Lecture 15 : Wavelet Bases
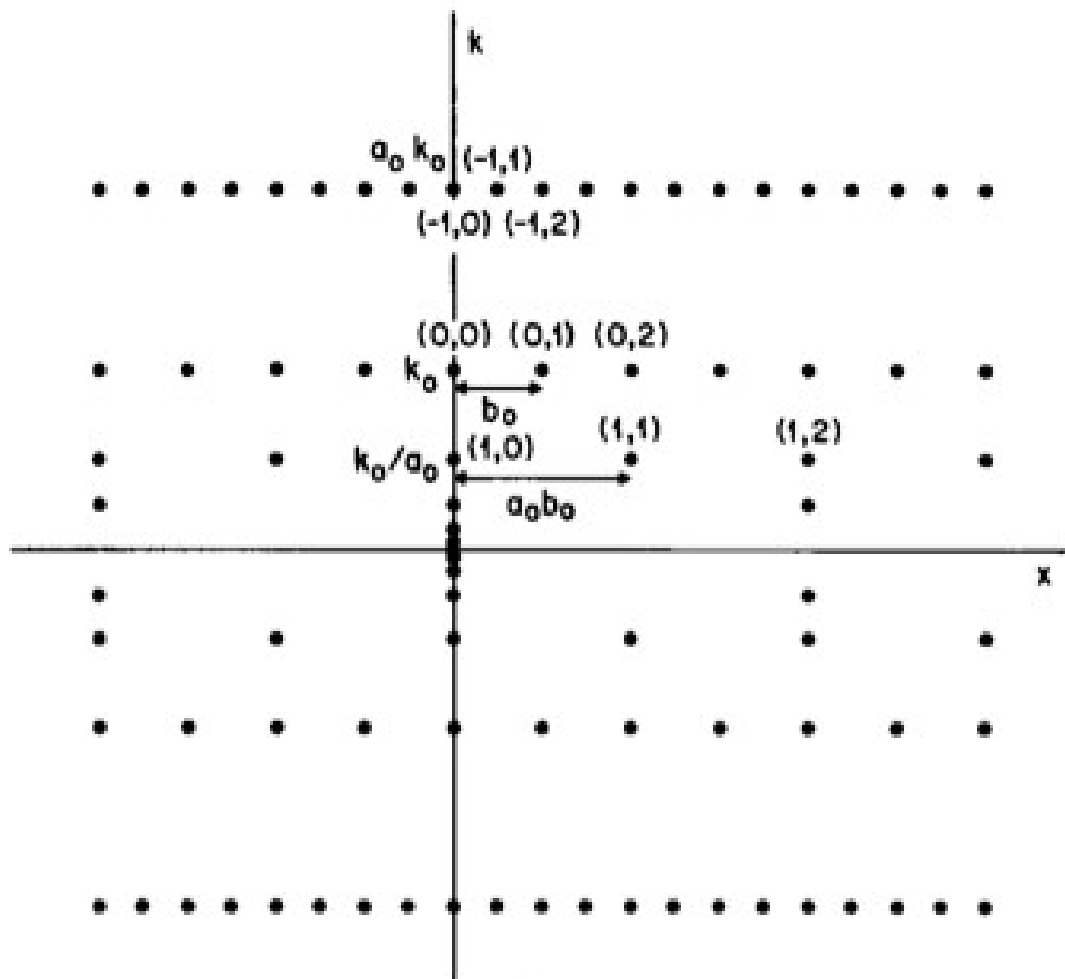
- Recall the sampling lattice pattern ("regular hyperbolic grid") to generate the discrete wavelet transf. from the continuous wavelet transf. $Wf(a,b)$.

$$(a,b) = (a_0^m, n a_0^m b_0), \quad m, n \in \mathbb{Z}$$

$$\psi_{m,n}(x) := a_0^{-m/2} \psi(a_0^{-m} x - n b_0), \quad m, n \in \mathbb{Z}$$

$$= \frac{1}{\sqrt{a_0^m}} \psi\left(\frac{x - n a_0^m b_0}{a_0^m}\right)$$

The most common/popular choice of (a,b) is $a = 2^j, \quad b = 2^j k$ i.e., $a_0 = 2, \quad b_0 = 1$.

$$\Rightarrow \psi_{j,k}(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x - 2^j k}{2^j}\right) = 2^{-j/2} \psi(2^{-j} x - k), \quad (j,k) \in \mathbb{Z}^2.$$

☆ Multiresolution Approximation

The parameter 2^{-j} represents **resolution** whereas 2^j represents **scale**, i.e.,

high resolution $\Longleftrightarrow$ fine (small) scale

low " coarse (large) "

For example, for a fixed j, consider the translated & dilated versions of a father wavelet : $\{\phi_{jk}\}_{k \in \mathbb{Z}}$, and the approximation (or projection) of $f \in L^2(\mathbb{R})$ as follows :

$$f_j(x) = \sum_{k \in \mathbb{Z}} <f, \phi_{jk}> \phi_{jk}(x)$$

Multiresolution Approximation with Haar Basis

$\phi_{j_0}(x)$

Data

S1

S2

S3

S4

S5

S6

0   200   400   600   800   1000

View the approximation of f at resolution 2^{-j} as an orthogonal projection P_j (or P_{V_j}) of f onto a subspace :

$$V_j := \overline{\operatorname{span}\{\phi_{jk}, k \in \mathbb{Z}\}} \subset L^2(\mathbb{R})$$

<u>Def.</u> (Multiresolution Approximation)

 a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ is called a *multiresolution approximation* (MRA) if the following six properties are satisfied :

(1) $\forall (j,k) \in \mathbb{Z}^2, \quad f(x) \in V_j \iff f(x - 2^j k) \in V_j$

j↓: high res. (2) $\forall j \in \mathbb{Z}, \quad V_{j+1} \subset V_j$

j↑: low res. (3) $\forall j \in \mathbb{Z}, \quad f(x) \in V_j \iff f(\frac{x}{2}) \in V_{j+1}$

(4) $\displaystyle\lim_{j \to \infty} V_j = \bigcap_{j=-\infty}^{\infty} V_j = \{0\}$

(5) $\displaystyle\lim_{j \to -\infty} V_j = \overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R})$

(6) $\exists \theta \in V_0$, s.t. $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ form a <u>Riesz basis</u> for V_0.

linearly indep. frame (i.e., a basis)
with $0 < A \le 1 \le B$ (if $\|\theta\| = 1$)

<u>Remark:</u>

$P_{V_j} f \in V_j, \quad P_{V_j}^2 = P_{V_j}, \quad P_{V_j}^* = P_{V_j}$

proj   orthogonal proj

$\displaystyle\lim_{j \to \infty} \|P_{V_j} f\| = 0, \quad \lim_{j \to -\infty} \|f - P_{V_j} f\| = 0.$

- Now $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_0
$\iff \forall f \in V_0,\ \exists!\ \{a_k\} \in \ell^2(\mathbb{Z})$ s.t.

$$f(x) = \sum_{k\in\mathbb{Z}} a_k\, \theta(x-k) \quad \text{with}$$

$$A \|f\|^2 \le \sum |a_k|^2 \le B\|f\|^2,\quad \exists A \not\ge 0\ \exists B > 0.$$

Then, we can adapt this $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ by dilation for V_j, i.e.,

$$\left\{ 2^{-j/2}\, \theta(2^{-j}x - k) \right\}_{k\in\mathbb{Z}} \quad \text{is a Riesz basis for } V_j$$

with the same frame bounds A, B.

<u>Prop.</u> A family $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for $V_0 = \overline{\text{span}\{\theta(x-k)\}}_{k\in\mathbb{Z}} \iff \exists\ B \ge A > 0$ s.t.

$$\forall \xi \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right],\quad \frac{1}{B} \le \sum_{k\in\mathbb{Z}} |\hat{\theta}(\xi-k)|^2 \le \frac{1}{A}$$

(Pf) ($\Leftarrow$) Take any $f \in V_0$.

$$f(x) = \sum_{k\in\mathbb{Z}} a_k\, \theta(x-k)$$

$$\downarrow \mathcal{F}$$

$$\hat{f}(\xi) = \hat{a}(\xi)\,\hat{\theta}(\xi),\qquad \hat{a}(\xi) := \sum_{k\in\mathbb{Z}} a_k\, e^{-2\pi i k\xi}$$

$\hookrightarrow$ 1-periodic!

Now, $\|f\|^2 = \|\hat{f}\|^2 = \displaystyle\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$

$$= \int_{-1/2}^{1/2} \sum_{\ell\in\mathbb{Z}} |\hat{a}(\xi+\ell)|^2\, |\hat{\theta}(\xi+\ell)|^2\, d\xi$$

$\hat{a}(\xi)$: 1-periodic $\longrightarrow$

$$= \int_{-1/2}^{1/2} |\hat{a}(\xi)|^2 \sum_{\ell\in\mathbb{Z}} |\hat{\theta}(\xi+\ell)|^2\, d\xi$$

Hence, together with the assumption, we have

$$\frac{1}{B} \int_{-1/2}^{1/2} |\hat{a}(\xi)|^2 d\xi \leq \|f\|^2 \leq \frac{1}{A} \int_{-1/2}^{1/2} |\hat{a}(\xi)|^2 d\xi$$

But $\int_{-1/2}^{1/2} |\hat{a}(\xi)|^2 d\xi = \sum_{k \in \mathbb{Z}} |a_k|^2$ via Parseval

Hence, $A \|f\|^2 \leq \sum_{k \in \mathbb{Z}} |a_k|^2 \leq B \|f\|^2$ ✓

($\Rightarrow$) If $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for V_0,
then $\exists \; B \geq A > 0$ s.t.

$$A \|f\|^2 \leq \underbrace{\sum_{k \in \mathbb{Z}} |a_k|^2}_{= \int_{-1/2}^{1/2} |\hat{a}(\xi)|^2 d\xi} \leq B \|f\|^2$$

Reversing the argument, we get

$$\frac{1}{B} \leq \sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi - k)|^2 \leq \frac{1}{A}, \quad \forall \xi \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

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Example 1. $\theta(x) = \chi_{[0,1)}(x)$

$\Rightarrow$ Piecewise constant approximation
$$\theta_{j,k}(x) = 2^{-j/2} \chi_{[k2^j, (k+1)2^j)}(x)$$
$\{\theta_{j,k}\}_{k \in \mathbb{Z}}$ form an ONB for V_j.

Example 2. $\theta(x) = \text{sinc}(x) = \dfrac{\sin \pi x}{\pi x}$
$$\Leftrightarrow \hat{\theta}(\xi) = \chi_{[-1/2, 1/2)}(\xi)$$

This is also called the Shannon scaling fcn.

$\theta_{j,k}(x) = 2^{-j/2} \text{sinc}(2^{-j}x - k) \Leftrightarrow \hat{\theta}_{j,k}(\xi) = 2^{j/2} e^{-2\pi i k 2^j \xi} \chi_{[-2^{-j-1}, 2^{-j-1})}(\xi)$

$\{ \theta_{j,k} \}_{k \in \mathbb{Z}}$ form an ONB for $V_j = BL_{2^{-j}}(\mathbb{R})$
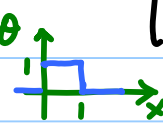
Band limited fcn
with band width 2^{-j}

Example 3. Spline of order m
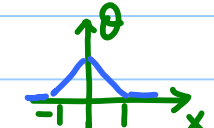(a.k.a. cardinal B-spline)

$$\theta(x) = \chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]}(x)$$

m times convolution

with centering at $x = \begin{cases} 0 & \text{if } m = \text{odd} \\ \frac{1}{2} & \text{if } m = \text{even} \end{cases}$
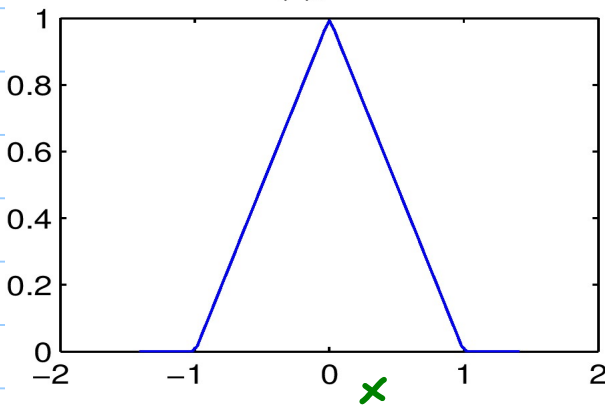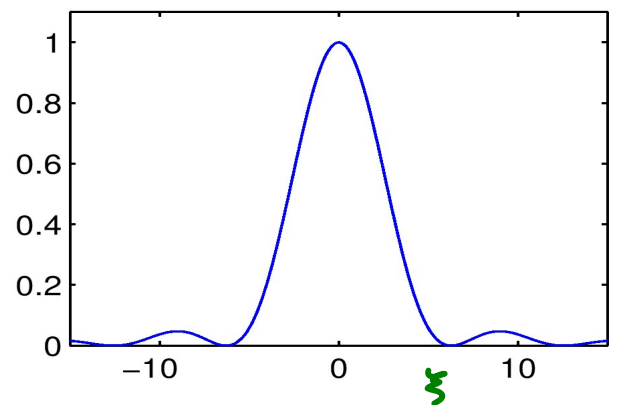
$m = 0$: Box spline

$m = 1$: linear spline (hat fcn)

$$\hat{\theta}(\xi) = \left(\frac{\sin \pi \xi}{\pi \xi} \right)^{m+1} e^{-i\varepsilon \pi \xi} \qquad \varepsilon = \begin{cases} 0 & m = \text{odd} \\ 1 & m = \text{even} \end{cases}$$
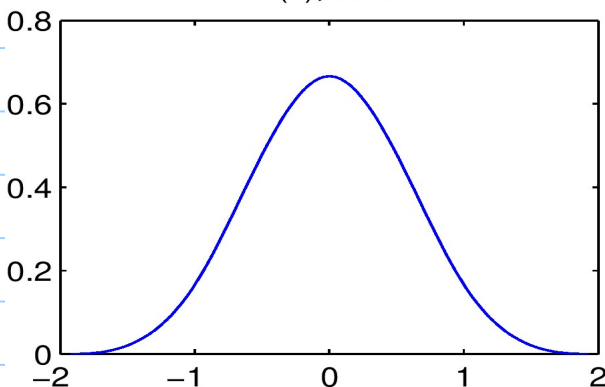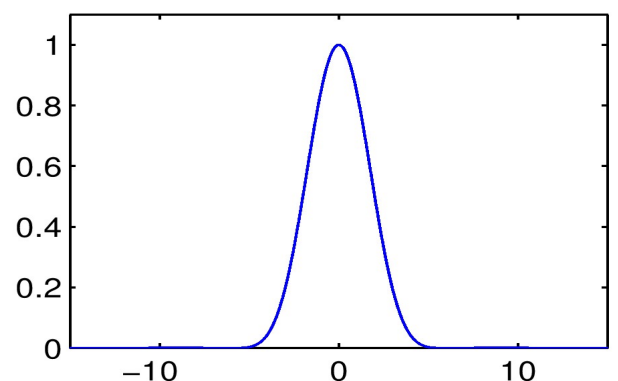
half-shift factor

$\mathcal{F}$

$\theta(x)$, m=1

$\Theta(\xi)$, m=1

x

ξ

$\theta(x)$, m=3

$\Theta(\xi)$, m=3

✱ Scaling Functions (Father Wavelets)

Can we construct an ONB from the Riesz basis $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ for V_0 ?

$\Rightarrow$ Yes !

Thm (Mallat '89)

Let $\{V_j\}_{j\in\mathbb{Z}}$ be an MRA of $L^2(\mathbb{R})$.
Let $\phi(x)$ be the scaling fcn whose Fourier transf. is

$$\hat{\phi}(\xi) = \frac{\hat{\theta}(\xi)}{\left(\sum_{k\in\mathbb{Z}}|\hat{\theta}(\xi-k)|^2\right)^{1/2}}$$

division by this quantity is often called the orthogonalization trick.

Then, $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ form an ONB for V_j, $\forall j\in\mathbb{Z}$.

(Pf) Let $\phi \in V_0$.

$\Leftrightarrow \quad \phi(x) = \sum_{k\in\mathbb{Z}} a_k\,\theta(x-k)$, $\exists\,\{a_k\}\in\ell^2(\mathbb{Z})$

$\Leftrightarrow \quad \hat{\phi}(\xi) = \hat{a}(\xi)\,\hat{\theta}(\xi)$, $\hat{a}$: 1-periodic Fourier series and $\hat{a}\in L^2[-\frac{1}{2},\frac{1}{2}]$.

We want to impose the orthogonality on ϕ, i.e.,

$$\langle \phi(\cdot-k), \phi(\cdot-\ell)\rangle = \delta_{k\ell}$$

But $\langle\phi(\cdot-k), \phi(\cdot-\ell)\rangle = \displaystyle\int_{-\infty}^{\infty} \phi(x-k)\,\overline{\phi(x-\ell)}\,dx$

$$= (\phi * \tilde{\phi})(\ell-k)$$

where $\tilde{\phi}(x) = \overline{\phi(-x)}$

Note: $\mathcal{F}[\phi * \tilde{\phi}](\xi) = |\hat{\phi}(\xi)|^2$

Thus, $\{\phi(x-k)\}_{k \in \mathbb{Z}}$: an ONB

$$\iff \phi * \tilde{\phi}(k) = \delta_{k0}$$

$$\iff \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi-k)|^2 \equiv 1 \quad \text{a.e. } \xi \in \mathbb{R} \quad \text{---}(*)$$

Hence if we choose

$$\hat{a}(\xi) = \left(\sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi-k)|^2 \right)^{-1/2}$$

<u>then</u> (*) is satisfied.

$$\Rightarrow \hat{\phi}(\xi) = \frac{\hat{\theta}(\xi)}{\left(\sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi-k)|^2 \right)^{1/2}}, \quad \text{and}$$

this is well defined since $\sqrt{A} \leq \hat{a}(\xi) \leq \sqrt{B}$ ///

Using these scaling fcn's, we can easily construct $\mathbb{P}_{V_j}$ as

$$\mathbb{P}_{V_j} f = \sum_{k \in \mathbb{Z}} \underbrace{\langle f, \phi_{j,k} \rangle}_{=: f_k^j} \phi_{j,k}$$

$$f_k^j := \langle f, \phi_{j,k} \rangle = \int f(x) \frac{1}{\sqrt{2^j}} \overline{\phi\left(\frac{x-2^j k}{2^j}\right)} \, dx$$

$$= (f * \tilde{\phi}_j)(2^j k)$$

$$\tilde{\phi}_j(x) = \frac{1}{\sqrt{2^j}} \overline{\phi\left(\frac{-x}{2^j}\right)} = \delta_{2^j} \tilde{\phi}(x)$$

So, $f * \tilde{\phi}_j \xrightarrow{\mathcal{F}} \hat{f} \cdot \delta_{2^{-j}} \hat{\tilde{\phi}} = \hat{f}(\xi) \cdot \sqrt{2^j} \overline{\hat{\phi}(2^j \xi)}$

$\Rightarrow f * \tilde{\phi}_j(2^j k)$ means low-pass filtering f followed by sampling at the rate 2^j.

Example 1. Haar's scaling fcn

Example 2. Shannon's scaling fcn

In these two cases, $\theta = \phi$ since $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ are already orthonormal. //

Example 3. Spline scaling fcn.

$$\hat{\phi}(\xi) = \frac{e^{-i\varepsilon\pi\xi}}{\xi^{m+1}\sqrt{S_{2m+2}(\xi)}}, \qquad \varepsilon = \begin{cases} 0 & m = \text{odd} \\ 1 & m = \text{even} \end{cases}$$

where $\quad S_n(\xi) := \sum_{k \in \mathbb{Z}} (\xi+k)^{-n}$

$\underline{m = 0}:\ S_2(\xi) = \sum_{k \in \mathbb{Z}} (\xi+k)^{-2} = \pi^2 \text{cosec}^2 \pi\xi = \left(\frac{\pi}{\sin\pi\xi}\right)^2$

$$\Rightarrow \hat{\phi}(\xi) = e^{-i\pi\xi} \frac{1}{\xi} \cdot \frac{\sin\pi\xi}{\pi} = e^{-i\pi\xi} \text{sinc}(\xi)$$
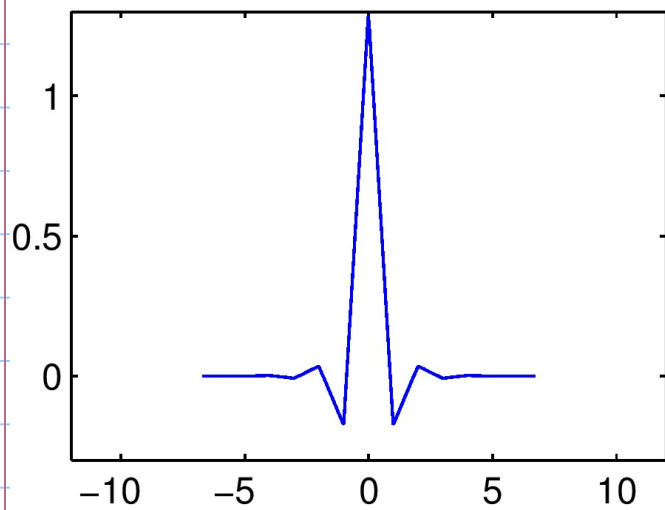
$\underline{m = 1}:\ S_4(\xi) = \sum (\xi+k)^{-4}$
how to get this?
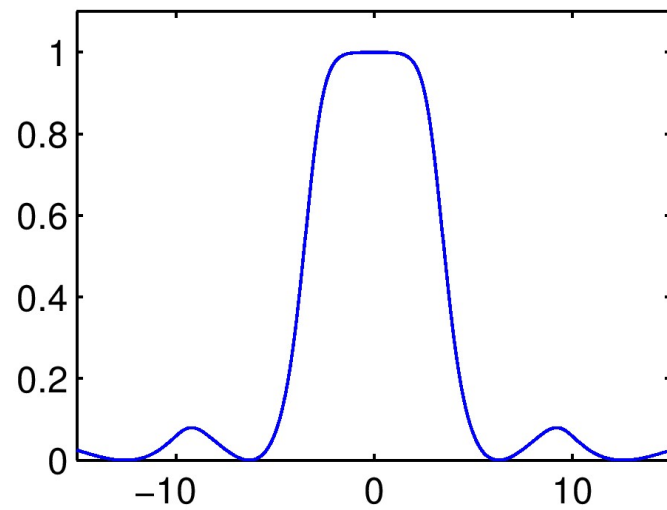
$\Rightarrow S_2''(\xi) = 6 \sum (\xi+k)^{-4}$, i.e.,

$S_4(\xi) = \frac{1}{6} S_2''(\xi) = \cdots = \frac{\pi^4}{3} \frac{1 + 2\cos^2\pi\xi}{\sin^4\pi\xi}$

$$\Rightarrow \hat{\phi}(\xi) = \frac{\sqrt{3}}{\sqrt{1 + 2\cos^2\pi\xi}} \left(\frac{\sin\pi\xi}{\pi\xi}\right)^2 \quad //$$
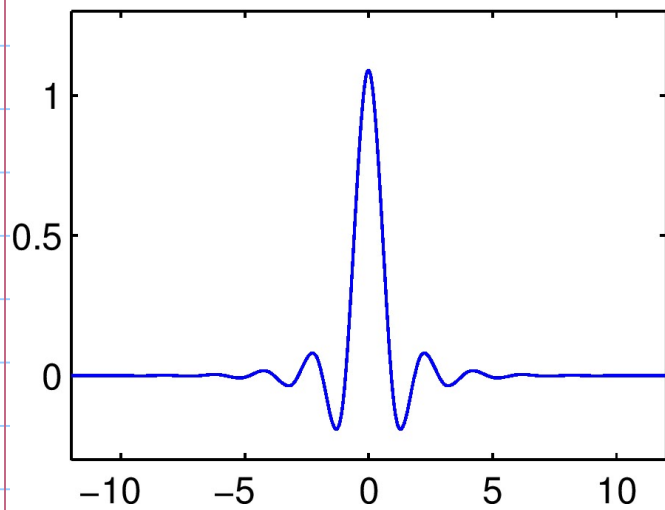
$\phi(x)$, m=1 $\Phi(\xi)$, m=1

$\phi(x)$, m=3 $\Phi(\xi)$, m=3