Lecture 17 : Wavelet Bases Ⅲ

Recall

Thm (Mallat , Meyer 1986)

Let ϕ be a scaling fcn (father wavelet)
and $\{h_k\}_{k \in \mathbb{Z}}$ be the corresponding C.M.F.
Let us define $\psi \in L^2(\mathbb{R})$ whose Fourier
transf. has

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right)$$

with $\hat{g}(\xi) = e^{-2\pi i \xi} \, \overline{\hat{h}\left(\xi + \frac{1}{2}\right)}$.

Let $\psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j}x - k)$.

Then, $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ form an ONB of W_j

for each $j \in \mathbb{Z}$, and $\{\psi_{j,k}\}_{(j,k) \in \mathbb{Z}^2}$ form

an ONB of $L^2(\mathbb{R})$.

Remarks :

(1) Let's check the relationship between
$\{g_k\}_{k \in \mathbb{Z}}$ and $\{h_k\}_{k \in \mathbb{Z}}$.

But, first, recall

$$\begin{cases} h_k = \langle \phi_{1,0}, \phi_{0,k} \rangle = \langle \frac{1}{\sqrt{2}} \phi\left(\frac{\cdot}{2}\right), \phi(\cdot - k) \rangle \\ g_k = \langle \psi_{1,0}, \phi_{0,k} \rangle = \langle \frac{1}{\sqrt{2}} \psi\left(\frac{\cdot}{2}\right), \phi(\cdot - k) \rangle \end{cases}$$

from the two-scale diff. eqn's. and $V_1 \oplus W_1 = V_0$.

Now,

via
$\mathcal{F}^{-1}$

$$\hat{g}(\xi) = e^{-2\pi i \xi} \, \overline{\hat{h}\left(\xi + \frac{1}{2}\right)}$$

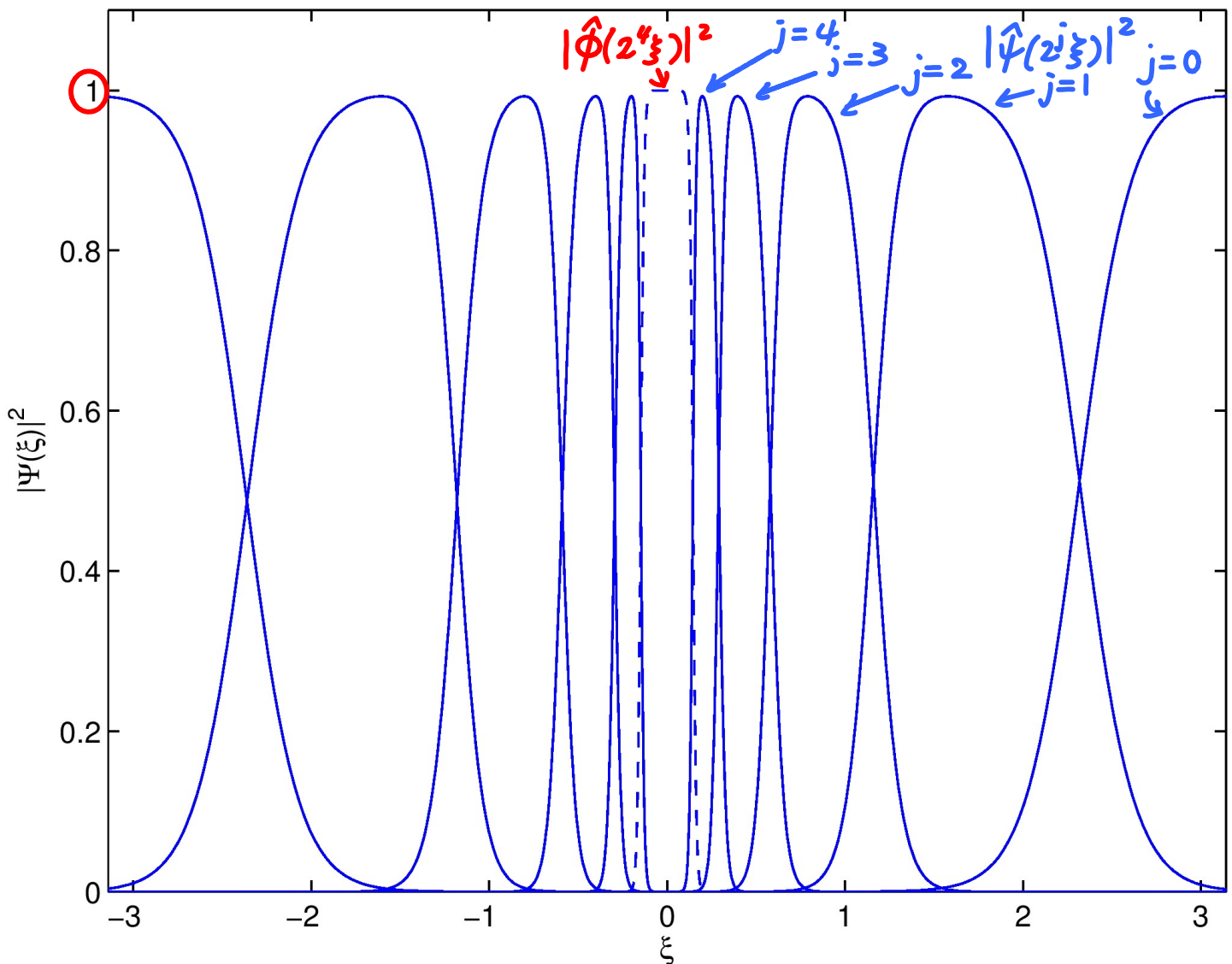This choice of $\hat{g}$ satisfies all the cond's as $\hat{g}$.

$$g_k = (-1)^{1-k} \overline{h_{1-k}} \, , \quad k \in \mathbb{Z}.$$

$$\hat{g}(\xi) = e^{-2\pi i \xi} \,\overline{\hat{h}\left(\xi + \tfrac{1}{2}\right)}$$
$$= e^{-2\pi i \xi} \sum h_\ell \, e^{+2\pi i \ell \left(\xi + \tfrac{1}{2}\right)}$$
$$= e^{-2\pi i \xi} \sum h_\ell \, e^{+2\pi i \ell \xi} \cdot \boxed{e^{\pi i \ell}} = (-1)^\ell$$
$$= \sum (-1)^\ell \, h_\ell \, e^{2\pi i (\ell - 1)\xi}$$

$1 - \ell = k \longrightarrow$
$$= \sum_k (-1)^{1-k} \, h_{1-k} \, e^{-2\pi i k \xi}$$
$$\underset{\sim\sim\sim\sim\sim\sim}{} = g_k$$

(2) $|\hat{\psi}(2^j \xi)|^2$ works as a **bandpass** filter.

Also $\displaystyle \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 \equiv 1 \qquad a.e. \; \xi \in \mathbb{R} \setminus \{0\}$
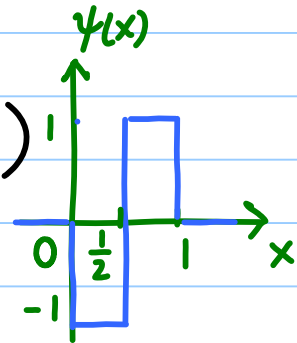
Example 1: Haar wavelet

$$\phi(x) = \chi_{[0,1)}(x) \implies \psi(x) = \begin{cases} -1 & 0 < x < \frac{1}{2} \\ 1 & \frac{1}{2} < x < 1 \\ 0 & o.w. \end{cases}$$

Recall $h_k = \begin{cases} \frac{1}{\sqrt{2}} & k = 0, 1 \\ 0 & o.w. \end{cases}$

$$\implies \frac{1}{\sqrt{2}} \psi\left(\frac{x}{2}\right) = \sum (-1)^{1-k} h_{1-k} \phi(x-k)$$

$$= -h_1 \phi(x) + h_0 \phi(x-1)$$

$$= \frac{1}{\sqrt{2}} \left(-\chi_{[0,1)}(x) + \chi_{[1,2)}(x) \right)$$

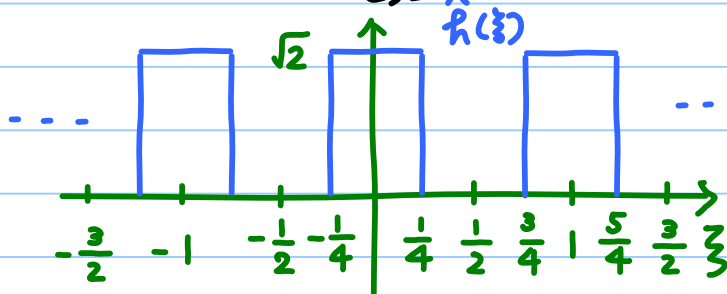$$\implies \psi(x) = -\chi_{[0,\frac{1}{2})}(x) + \chi_{[\frac{1}{2},1)}(x)$$

$$= \frac{1}{\sqrt{2}} \left[\sqrt{2} \phi(2x-1) - \sqrt{2} \phi(2x) \right]$$

Example 2: Shannon wavelet

Recall: $\hat{\phi}(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi)$, $\hat{h}(\xi) = \sqrt{2} \chi_{[-\frac{1}{4},\frac{1}{4})}(\xi)$
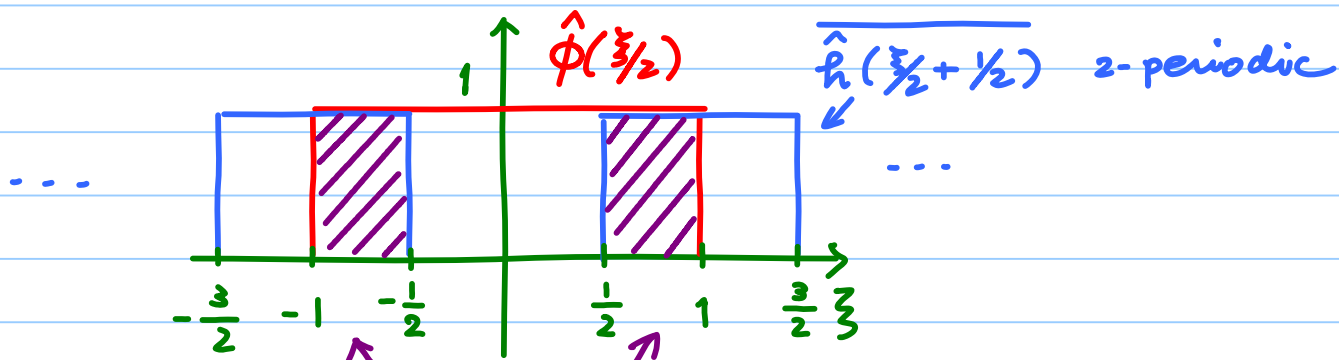
1 - periodic

$$= \sqrt{2} \sum_k \chi_{[-\frac{1}{4},\frac{1}{4})}(\xi+k)$$

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right) = \frac{1}{\sqrt{2}} e^{-i\pi\xi} \overline{\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}\right)} \hat{\phi}\left(\frac{\xi}{2}\right)$$

$$= e^{-i\pi\xi} \chi_{[-\frac{3}{2},-\frac{1}{2})}(\xi) \, \chi_{[-1,1)}(\xi) \qquad \overset{"}{=} \sqrt{2} \chi_{[-\frac{3}{2},-\frac{1}{2})}(\xi)$$

2 - periodic   2 - periodic

$$= e^{-i\pi\xi} \left(\chi_{[-1,1)}(\xi) - \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi) \right)$$

$\hat{\phi}(\xi/2)$

$\overline{\hat{h}(\xi/2 + 1/2)}$ 2-periodic

$|\hat{\psi}(\xi)|$ or $e^{i\pi\xi} \hat{\psi}(\xi)$
ideal (or pure) bandpass
filter (could be too sharp
in practice).

Hence via $\mathcal{F}^{-1}$, we have
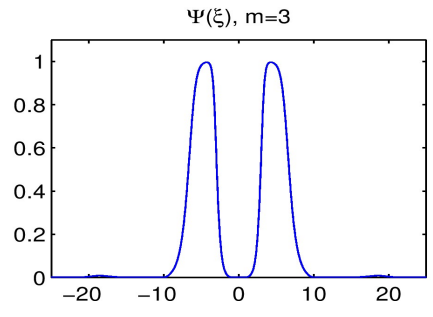
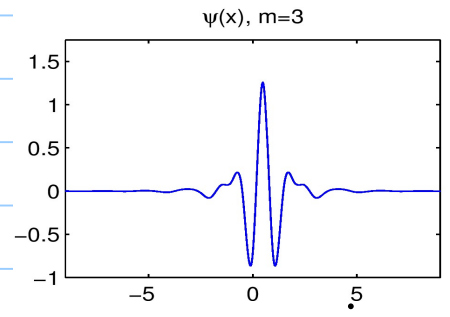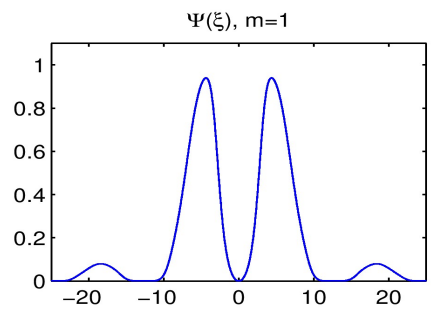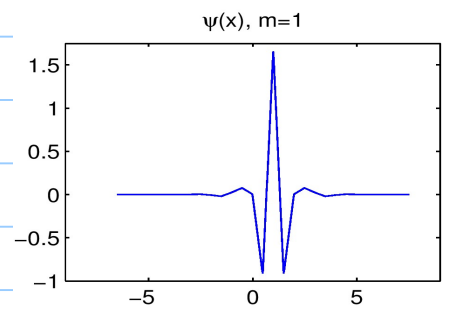$$\psi(x) = 2\,\text{sinc}(2x-1) - \text{sinc}(x-\tfrac{1}{2})$$

$$= \sqrt{2}\left[\sqrt{2}\,\phi(2x-1) - \frac{1}{\sqrt{2}}\,\tau_{\frac{1}{2}}\phi(x)\right]$$

Example 3: Spline (a.k.a. Battle–Lemarié) wavelet

$m = 0$: the Haar wavelet

For general $m \in \mathbb{N}$

$$\hat{\psi}(\xi) = \frac{e^{-i\pi\xi}}{\xi^{m+1}}\sqrt{\frac{S_{2m+2}(\xi/2 + 1/2)}{S_{2m+2}(\xi)\,S_{2m+2}(\xi/2)}}$$

☆ Basic Properties of Wavelets
⇒ Important in choosing/designing wavelets.

(1) Vanishing Moments

Def. A mother wavelet ψ is said to have p **vanishing moments** if

$$\int_{-\infty}^{\infty} x^m \psi(x)\, dx = 0, \quad m = 0, 1, \cdots, p-1$$

i.e., $\psi \perp P_{p-1}[x]$ (any polynomial with deg. $\leq p-1$).

If your fcn $f(x)$ is locally C^m, **then** f is well-approximated by a Taylor polynomial of deg m over some short interval.

⇒ $\langle f, \psi_{j,k} \rangle \approx 0$ for small j if $m < p$.

So, in general, the larger p, the better.

Thm (Vanishing moments)

Let ϕ & ψ be father & mother wavelets generating a wavelet ONB for $L^2(\mathbb{R})$ with $|\phi(x)| = O((1+x^2)^{-p/2-1})$, $|\psi(x)| = O((1+x^2)^{-p/2-1})$.

Then the following statements are equivalent:

(a) ψ has p vanishing moments.

(b) $\hat{\psi}^{(m)}(0) = 0, \quad m = 0, 1, \cdots, p-1$.

(c) $\hat{h}^{(m)}(\frac{1}{2}) = 0, \quad m = 0, 1, \cdots, p-1$.

(d) $\forall\ 0 \leq m < p$,

$$g_m(x) := \sum_{k \in \mathbb{Z}} k^m \phi(x-k) \text{ is a polynomial of deg. } m.$$

(Proof) See Mallat's book sec. 7.2. for the detail.

Note that

$$\hat{\psi}^{(m)}(\xi) = \mathcal{F}\left[(-2\pi i x)^m \psi(x)\right] \Rightarrow \hat{\psi}^{(m)}(0) = (-2\pi i)^m \int_{-\infty}^{\infty} x^m \psi(x)\, dx.$$

(2) Support Size

If your f has an isolated singularity at $x = x_0$ and if $x_0 \in \text{supp } \psi_{j,k}$, then $|<f, \psi_{j,k}>|$ may be large.

$\Rightarrow$ better to have $\text{supp } \psi :$ compact !

Thm (Compact Support)

The father wavelet ϕ has a compact support $\iff \{h_k\}$ has a compact support (i.e., finite taps). Their supports are equal.

a slight abuse of def. of supp.

If $\underline{\text{supp } \phi} \overset{*}{=} \underline{\text{supp } h} = [K_1, K_2]$, $K_1, K_2 \in \mathbb{Z}$, then $\text{supp } \psi = [(K_1 - K_2 + 1)/2, (K_2 - K_1 + 1)/2]$.

interval

list of indices i.e., $\{k \in \mathbb{Z} \mid h_k \neq 0\}$

(Proof) ($\Rightarrow$) Recall $h_k = <\phi_{1,0}, \phi_{0,k}>$.

Since both $\phi_{1,0}$ & $\phi_{0,0}$ have compact supports, $\{h_k\}$ must be of finite taps.

($\Leftarrow$) See Daubechies (1988, Comm. Pure & Appl. Math.)

Now, check the supports of ϕ and $\{h_k\}$.

Suppose $h_k \neq 0$ for $K_1 \leq k \leq K_2$, and $\text{supp } \phi = [N_1, N_2]$. Then, $\text{supp } \phi_{1,0} = [2N_1, 2N_2]$

On the other hand,

$$\phi_{1,0}(x) = \frac{1}{\sqrt{2}} \phi\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} h_k \phi(x-k)$$

$$\Rightarrow \text{supp } \phi_{1,0} = [K_1 + N_1, K_2 + N_2]$$

Hence $2N_1 = K_1 + N_1$, $2N_2 = K_2 + N_2$

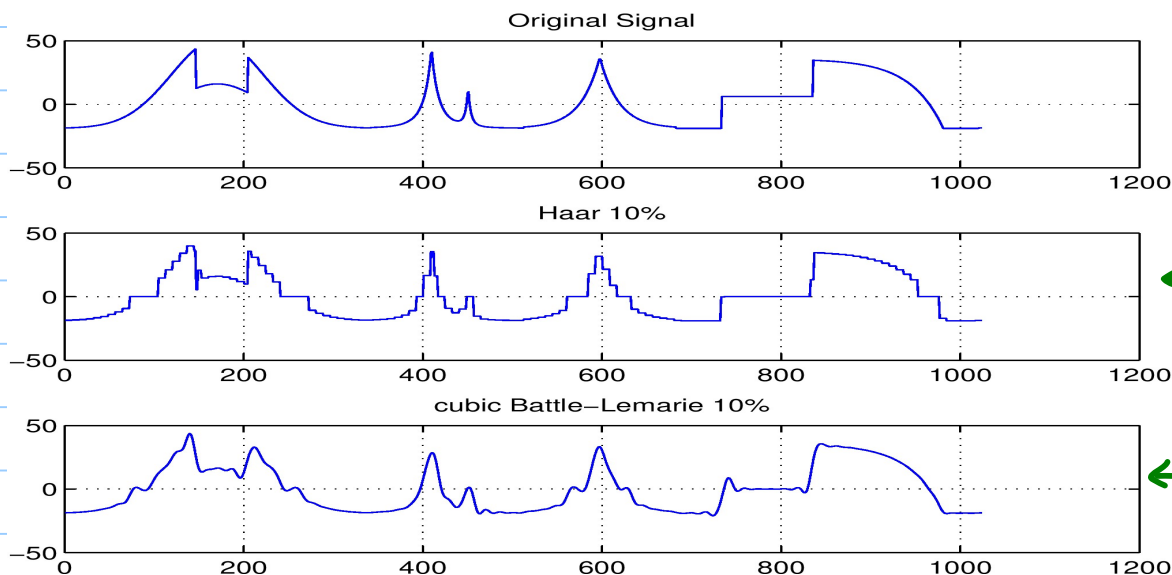i.e., $N_1 = K_1$, $N_2 = K_2$. ✓

As for $\text{supp } \psi$, recall

$$\psi_{1,0}(x) = \frac{1}{\sqrt{2}} \psi\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} g_k \phi(x-k) = \sum_{k \in \mathbb{Z}} (-1)^{1-k} h_{1-k} \phi(x-k)$$

Since supp ϕ = supp h = $[K_1, K_2]$, the support
of the above two-scale diff. eqn. is
$\;\;[K_1 - K_2 + 1, K_2 - K_1 + 1]$, $\qquad 1-k \in [K_1, K_2]$
which is supp $\psi_{1,0}$. $\qquad\qquad \Leftrightarrow k \in [1-K_2, 1-K_1]$
Hence, supp ψ = $[(K_1 - K_2 + 1)/2, (K_2 - K_1 + 1)/2]$ ///

Remark: The support size & the number of
vanishing moments of ψ seems independent.
However, Daubechies showed that for
orthonormal wavelets, ψ has p vanishing
moments $\Rightarrow$ |supp ψ| $\geq 2p-1$.
$\exists$ a trade-off between these two concepts.
Daubechies's orthonormal wavelets achieve
|supp ψ| = $2p-1$ with p vanishing moments.

(3) Regularity
When you reconstruct (or approximate) f
from its compressed representation (e.g.,
from a finite subset of $\{<f, \psi_{j,k}>\}$,
the regularity (i.e., smoothness) of $\psi_{j,k}$ matters.

Original Signal

Haar 10%

← basis fcns are
← visible.

cubic Battle–Lemarie 10%

Thm (Tchamitchian 1987)

Let $\hat{h}(\xi)$ be a CMF with p zeros at $\xi = \frac{1}{2}$ which satisfies the sufficient cond.'s of the Mallat-Meyer Thm, i.e.,

1) 1-periodic;

2) C^1 in the neighborhood of $\xi = 0$; and

3) $|\hat{h}(\xi)|^2 + |\hat{h}(\xi + \frac{1}{2})|^2 \equiv 2$; $\hat{h}(0) = \sqrt{2}$;

$\quad \inf\limits_{\xi \in [-\frac{1}{4}, \frac{1}{4}]} |\hat{h}(\xi)| > 0$.

Now, let's perform the factorization:

$$\hat{h}(\xi) = \sqrt{2} \left(\frac{1 + e^{i 2\pi \xi}}{2} \right)^p \hat{l}(\xi)$$

<u>If</u> $\sup\limits_{\xi \in \mathbb{R}} |\hat{l}(\xi)| = B$, <u>then</u> ϕ & ψ are

uniformly Lipschitz (or Hölder) α for

$$\alpha < \alpha_0 := p - \log_2 B - 1.$$

(Proof) See Mallat's book Sec. 7.2.

Note that we refine the definition of Lipschitz continuity in Lecture 5 as follows:

<u>Def</u>. A fcn f is pointwise Lipschitz $\alpha \geq 0$ at x_0 if $\exists K > 0$, $\exists p_{x_0} \in P_{\lfloor \alpha \rfloor}[x]$ s.t.

(*) $\quad |f(x) - p_{x_0}(x)| \leq K |x - x_0|^\alpha$, $\forall x \in \mathbb{R}$.

A fcn f is uniformly Lipschitz α over $[a, b]$ if it satisfies (*) for $\forall x_0 \in [a, b]$ and K is independent of x_0.

The space of uniformly Lipschitz continuous fcns with exponent α is often written as $\text{Lip}_\alpha [a, b]$. But if $\alpha = m + \beta$, $m \in \mathbb{N} \cup \{0\}$, $0 < \beta < 1$,

it is also written as $C^{m,\beta}[a,b]$, i.e.,
$$f \in C^{m,\beta}[a,b] \Longleftrightarrow f \in C^m[a,b] \text{ \& } f^{(m)} \in C^{0,\beta}[a,b]$$
$$= Lip_\beta[a,b].$$

Ex. $f(x) = x^\gamma \in C[0,1]$, $0 < \gamma \leq 1$
$$\not\in Lip_\alpha[0,1] \text{ with } \gamma < \alpha \leq 1, \text{ but}$$
$$\in Lip_\alpha[0,1] \text{ with } 0 < \alpha \leq \gamma$$

Let's check these properties in Examples 1, 2, 3.

Example 1 : Haar Wavelets
- Vanishing moments $p=1$.
- Supp $\psi = [0,1)$
- Regularity : discontinuous
- Antisymmetric

Example 2 : Shannon Wavelets
- Vanishing moments $p = +\infty$ (because
- Supp $\psi = \mathbb{R}$, slow decay $\sim \frac{1}{x}$ $\hat{\psi}^{(m)}(0) \equiv 0$
- Regularity : $C^\infty(\mathbb{R})$ $\forall m \in \mathbb{N}$.)
- Symmetric

Example 3 : Battle-Lemarié Wavelets
- Vanishing moments $p = m+1$
- Supp $\psi = \mathbb{R}$, exponential decay
- Regularity : $C^{m-1}(\mathbb{R})$
- Symmetric

☆ **A Brief Intro. to Compactly Supported Wavelets of Daubechies**

Daubechies's wavelets have the following properties:

- Vanishing moments $p \in \mathbb{N}$.
- $|\text{Supp } \psi| = 2p - 1$ minimal supp for a given p.
- Regularity : $\text{Lip}_\alpha(\mathbb{R})$, $\alpha < p - \log_2 B - 1$
- Neither symmetric $\approx P/5$ for nor antisymmetric large p.
  if $p > 1$ ($p = 1$ reduces to the Haar case).

The name of the game here is how to find (or design) a trigonometric polynomial $R(e^{-i2\pi\xi})$ of minimum degree s.t.

$$\hat{h}(\xi) = \sqrt{2} \left(\frac{1 + e^{-i2\pi\xi}}{2} \right)^p R(e^{-2\pi i\xi})$$

with $|\hat{h}(\xi)|^2 + |\hat{h}(\xi + \tfrac{1}{2})|^2 \equiv 2$ a.e. $\xi \in \mathbb{R}$
$\hat{h}(0) = \sqrt{2}$.

Daubechies proved the minimum $\deg = p - 1$.

Thm (Daubechies 1988)

A real-valued CMF $\{h_k\}$ with $\hat{h}(\xi)$ having p zeros at $\xi = \tfrac{1}{2}$ has at least $2p$ nonzero coefficients (i.e., $|\text{supp } h| \geq 2p$).
$|\text{supp } \psi| \geq 2p - 1$
Daubechies's wavelet filters have $2p$ taps.

(Proof) See Daubechies 1988 or Mallat's book Sec. 7.2.
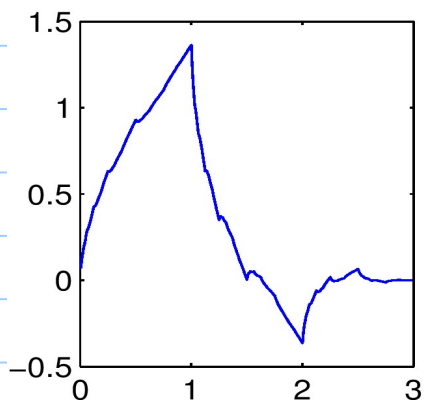
Thm (Daubechies 1988)

If ψ is a mother wavelet with p vanishing moments that generates an ONB of $L^2(\mathbb{R})$, then $|\text{supp }\psi| \geq 2p-1$.
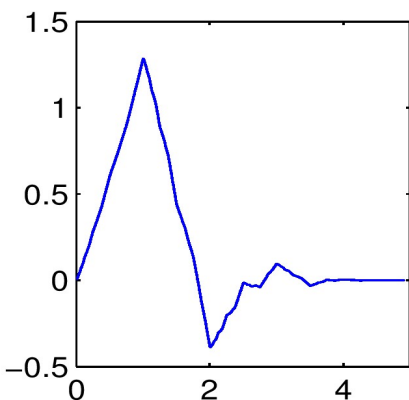
A Daubechies mother wavelet has a minimum size support $\text{supp }\psi = [-p+1, p]$ and the corresponding father wavelet has $\text{supp }\phi = [0, 2p-1]$.
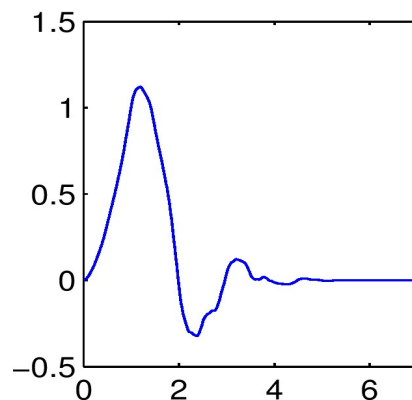
(Proof) See Daubechies 1988 or Mallat's book Sec. 7.2.
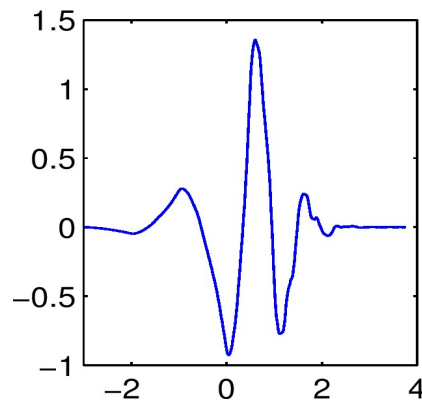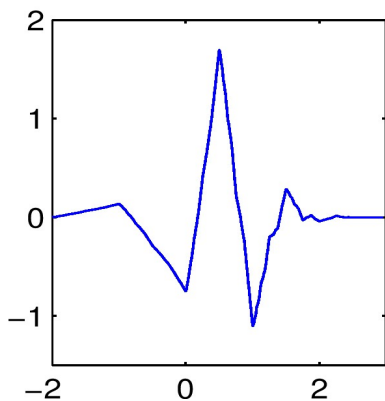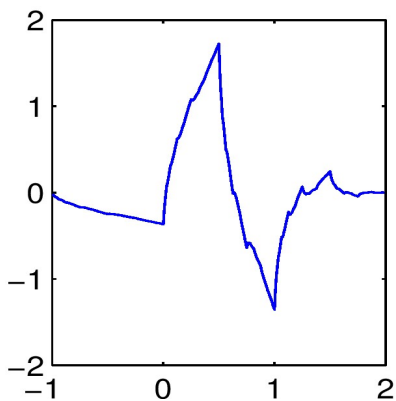
$\phi(x)$

$p = 2$ $p = 3$ $p = 4$

$\psi(x)$

$h_0 =$ 0.482962913144831
$h_1 =$ 0.836516303737708
$h_2 =$ 0.224143868041922
$h_3 =$ -0.129409522550955

$p = 2$ case.