Lecture 18: Fast Wavelet Transform; Various Extensions

Note Title

☆ Assumptions on an input signal

- In all practical applications, $\exists$ a finest and a coarsest scale of interest.
- For simplicity, let's assume that an input signal is a vector $f = (f_0, \cdots, f_{N-1})^T$, $N = 2^n$, and periodic.
- Let's assume the finest scale is $2^0 = 1$, and we view an input signal $\in V_0$, $\dim(V_0) = N$.
- also assume the coarsest scale is 2^J with $1 \le J \le n$. This implies that $\dim(V_J) = 2^{n-J} = N/2^J$, and

$$V_0 = V_J \oplus \bigoplus_{j=1}^{J} W_j .$$

- Finally assume the given samples $f_0, \cdots, f_{N-1}$ are the finest scale coefficients, i.e.,

$$f_k = \langle f, \phi_{0,k} \rangle =: s_k^0$$

$\{f_k\}$ are given. So, we implicitly deal with "fictitious" $f = \sum_{k=0}^{N-1} f_k \phi_{0,k} = \sum_{k=0}^{N-1} s_k^0 \phi_{0,k}$.

Hence in this case, $f = P_{V_0} f$.

- If you know $f(x)$ over $[0,1]$, and want to have $f_k \approx f(\frac{k}{N})$, then you need to design ϕ with high vanishing moments $\Rightarrow$ "coiflets".

  Σ normally ϕ does not have vanishing moments.

☆ Fast Orthogonal Wavelet Transform

Let us write

$$P_{V_j} f = \sum_{k=0}^{2^{n-j}-1} s_k^j \phi_{j,k}, \quad P_{W_j} f = \sum_{k=0}^{2^{n-j}-1} d_k^j \psi_{j,k},$$

where $s_k^j := \langle f, \phi_{j,k} \rangle$, $d_k^j := \langle f, \psi_{j,k} \rangle$

sum difference

Forward transf: Given $P_{V_0} f$, compute $P_{W_1} f, P_{W_2} f, \cdots, P_{W_J} f, P_{V_J} f$.

$\Longleftrightarrow$ Given $\{s_k^0\}_{k=0}^{N-1}$, compute $\{d_k^j\}_{k=0}^{2^{n-j}-1}$, $j=1,\cdots,J$ and $\{s_k^J\}_{k=0}^{2^{n-J}-1}$.

Inverse transf: Given $P_{W_1} f, \cdots, P_{W_J} f, P_{V_J} f$, reconstruct $P_{V_0} f$.

$\Longleftrightarrow$ Reconstruct $\{s_k^0\}_{k=0}^{N-1}$ from $\{d_k^j\}_{k=0}^{2^{n-j}-1}$, $j=1,\cdots,J$ and $\{s_k^J\}_{k=0}^{2^{n-J}-1}$.

discrete convolution

Thm (Mallat 1989)

Forward transf.

$$\begin{cases} s_k^{j+1} = \sum_{\ell\in\mathbb{Z}} h_{\ell-2k} s_\ell^j = (s_\cdot^j * \tilde{h})_{2k} \\ d_k^{j+1} = \sum_{\ell\in\mathbb{Z}} g_{\ell-2k} s_\ell^j = (s_\cdot^j * \tilde{g})_{2k} \end{cases} \quad k=0,\cdots,2^{n-j-1}-1.$$

subsampling (every other samples)

where $\tilde{h}_\ell := h_{-\ell}$

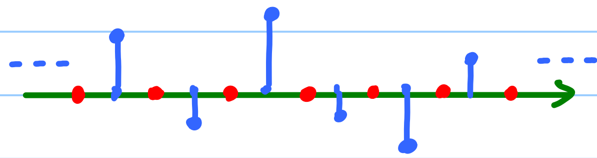
Inverse transf.

$$s_k^j = \sum_{\ell\in\mathbb{Z}} h_{k-2\ell} s_\ell^{j+1} + \sum_{\ell\in\mathbb{Z}} g_{k-2\ell} d_\ell^{j+1}, \quad k=0,\cdots,2^{n-j}-1.$$

$$= \left(\check{S}^{j+1}_{\bullet} * h \right)_k + \left(\check{d}^{j+1}_{\bullet} * g \right)_k$$

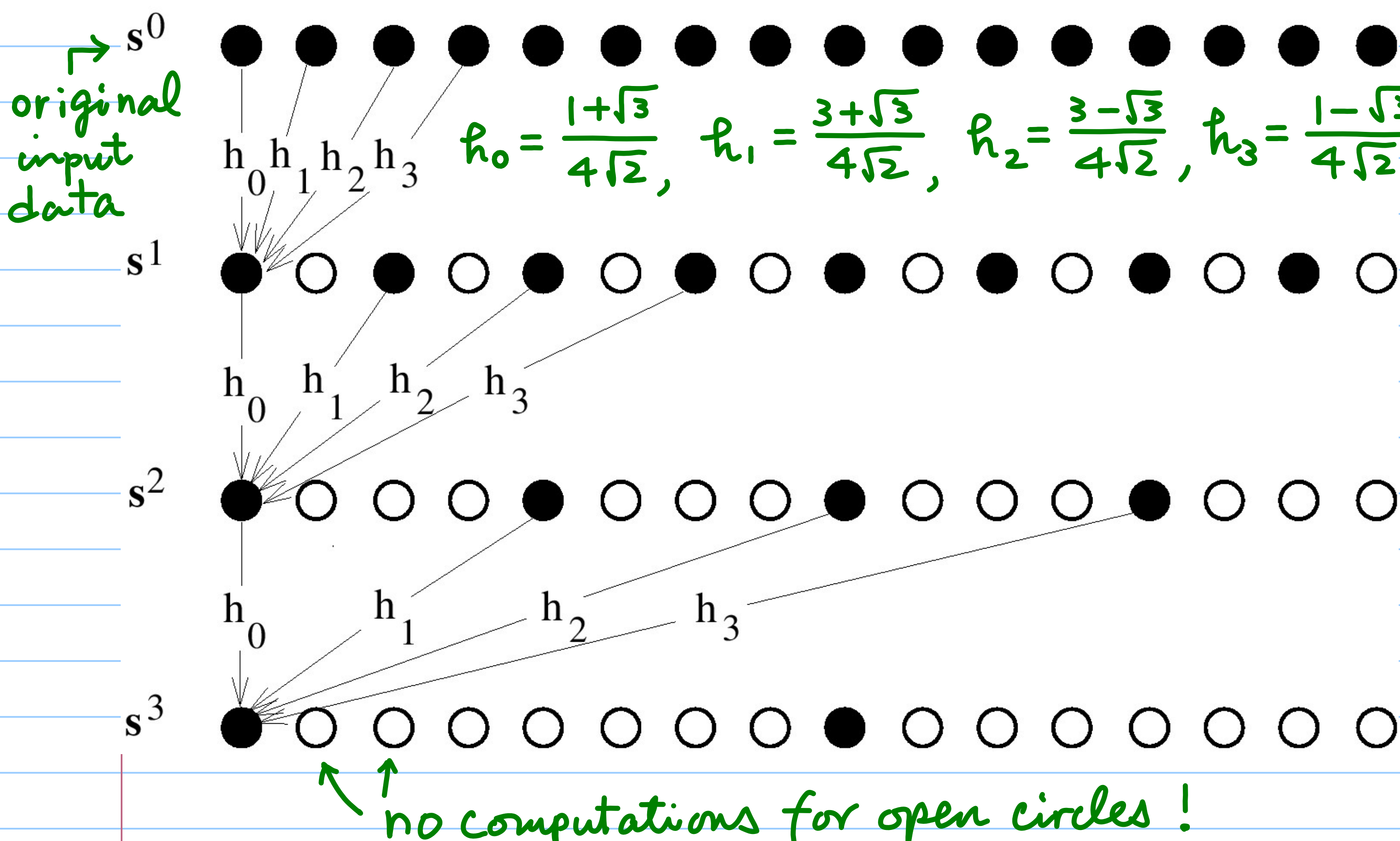
where $\vee$ is an up sampling operation (with 0s):

for $\{x_\ell\}_{\ell \in \mathbb{Z}}$, $\quad \check{x}_\ell := \begin{cases} x_k & \text{if } \ell = 2k \,; \\ 0 & \text{if } \ell = 2k+1 \,. \end{cases}$

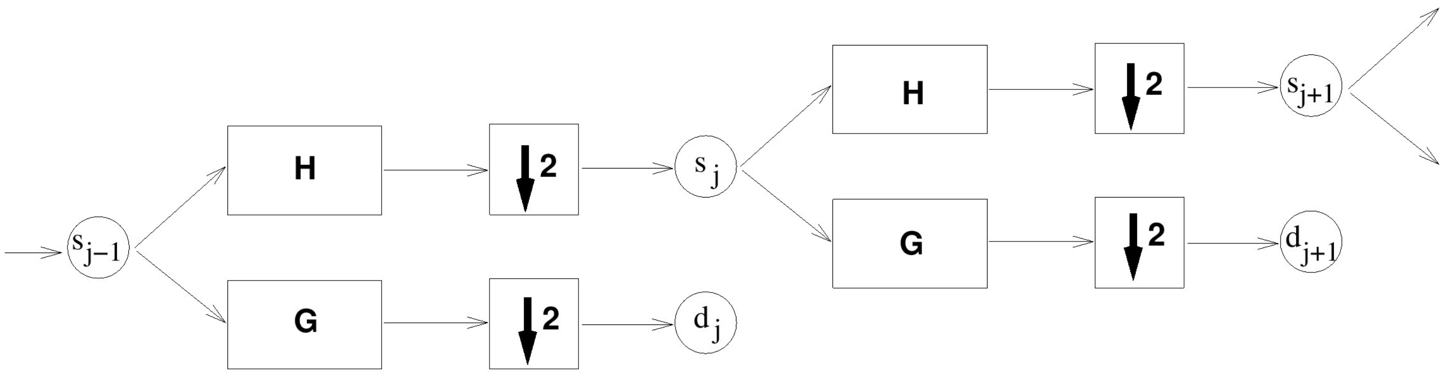


Note that for compactly supported wavelets, only finite numbers of $\{h_k\}, \{g_k\}$ are nonzeros.

Ex. Daubechies's wavelet $p = 2$



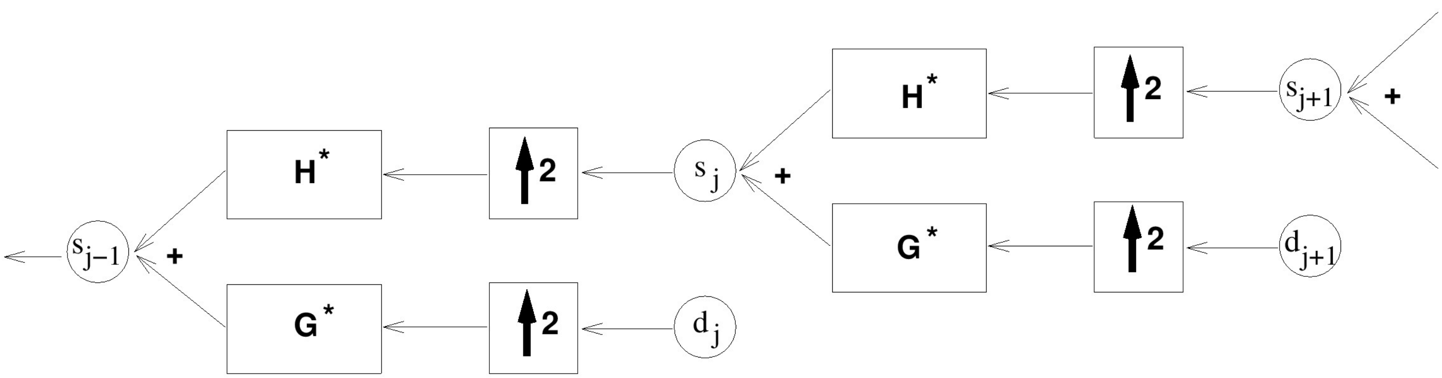
$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$



Decomposition by QMFs

Quadrature Mirror Filters

$\supset$ CMF



Reconstruction by QMFs

- <u>Computational Complexity</u>

Recall FFT's cost $O(N \log N)$

<u>If</u> $|\operatorname{supp} h| = |\operatorname{supp} g| = K$ (taps), <u>then</u> the cost for the forward/inverse transf. is at most $2KN$, i.e., $O(KN)$ or even you can say $O(N)$.

(Proof of the Thm)

Since $\phi_{j+1,\ell} \in V_{j+1} \subset V_j$,

$$(*) \quad \phi_{j+1,k} = \sum_{\ell\in\mathbb{Z}} \langle \phi_{j+1,k}\ \phi_{j,\ell} \rangle \phi_{j,\ell}$$

$$\langle \phi_{j+1,k}\ \phi_{j,\ell} \rangle = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2^{j+1}}}\phi\left(\frac{x-2^{j+1}k}{2^{j+1}}\right) \overline{\frac{1}{\sqrt{2^j}}\phi\left(\frac{x-2^j\ell}{2^j}\right)}dx$$

$$\overset{t=2^{-j}x-k}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\phi\left(\frac{t}{2}\right) \overline{\phi(t-\ell+2k)}\,dt$$

$$= \langle \phi_{1,0}, \phi_{0,\ell-2k} \rangle = h_{\ell-2k} \quad (**)$$

So, (*) is in fact

$$\phi_{j+1,k} = \sum_{\ell\in\mathbb{Z}} h_{\ell-2k}\phi_{j,\ell}$$

$$\Rightarrow \quad s_k^{j+1} = \langle f, \phi_{j+1,k} \rangle = \sum_{\ell\in\mathbb{Z}} h_{\ell-2k} \langle f, \phi_{j,\ell}\rangle$$

$$= \sum_{\ell\in\mathbb{Z}} h_{\ell-2k}\, s_\ell^j \quad \checkmark$$

Similarly, it's easy to derive

$$d_k^{j+1} = \sum_{\ell\in\mathbb{Z}} g_{\ell-2k}\, s_\ell^j \quad \checkmark$$

As for the inverse transf., note that

$$V_{j+1} \oplus W_{j+1} = V_j$$

Hence $$\phi_{j,k} = \sum_{\ell\in\mathbb{Z}} \langle \phi_{j,k}, \phi_{j+1,\ell}\rangle \phi_{j+1,\ell} + \sum_{\ell\in\mathbb{Z}} \langle \phi_{j,k}, \psi_{j+1,\ell}\rangle \psi_{j+1,\ell}.$$

$$\overset{(**)}{=} \sum_\ell \overline{h}_{k-2\ell}\,\phi_{j+1,\ell} + \sum_\ell \overline{g}_{k-2\ell}\,\psi_{j+1,\ell}$$

$$\overset{h_k,\, g_k \in \mathbb{R}}{=} \sum_\ell h_{k-2\ell}\,\phi_{j+1,\ell} + \sum_\ell g_{k-2\ell}\,\psi_{j+1,\ell} \quad ///$$

☆ Other potential problems of fast discrete wavelet transforms with compactly supported wavelets

- Boundary treatment
- Lack of translation invariance
- Lack of symmetry/antisymmetry
- Lack of high frequency resolution
- Lack of orientation sensitivity in 2D & higher

(1) Boundary treatment

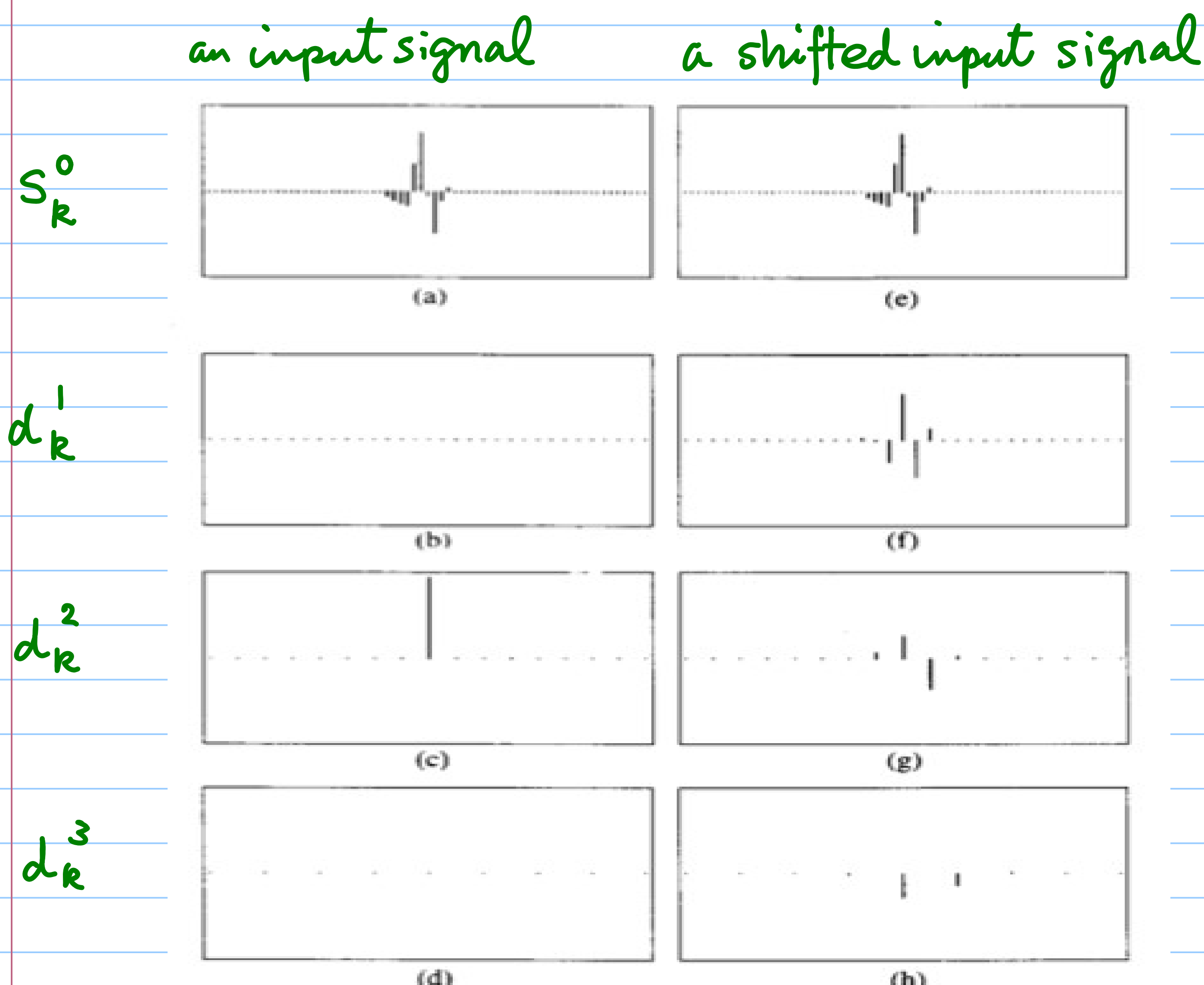
DWT requires information of the outside of the input signal $f = [f_0, \cdots, f_{N-1}]^T$, i.e., needs f_j for some $j < 0$ and $j \geq N$, due to the convolution operations with $\{h_k\}$ & $\{g_k\}$.

Possible solutions:

- Periodize f
  $\Rightarrow$ creates artifitial discontinuity because in general, the head and tail of f may be quite different.
  $\Rightarrow$ creates large wavelet coeff's, i.e., no good although it's easy to implement
- Even-reflect f at the boundary
  $\Rightarrow$ no artifitial discont., recommended!
- Design the "boundary" wavelets, i.e., use different ϕ & ψ toward the boundary (Cohen, Daubechies, Vial, 1993)
  $\Rightarrow$ Great, but cumbersome to implement.

(2) Lack of translation invariance

Due to the subsampling operations in DWT, the wavelet coef's of f and those of the shifted version of f are completely different, i.e., they are very sensitive to translations of an input signal.
It's quite a contrast to DFT where a translation amounts to a simple phase factor, i.e., $\mathcal{D}_N[\tau_\ell f](k) = \omega_N^{-k\ell}\,\mathcal{D}_N[f]$.

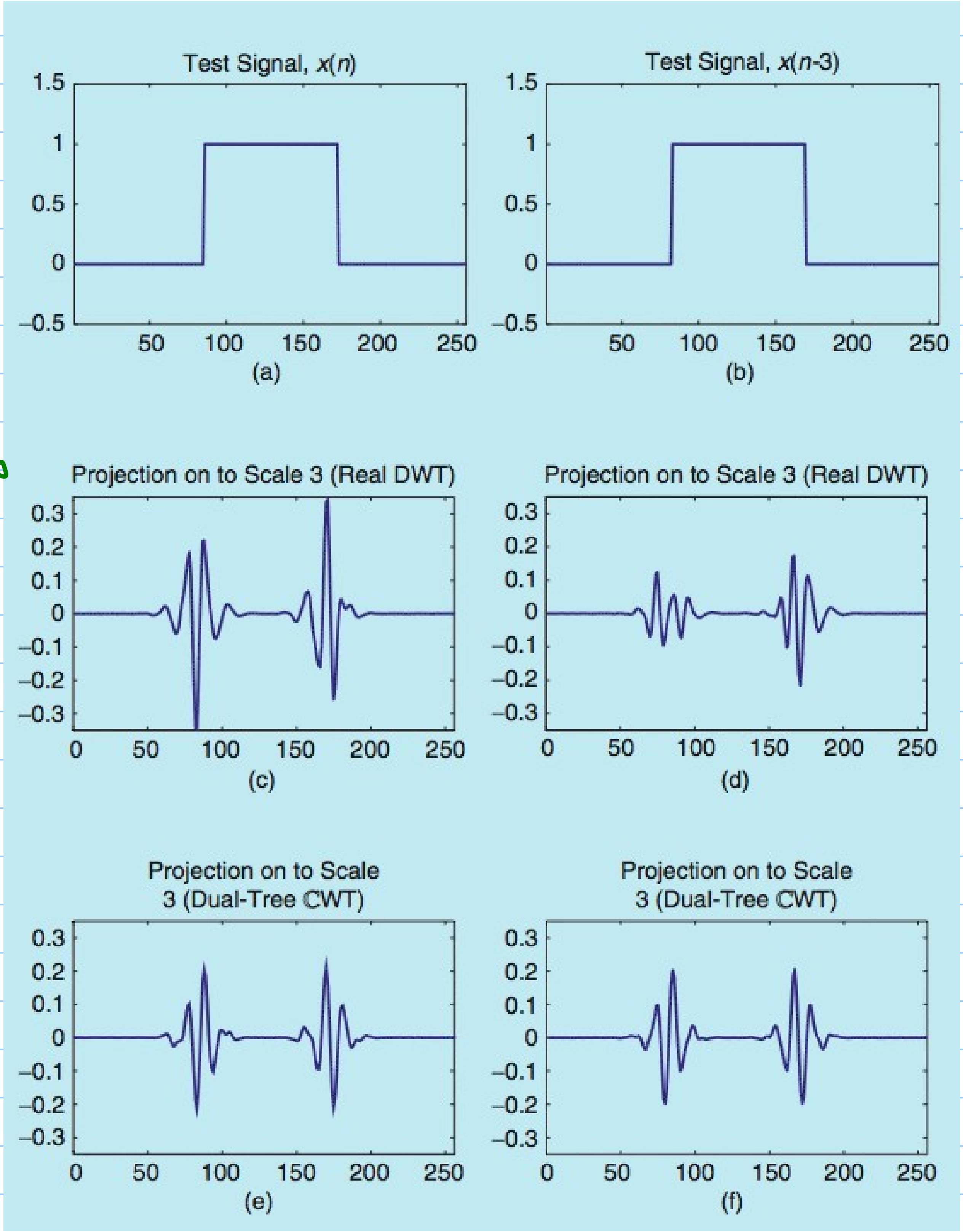


Possible solutions:

- Abandon the basis (nonredundancy) and use the special frame (stationary wavelet transform) ⇒ no subsampling at each level. Beylkin (1992), Nason & Silverman (1995)
  Redundancy factor: $J+1$ where $J = \#$ levels / scales

- Abandon the exact translation invariance but shoot for near trans. invariance in the magnitude of the wavelet coef's.
  ⇒ Shiftable multiscale transf.
  Simoncelli, Freeman, Adelson, & Heeger (1992)
  Here, the energy of each subspace is trans. inv.
  They also developed such 'shiftability' in orientation & scale for 2D transf.
  It's a tight frame with redundancy factor $\propto \#$ orientations $\times 4/3$
  ⇒ Dual-tree complex wavelet transf. ($\mathbb{C}$WT)
  Kingsbury (2001), Selesnick, Baraniuk & Kingsbury (2005). Can have some oriented basis fcn's and near translation invariance.
  Redundancy factor: 2^d $d = 1$ for 1D signal
  $= 2$ for 2D images.

Daubechies wavelets with $p=7$



(3) Lack of symmetry/antisymmetry

ϕ & ψ of Daubechies's cannot have symmetry/antisymmetry for $p>1$.

$$\begin{cases} p=1 \Rightarrow \text{Haar, so } \phi\text{: symmetric, } \psi\text{: antisymmetric} \\ p \to \infty \Rightarrow \text{Shannon, so both } \phi \,\&\, \psi\text{: symmetric} \\ \qquad\qquad \text{but not compactly supported!} \end{cases}$$

The source of the problem is the difficulty in finding symmetric/antisymmetric CMF coef's $\{h_k\}$ of finite taps.

Possible solutions:

- Abandon true symmetry/antisymmetry and seek near linear phase CMF $\{h_k\}$.
  If $\{h_k\}$ is symmetric at $k=\ell$, say, $h_k = h_{2\ell-k}$, $k \in \mathbb{Z}$, then we can show

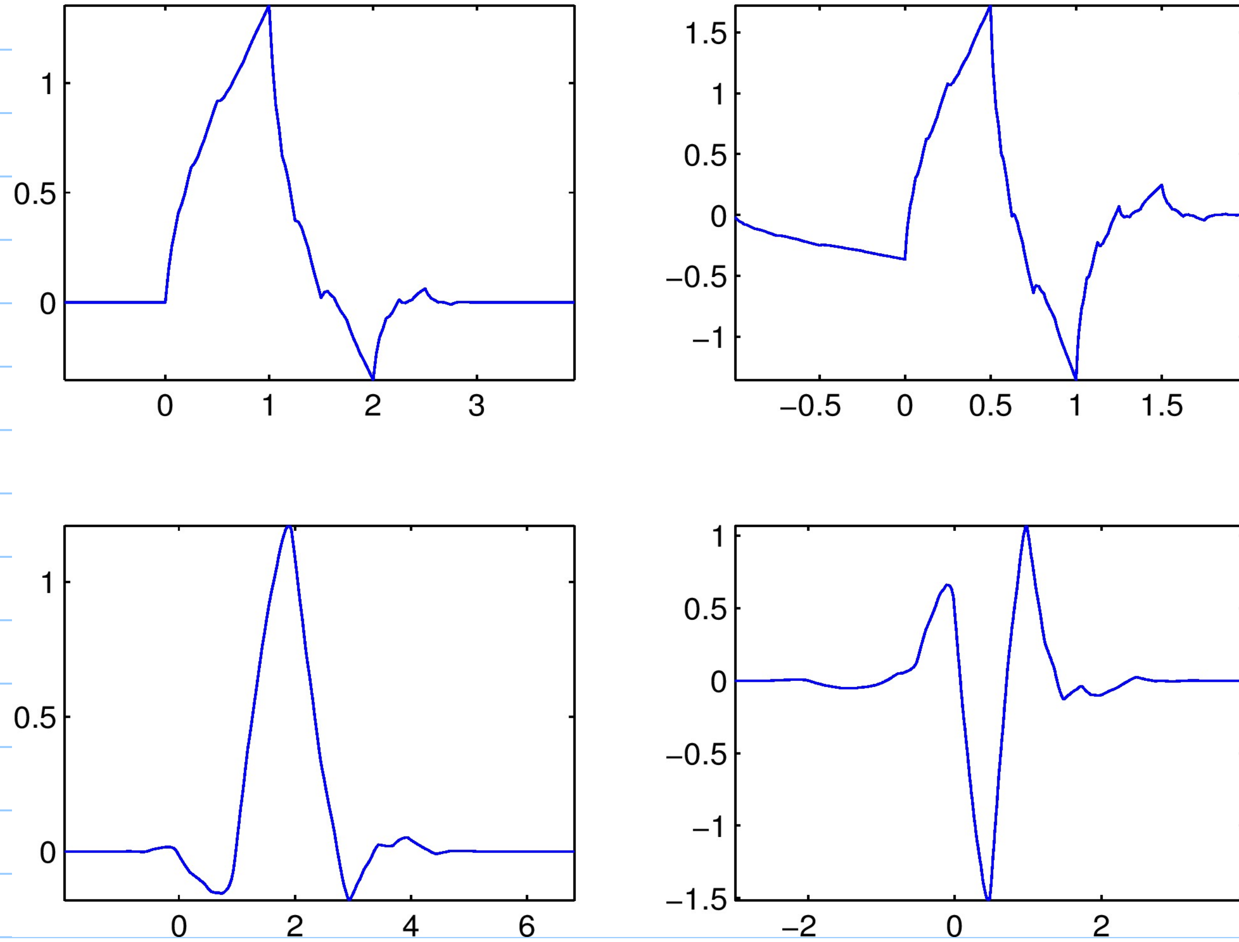
  $$\hat{h}(\xi) = \underbrace{e^{-2\pi i \ell \xi}}_{\text{linear phase}} |\hat{h}(\xi)|$$

  To allow symmetry at half integers, we need to extend the definition of linear phase by including piecewise linear phase with constant slope whose discontinuities occur only at zeros of $|\hat{h}(\xi)|$ (e.g., the Haar case).
  Daubechies (1990) found a way to optimize the choice of $\{h_k\}$ to have almost linear phase with $\operatorname{supp} h = [-p+1, p]$.
  $\Longrightarrow$ 'Symmlets'

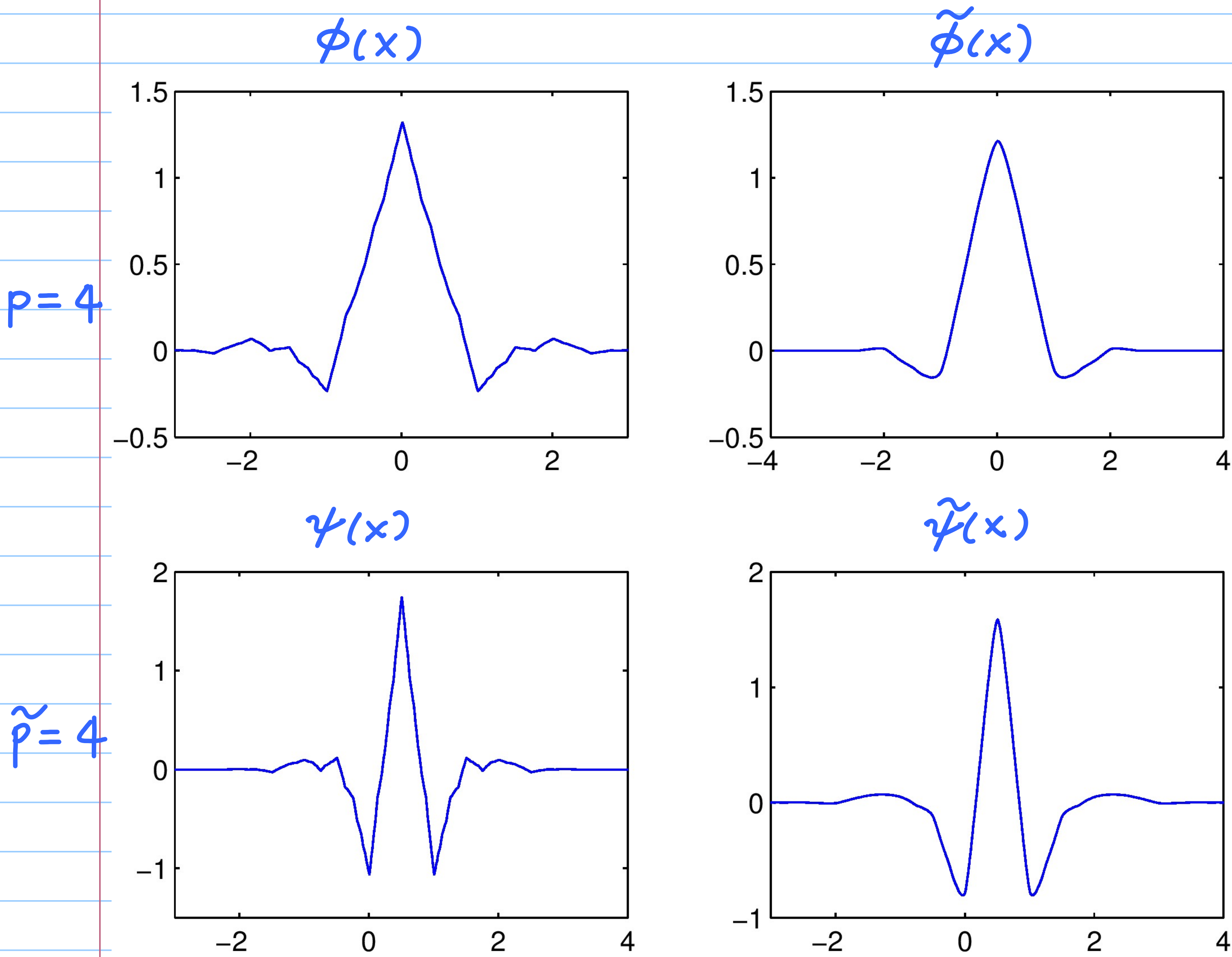
Father Mother

Default Daubechies's wavelets $p = 2$

Symmlets $p = 2$



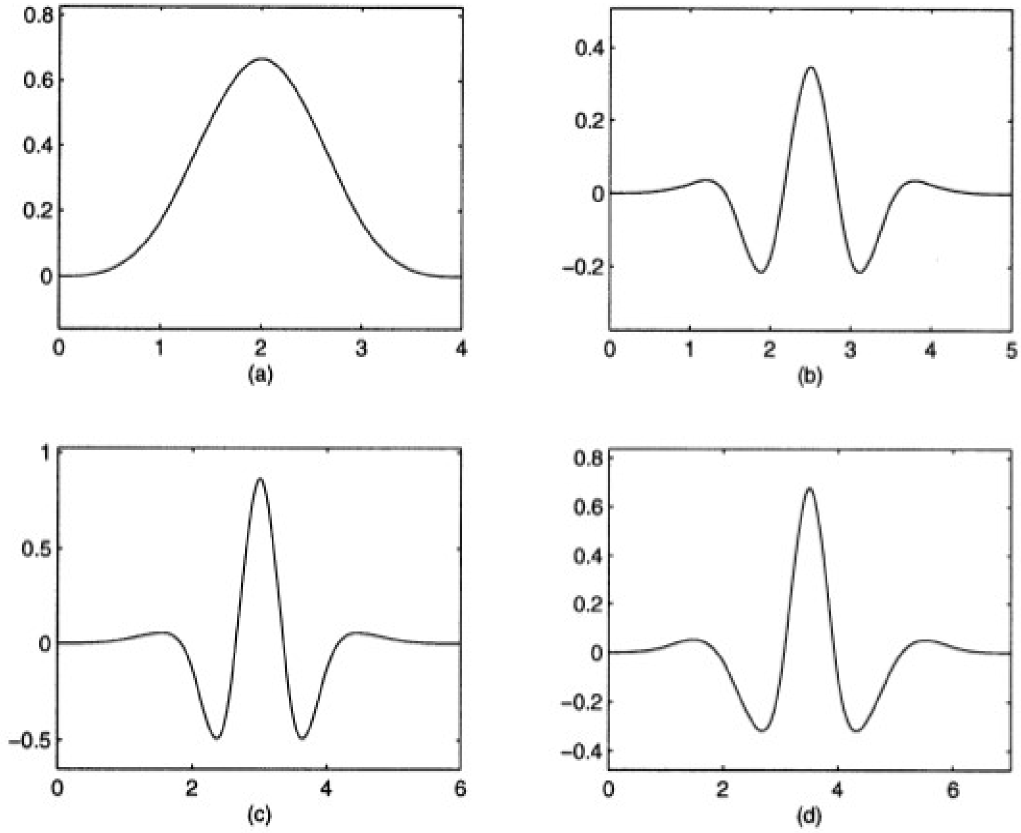
- Abandon the orthogonality for symmetry $\Rightarrow$ Biorthogonal wavelet bases
  Cohen, Daubechies, & Feauveau (1992)
  Needs to use two sets of families
  $\{\phi_{j,k}, \psi_{j,k}\}$ for analysis and
  $\{\tilde{\phi}_{j,k}, \tilde{\psi}_{j,k}\}$ for synthesis (or viceversa)
  Quite flexible in terms of filter design, e.g., vanishing moments for ψ & $\tilde{\psi}$ can be different as well as their support.
  JPEG 2000 standard recommends the following biorthogonal wavelets:



They are referred to as "9/7" biorthogonal wavelets since $|\text{supp } h| = 9$, $|\text{supp } \tilde{h}| = 7$.

- Use frame, e.g., use more than one pair of father & mother wavelets
  $\Rightarrow$ wavelet frames (framelets)
  Ron & Shen (1997),
  Benedetto & Li (1998),
  Daubechies, Han, Ron, & Shen (2003)
  and many others ...

Tight framelets system's father and 3 mother wavelets ($p=4$)



- Use the autocorrelations of father & mother wavelets of Daubechies

  $\Rightarrow$ Autocorrelation shell

  Saito & Beylkin (1993)

  This is a special frame, i.e., redundant, nonorthogonal, but translation invariant & symmetric

$p=1$: autocorr of boxcar = hat fcn $\to$ linear interpol.

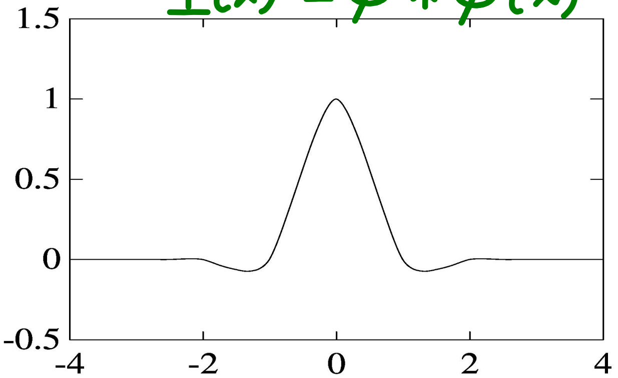
$\vdots$

$p=2$: " of Daub. 4 = Deslauriers-Dubuc interpolation

$\vdots$

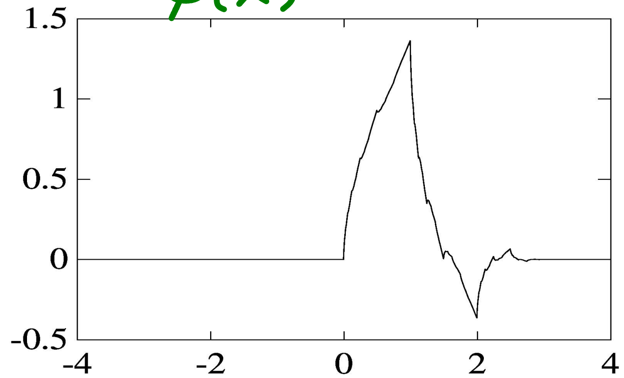
$p\to\infty$: " of sinc = sinc $\to$ BL interpol.

$\Phi(x) = \phi * \tilde{\phi}(x)$



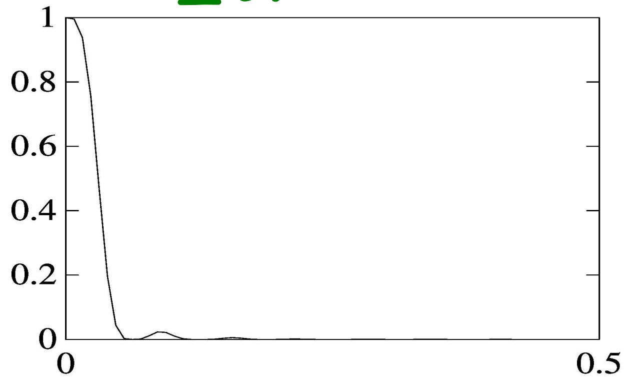
(a)

$\phi(x)$



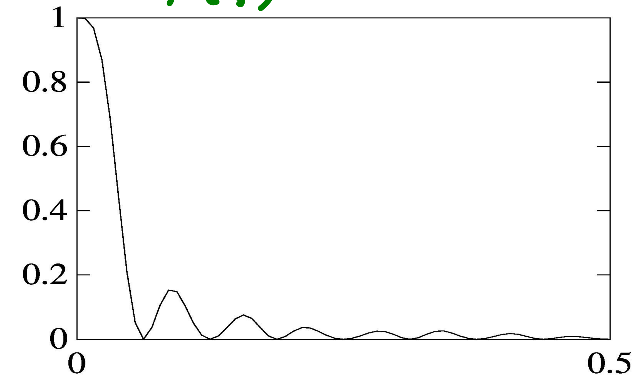
(b)

$\hat{\Phi}(\xi)$



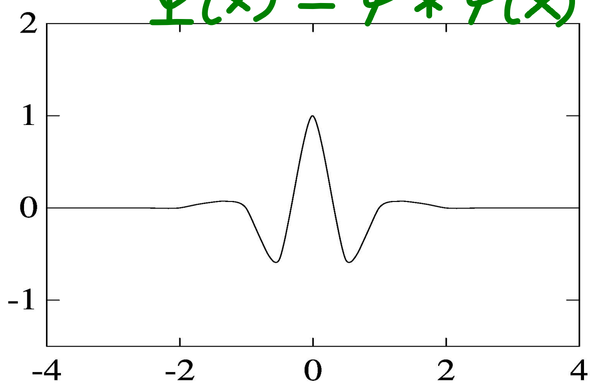
(c)

$\hat{\phi}(\xi)$



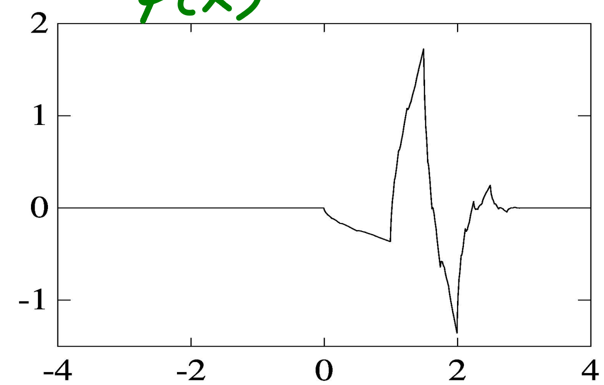
(d)

$\Psi(x) = \psi * \tilde{\psi}(x)$



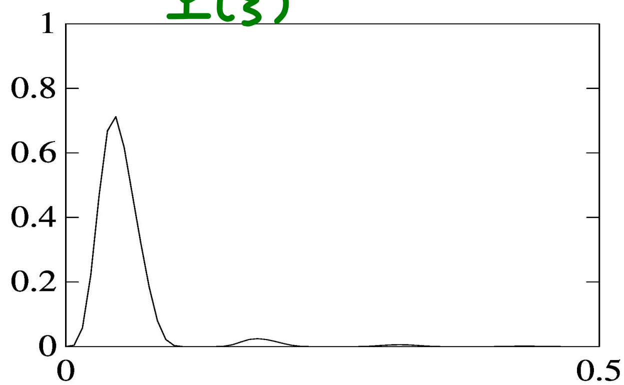
(a)

$\psi(x)$



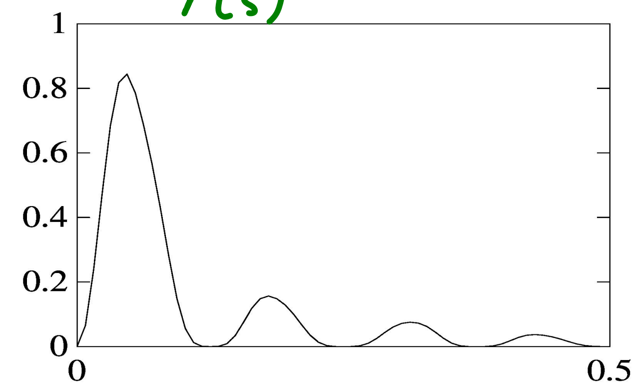
(b)

$\hat{\Psi}(\xi)$



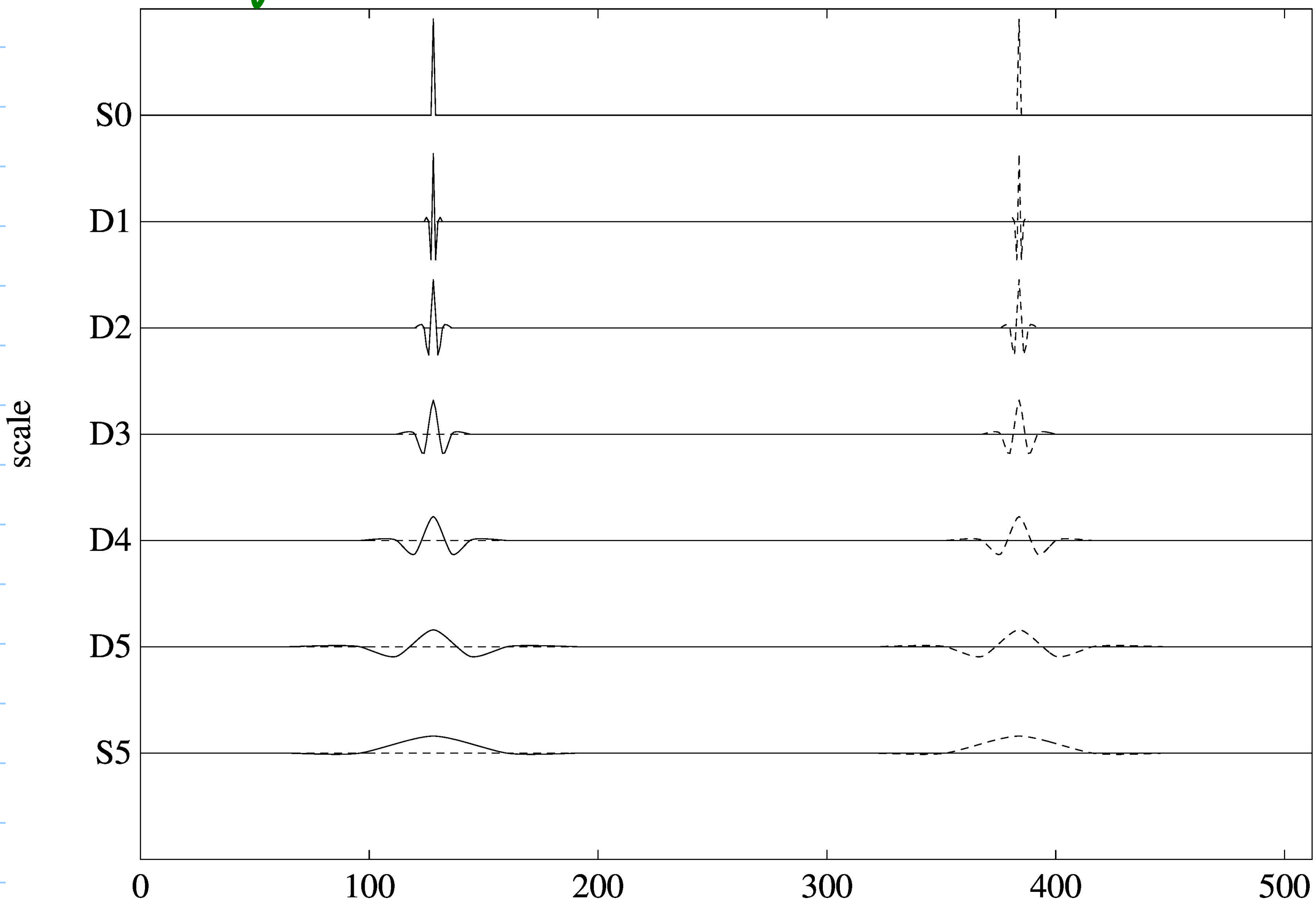
(c)

$\hat{\psi}(\xi)$

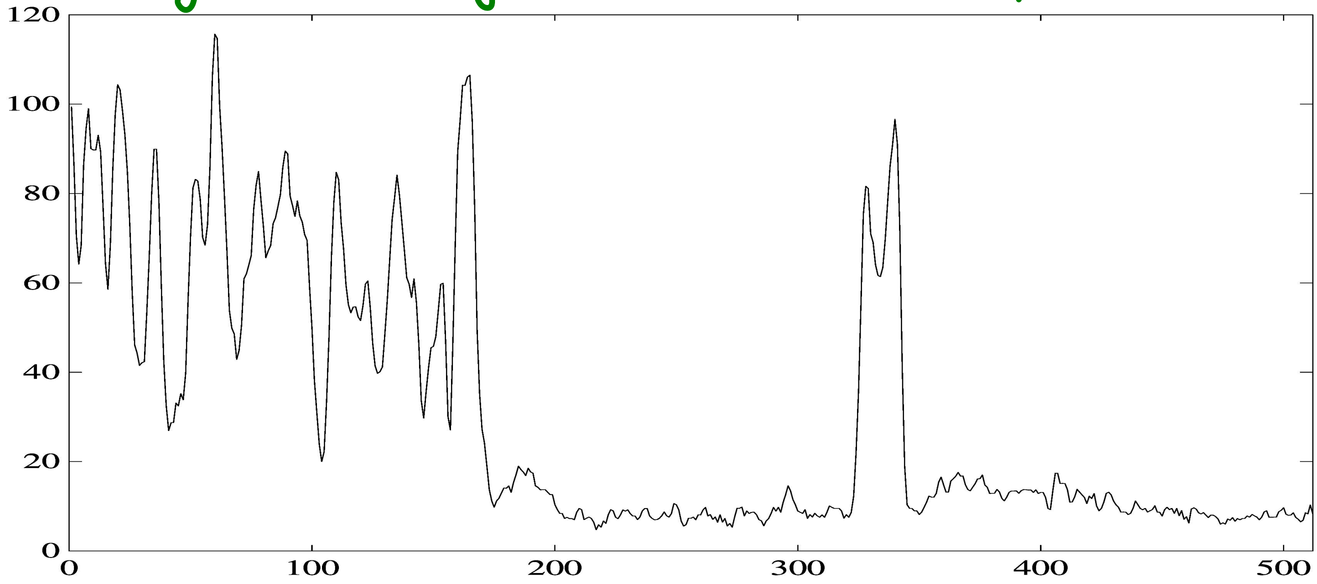


(d)

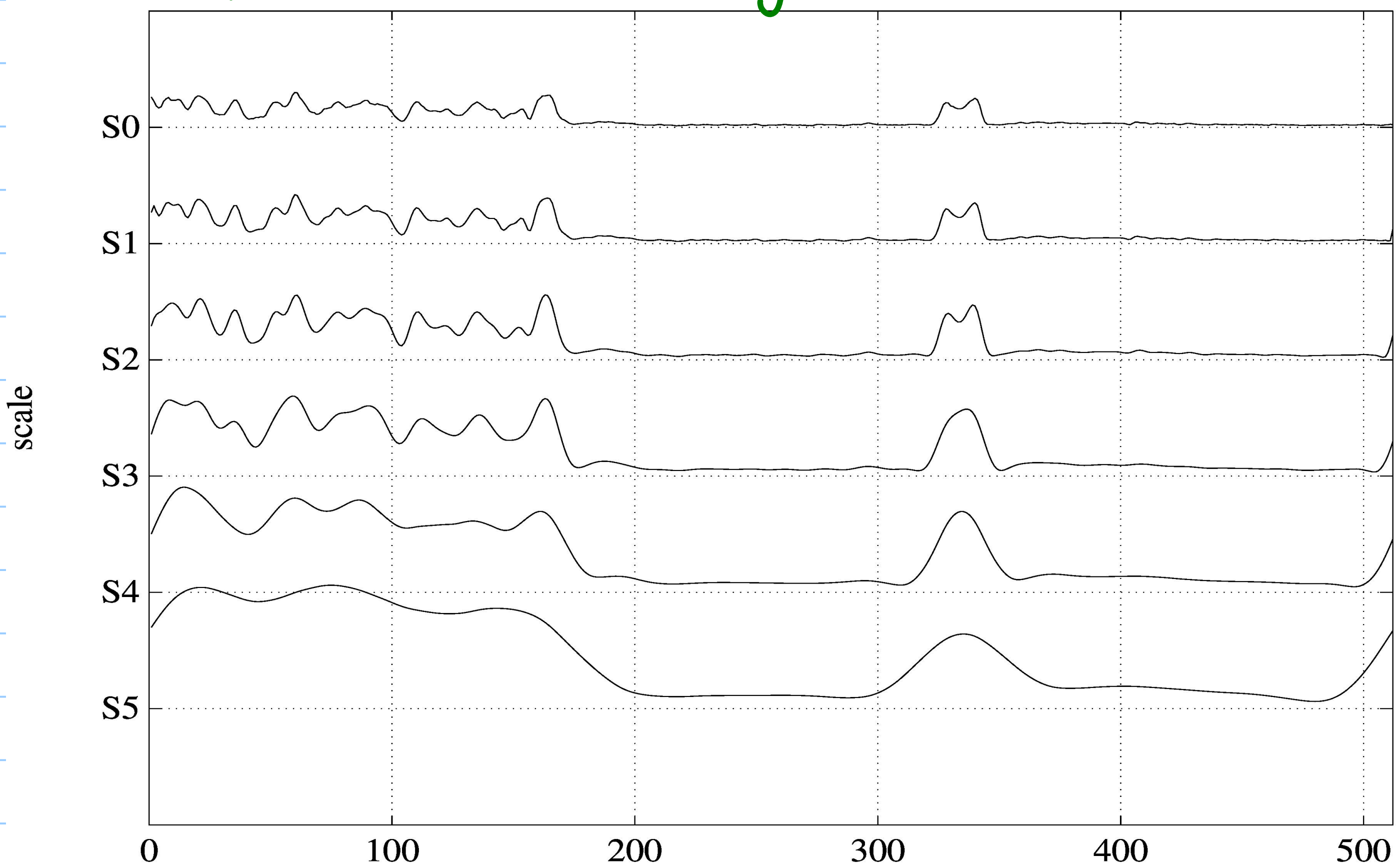
Demonstration of translation invariance of Auto correlation Shell



Original Signal to be decomposed:



Multiscale Averages in Autocorr. Shell



Multiscale Differences in Autocorr. Shell