MAT 280: Harmonic Analysis on Graphs & Networks
Lecture 1: Why Graphs & Networks? Our Course Plan

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Outline

1. Motivations: Why Graphs?

2. Our Course Plan

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Motivations: Why Graphs?

- More and more data are collected in a distributed and irregular manner; they are not organized such as familiar digital signals and images sampled on regular lattices. Examples include:
  - Data from sensor networks
  - Data from social networks, webpages, ...
  - Data from biological networks
  - ...
- It is quite important to analyze:
  - Topology of graphs/networks (e.g., how nodes are connected, etc.)
  - Data measured on nodes (e.g., a node = a sensor, then what is an edge?)

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Motivations: Why Graphs?

- Fourier analysis/synthesis and wavelet analysis/synthesis have been 'crown jewels' for data sampled on the regular lattices.
- Hence, we need to lift such tools for unorganized and irregularly-sampled datasets including those represented by graphs and networks.
- Moreover, constructing a graph from a usual signal or image and analyzing it can also be very useful! E.g., Nonlocal means image denoising of Buades-Coll-Morel.

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An Example of Sensor Networks

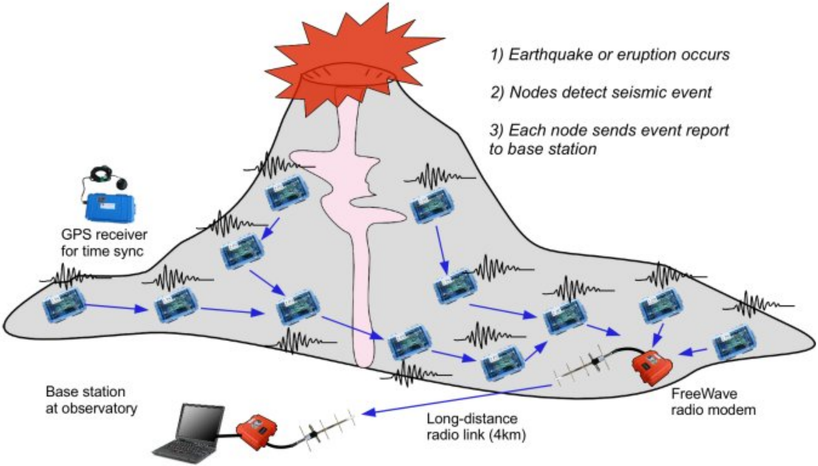


Figure: Volcano monitoring sensor network architecture of Harvard Sensor Networks Lab

An Example of Social Networks

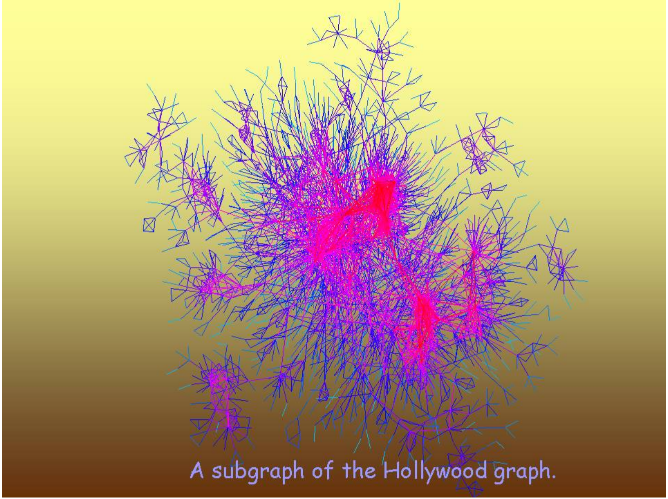


Figure: Through the courtesy of Prof. Fan Chung, UC San Diego

An Example of Biological Networks

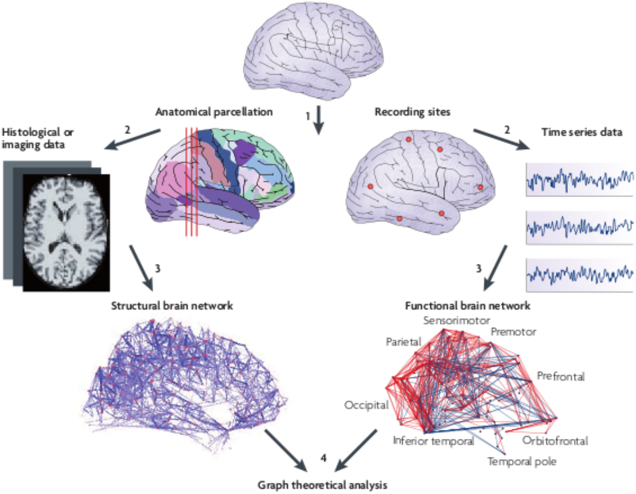
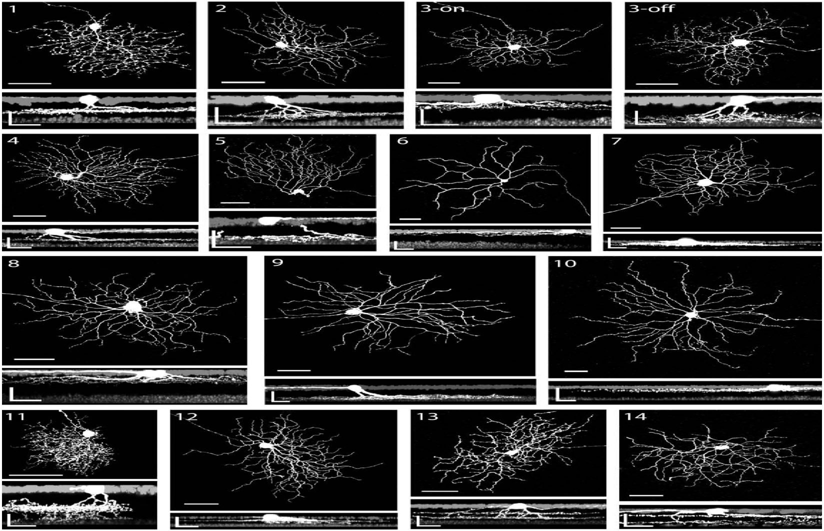
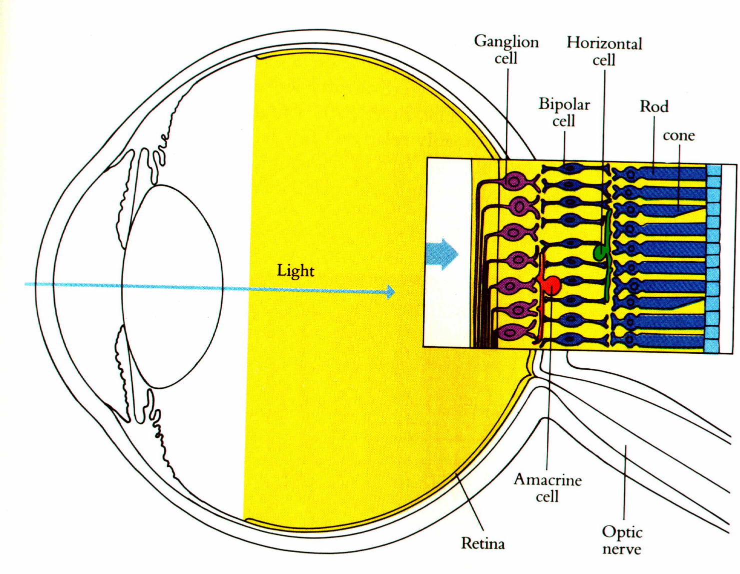


Figure: From E. Bullmore and O. Sporns, *Nature Reviews Neuroscience*, vol. 10, pp.186–198, Mar. 2009.

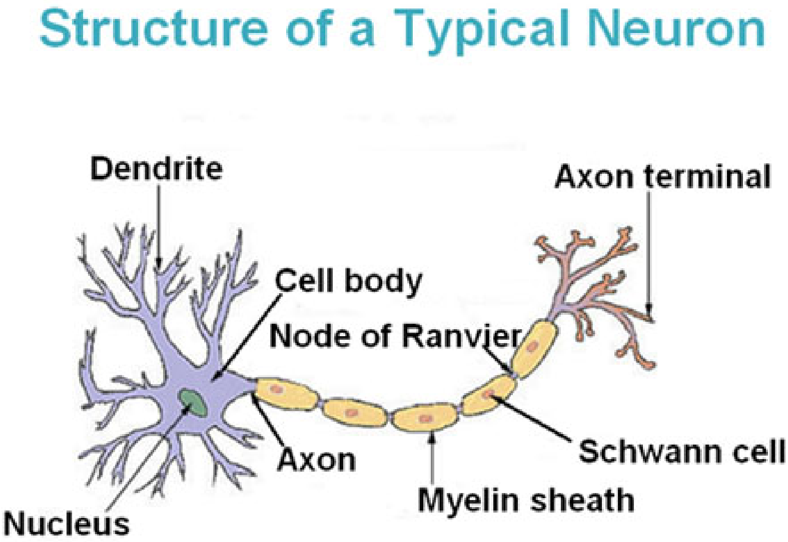
Another Biological Example: Retinal Ganglion Cells



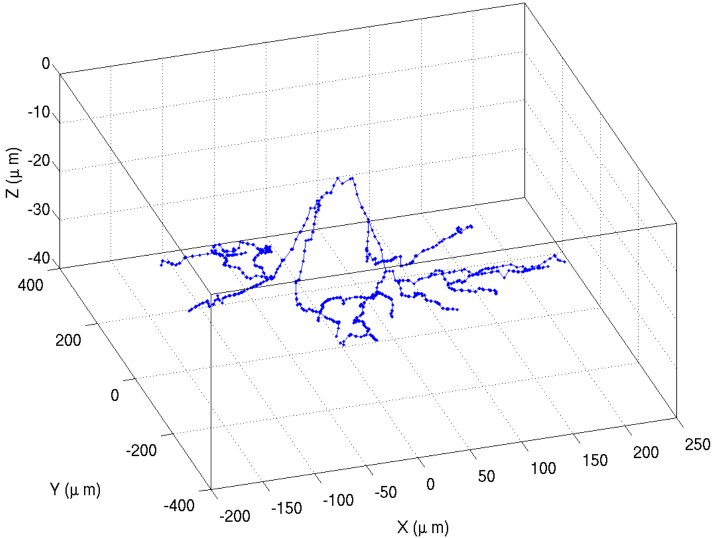
Retinal Ganglion Cells (D. Hubel: *Eye, Brain, & Vision*, '95)



A Typical Neuron (from Wikipedia)



Mouse's RGC as a Graph



Representing a Regular Image as a Graph

often turns out to be quite useful for various purposes. In particular, Nonlocal Means Denoising Algorithm of Buades-Coll-Morel is quite impressive.

- Construct a graph each of whose vertices represents $k \times k$ patch of a given image (k may be $3, 5, \ldots$, etc.) So each vertex represents a point in $\mathbb{R}^{k^2}$.
- Connect every pair of vertices with the weight $W_{ij} = \exp(-\|\text{patch}_i - \text{patch}_j\|^2/\epsilon^2)$ with *appropriately chosen* scale parameter $\epsilon > 0$.
- Compute the weighted average of the center pixel of each patch using the normalized weights $W_{ij}/\sum_l W_{il}$. More precisely, the average of the center of the ith patch, $\bar{c}_i = \sum_j W_{ij} c_j / \sum_l W_{il}$.
- See also an interesting work by Daitch-Kelner-Spielman: "Fitting a Graph to Vector Data," *Proc. 26th Intern. Conf. Machine Learning*, 2009.

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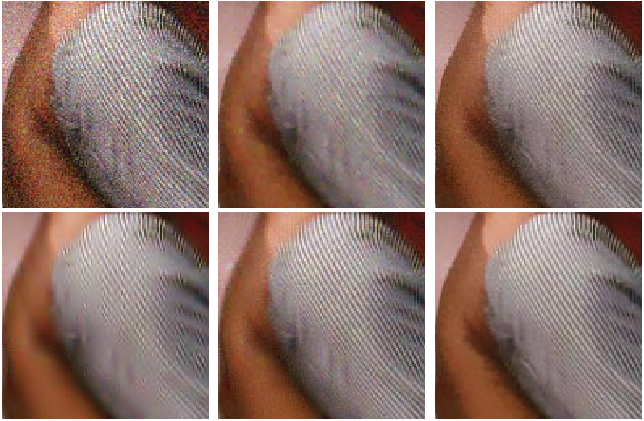
From: A. Buades, B. Coll, and J.-M. Morel, *SIAM Review*, vol. 52, no. 1, pp. 113–147, 2010.

Noisy Image; Total Variation Denoising; Neighborhood Filter



Trans. Inv. Wavelets; Empirical Wiener; Nonlocal Means

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