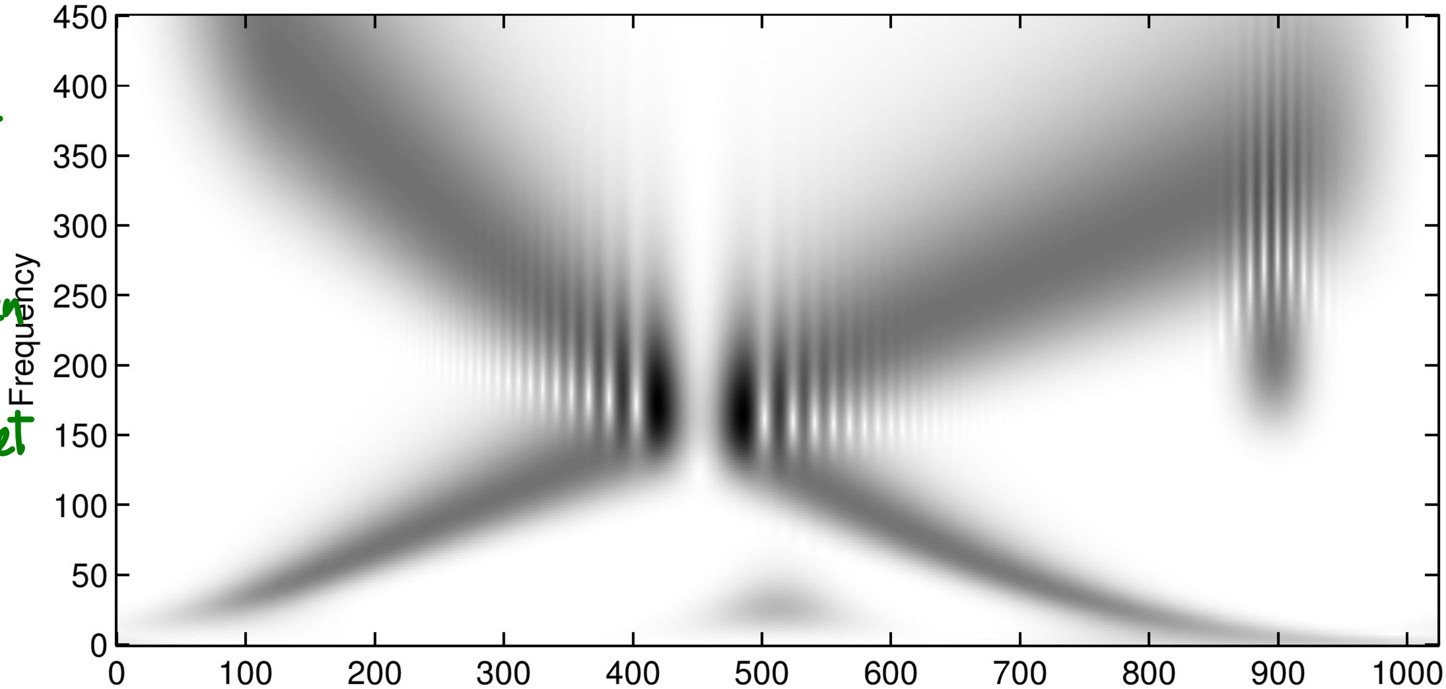
<u>Warning</u>: $|Wf(a,b)|^2$ is sometimes called the scalogram too.



Windowed Fourier Transf



Scalo-gram with the Mexican hat wavelet



Remarks :

(1) Using the truly analytic wavelets (e.g. generalized Morse wavelets), the scalogram should become more focussed & less artifacts.

(2) $\exists$ a sharpening technique called "synchrosqueezing" wavelet transform

$\Rightarrow$ A possible final project

(3) $\mathbb{C}$-valued wavelets have gained popularity among discrete wavelet transforms! $\Rightarrow$ The Dual Tree $\mathbb{C}$WT.

(4) What is the extension of "analyticity" in higher dimensions?

$\Rightarrow$ monogenicity