MAT 280: Harmonic Analysis on Graphs & Networks
Lecture 4: Graph Laplacian Eigenvalues

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Outline

1. Properties of Graph Laplacian Eigenvalues
2. Algebraic Connectivity $a(G) := \lambda_1(G)$
3. Wiener Index

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Notations and Definitions

- In this lecture, we only consider *undirected* and *unweighted* graphs and their *unnormalized* Laplacians $L(G) = D(G) - A(G)$.
- It is your exercise to see how the statements change for the normalized or symmetrically-normalized graph Laplacians.
- Let $|V(G)| = n$, $|E(G)| = m$, and assign each edge an arbitrary orientation to turn G into a directed graph temporarily. Then let us define the *directed incidence matrix* $R = R(G) = (r_{ij}) \in \mathbb{R}^{n \times m}$ of G by

$$r_{ij} = \begin{cases} 1 & \text{if } e_j = [v_{i'}, v_i] \text{ for some } i'; \\ -1 & \text{if } e_j = [v_i, v_{i'}] \text{ for some } i'; \\ 0 & \text{otherwise.} \end{cases}$$

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- Hence, we can sort the eigenvalues of $L(G)$ as $0 = \lambda_0(G) \le \lambda_1(G) \le \cdots \le \lambda_{n-1}(G)$ and denote the set of these eigenvalues by $\Lambda(G)$.
- $m_G(\lambda) :=$ the multiplicity of λ.
- Let $I \subset \mathbb{R}$ be an interval of the real line. Then define $m_G(I) := \#\{\lambda_k(G) \in I\}$.

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General Properties of Graph Laplacian Eigenvalues

- Graph Laplacian matrices of the same graph are *permutation-similar*. In fact, graphs G_1 and G_2 are isomorphic iff there exists a permutation matrix P such that $L(G_2) = P^{\mathsf{T}} L(G_1) P$.
- $\operatorname{rank} L(G) = n - m_G(0)$ where $m_G(0)$ turns out to be the number of connected components of G. Easy to check that $L(G)$ becomes $m_G(0)$ diagonal blocks, and the eigenspace corresponding to the zero eigenvalues is spanned by the *indicator* vectors of each connected component.
- In particular, $\lambda_1 \neq 0$ iff G is connected.
- This led M. Fiedler (1973) to define the *algebraic connectivity* of G by $a(G) := \lambda_1(G)$, viewing it as a quantitative measure of connectivity.

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Miroslav Fiedler (1926–2015)

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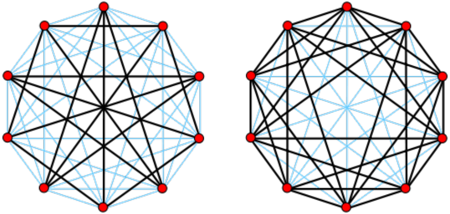

The Petersen graph and its complement in K_{10} (from Wikipedia)

- Then, we have

$$L(G) + L(G^c) = L(K_n) = nI_n - J_n,$$

where J_n is the $n \times n$ matrix whose entries are all 1. Moreover, one can easily show: $\lambda_0(K_n) = 0$, $\lambda_j(K_n) \equiv n$, $1 \le j \le n-1$.

- We also have:

$$\Lambda(G^c) = \{0, n - \lambda_{n-1}(G), n - \lambda_{n-2}(G), \ldots, n - \lambda_1(G)\}.$$

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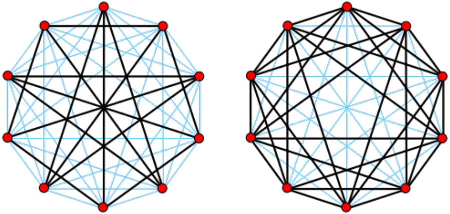

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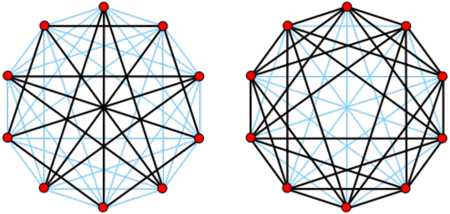

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- From the above, we can see that

$$\lambda_{\max}(G) = \lambda_{n-1}(G) \leq n,$$

and $m_G(n) = m_{G^c}(0) - 1$.

- On the other hand, Grone and Merris showed in 1994

$$\lambda_{\max}(G) = \lambda_{n-1}(G) \geq \max_{1 \leq j \leq n} d_j + 1.$$

- Let G be a connected graph and suppose $L(G)$ has exactly k distinct eigenvalues. Then

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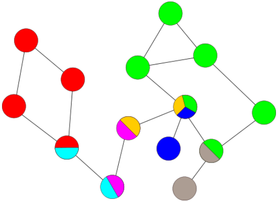
General Properties of Graph Laplacian Eigenvalues ...

- Now define a *cut vertex* by any vertex that increases the number of connected components of G when removed.



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- Let u be a cut vertex of the connected graph G. If the largest component of $G \setminus \{u\}$ contains k vertices, then $\lambda_{n-2}(G) \leq k+1$.

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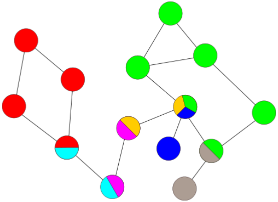

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- A vertex of degree 1 is called a *pendant* vertex; a vertex adjacent to a pendant vertex is called *pendant neighbor*.
- Let $p(G)$ and $q(G)$ be the number of pendant vertices and that of pendant neighbors, respectively.
- The number of pendant neighbors of G is bounded as:

$$p(G) - m_G(1) \leq q(G) \leq m_G(2, n],$$

where the second inequality holds if G is connected and satisfies $2q(G) < n$.

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Outline

Some Graph Operations

- G is said to be *k-vertex-connected* if k is the size of the smallest subset of vertices such that the graph becomes disconnected if they are deleted.
- A 1-vertex-connected graph is called *connected* while a 2-vertex-connected graph is said to be *biconnected*.
- The *vertex-connectivity* $\kappa(G)$ of G is the largest k for which G is k-vertex-connected.
- Similarly we can define the *k-edge-connectedness* and the *edge-connectivity* $\epsilon(G)$.

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- The *Edge-union* $G(V,E)$ of $G_1(V,E_1)$ and $G_2(V,E_2)$ is defined as $E = E_1 \cup E_2$ and V is common among G, G_1, and G_2.
- The *Cartesian product* $G = G_1 \times G_2$ (or also written as $G = G_1 \square G_2$):
- $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ are said to be obtained from a *vertex decomposition* of $G(V,E)$ if $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- If $L(G) = \begin{bmatrix} L(G_1) & O \\ O & L(G_2) \end{bmatrix}$, then G is said to be the *direct sum* of G_1 and G_2 and written as $G = G_1 \oplus G_2$.

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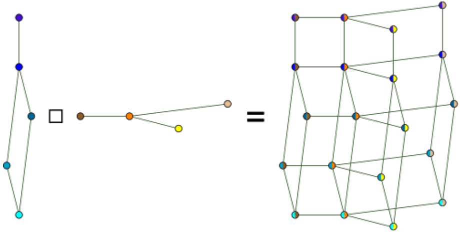

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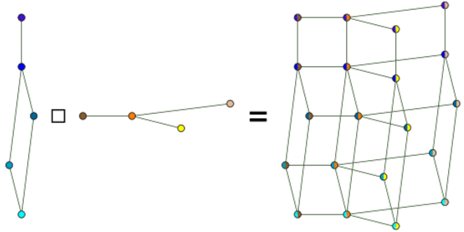

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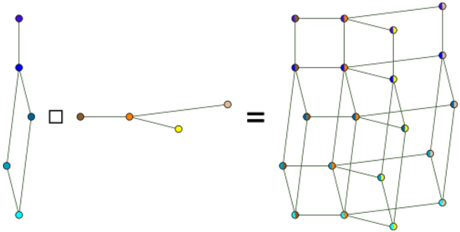

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Algebraic Connectivity and Graph Operations (de Abreu, 2007)

Operations	Relations of $a(G)$, $a(G_i)$, $i = 1,2$
G^c	$a(G^c) = n - \lambda_{n-1}$
$G_1 = G \setminus \{e\}$	$a(G_1) \le a(G)$
$G_1 = G \setminus \{v_{i_1}, \dots, v_{i_k}\}$	$a(G) \le a(G_1) + k$
$G_1 = G \cup \{e\}$	$a(G) \le a(G_1) \le a(G) + 2$
G: edge-union of G_1, G_2	$a(G_1) + a(G_2) = a(G)$
$G = G_1 \times G_2$	$a(G) = \min\{a(G_1), a(G_2)\}$
G_1, G_2: vertex decomposition of G	$a(G) \le \min\{a(G_1) + \|V_2\|, a(G_2) + \|V_1\|\}$
$G = G_1 \oplus G_2$	$a(G_1) + a(G_2) \le a(G_1 \oplus G_2)$

Algebraic Connectivities of Specific Graphs (de Abreu, 2007)

Graph G	Algebraic Connectivity $a(G)$
Complete graph K_n	$a(K_n) = n$
Path P_n	$a(P_n) = 2\left(1 - \cos\frac{\pi}{n}\right)$
Cycle C_n	$a(C_n) = 2\left(1 - \cos\frac{2\pi}{n}\right)$
Bipartite complete graph $K_{p,q}$	$a\left(K_{p,q}\right) = \min\{p, q\}$
Star $K_{1,q}$	$a\left(K_{1,q}\right) = 1$
Cube m-dimension Cb_m	$a(Cb_m) = 2$
Petersen Graph P	$a(P) = 2$

Bounds to Algebraic Connectivity

- Fiedler showed in 1973 the following bounds to $a(G)$:
- For $G \neq K_n$, $a(G) \leq n-2$;
- $2\min_j d_j - n + 2 \leq a(G) \leq \frac{n}{n-1} \min_j d_j$;
- $a(G) \leq \kappa(G) \leq \epsilon(G) \leq \min_j d_j$;
- $2\epsilon(G)\left(1 - \cos\frac{\pi}{n}\right) \leq a(G)$;
- $2\left(\cos\frac{\pi}{n} - \cos\frac{2\pi}{n}\right)\kappa(G) - 2\cos\frac{\pi}{n}\left(1 - \cos\frac{\pi}{n}\right)\max_j d_j \leq a(G)$.

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- $2\left(\cos\frac{\pi}{n} - \cos\frac{2\pi}{n}\right)\kappa(G) - 2\cos\frac{\pi}{n}\left(1-\cos\frac{\pi}{n}\right)\max_j d_j \le a(G)$.

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- $2\epsilon(G)\left(1-\cos\frac{\pi}{n}\right) \leq a(G)$;
- $2\left(\cos\frac{\pi}{n} - \cos\frac{2\pi}{n}\right)\kappa(G) - 2\cos\frac{\pi}{n}\left(1-\cos\frac{\pi}{n}\right)\max_j d_j \leq a(G)$.

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Algebraic Connectivity of Trees

- A *cycle* is a connected graph where every vertex has exactly two neighbors.
- A *tree* T is a connected graph without cycles.
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- They also showed: if $T \neq K_{1,n-1}$ with $n \ge 6$, then $a(T) < 0.49$.

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Isoperimetric Number

- Let $S \subset V(G)$ be a nonempty subset of vertices of G.
- $\partial S := \{e = (u,v) \in E(G) \mid u \in S, v \notin S\}$, which is called the *boundary* of S.
- The *isoperimetric number* of G is defined as

$$i(G) := \inf\left\{ \frac{|\partial S|}{|S|} \,\middle|\, \emptyset \neq S \subset V, |S| \leq \frac{n}{2} \right\},$$

which is closely related to the *conductance* of a graph, i.e., how fast a random walk on G converges to a stationary distribution.
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Outline

Wiener Index

- The *distance matrix* $\Delta(G)$ of G represents "distances" among the vertices, i.e., $\Delta(G)_{i,j} = d(v_i, v_j)$ is the length (or cost) of the shortest path from vertex v_i to vertex v_j.
- The *Wiener index*[1] $W(G)$ of a graph G is the sum of the entries in the upper triangular part of the distance matrix $\Delta(G)$.
- The Wiener index of a molecular graph has been used in chemical applications because it may exhibit a good correlation with physical and chemical properties (e.g., the boiling point, density, viscosity, surface tension, ...) of the corresponding molecule/material.
- Let G be a tree. Then

$$W(G) = \sum_{k=1}^{n-1} \frac{n}{\lambda_k}.$$

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