MAT 280: Harmonic Analysis on Graphs & Networks
Lecture 6: Graph Laplacian Eigenfunctions

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Outline

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Why Graph Laplacian Eigenfunctions?

- Provide an *orthonormal basis* on a graph:
  - can expand functions defined on a graph
  - can perform spectral analysis/synthesis/filtering of data measured on vertices of a graph
  - ...
- Can be used for graph partitioning, graph drawing, data analysis, clustering, ... $\Longrightarrow$ *Graph Cut*, *Spectral Clustering*
- But, less studied than graph Laplacian eigenvalues
- In this course, I will use the terms "eigenfunctions" and "eigenvectors" interchangeably.
- Also, an eigenvector/function is denoted by $\boldsymbol{\phi}$, and its value at vertex $x \in V$ is denoted by $\boldsymbol{\phi}(x)$.

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Basic Properties of GL Eigenfunctions

- If $G = (V, E)$, $|V| = n$, is connected, then $\lambda_0 = 0$, $a(G) = \lambda_1 > 0$.
- We already know that the eigenfunction corresponding to $\lambda_0 = 0$ is $\boldsymbol{\phi}_0 = \frac{1}{\sqrt{n}} \mathbf{1}_n$.
- Hence, $\boldsymbol{\phi}_j$ corresponding to $\lambda_j > 0$, $j = 1, \dots, n-1$, must be orthogonal to $\mathbf{1}_n$: $\sum_{x \in V} \boldsymbol{\phi}_j(x) = 0$, i.e., it must *oscillate*.
- If $\boldsymbol{\phi}(x) = 0$, then $(L\boldsymbol{\phi})(x) = \lambda \boldsymbol{\phi}(x) = 0$. Hence, $\sum_{y \sim x} L_{xy} \boldsymbol{\phi}(y) = 0$.

Theorem (Grover (1990); Gladwell & Zhu (2002))

An eigenfunction of $L(G)$ cannot have a nonnegative local minimum or a nonpositive local maximum.

Proof. Suppose $\boldsymbol{\phi}(x)$ is a local minimum of $\boldsymbol{\phi}$ with $\boldsymbol{\phi}(x) \geq 0$. Then, $\forall y \sim x$, $\boldsymbol{\phi}(x) - \boldsymbol{\phi}(y) < 0$. Now, recall $L\boldsymbol{\phi}(x) = \sum_{y \sim x} a_{xy}(\boldsymbol{\phi}(x) - \boldsymbol{\phi}(y)) = \lambda \boldsymbol{\phi}(x) \geq 0$ where $a_{xy} \geq 0$ is the xy-th entry of the adjacency matrix $A(G)$. These contradicts each other. □

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Basic Properties of *Unweighted* GL Eigenfunctions

Theorem (Merris (1998))

If $0 \lneqq \lambda < n$ is an eigenvalue of $L(G)$, then any eigenfunction affording λ takes the value 0 on every vertex of degree $n-1$.

Proof. Let $v \in V$ be a vertex with $d(v) = n-1$. Then, $L\phi(v) = (n-1)\phi(v) - \sum_{u \neq v} \phi(u) = \lambda\phi(v)$. But, $\phi \perp \mathbf{1}_n$, so $\sum_{u \neq v} \phi(u) = -\phi(v)$. This leads to: $n\phi(v) = \lambda\phi(v)$. Since $0 \lneqq \lambda \lneqq n$, we must have $\phi(v) = 0$. □

Theorem (Merris (1998))

Let (λ, ϕ) be an eigenpair of $L(G)$. If $\phi(u) = \phi(v)$, then (λ, ϕ) is also an eigenpair of $L(G')$ where G' is the graph obtained from G by either deleting or adding the edge $e = (u, v)$ depending on whether or not $e \in E(G)$.

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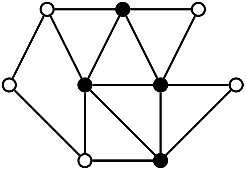
Let W be a nonempty subset of $V(G)$. Then, the *reduced graph* $G\{W\}$ is obtained from G by deleting all vertices in $V \setminus W$ that are not adjacent to a vertex of W and subsequent deletion of any remaining edges that are not incident with a vertex in W.



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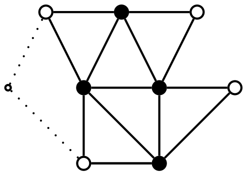

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Basic Properties of *Unweighted* GL Eigenfunctions ...

Theorem (Merris (1998))

Fix a nonempty subset $W \subset V$. Suppose $\boldsymbol{\phi}$ is an eigenfunction of the reduced graph $G\{W\}$ that affords λ and is supported by W in the sense that if $\boldsymbol{\phi}(u) \neq 0$, then $u \in W$. Then the extension $\widetilde{\boldsymbol{\phi}}$ with $\widetilde{\boldsymbol{\phi}}(v) = \boldsymbol{\phi}(v)$ for $v \in W$ and $\widetilde{\boldsymbol{\phi}}(v) = 0$ for $v \in V \setminus W$ is an eigenfunction of G affording λ.

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Let $\boldsymbol{\phi}$ be an eigenfunction affording λ of G. Let N_v be the set of neighbors of v. Suppose $\boldsymbol{\phi}(u) = \boldsymbol{\phi}(v) = 0$, where $N_u \cap N_v = \emptyset$. Let G' be the graph on $n-1$ vertices obtained by coalescing u and v into a single vertex, which is adjacent in G' to precisely those vertices that are adjacent in G to u or to v. Then, the function $\boldsymbol{\phi}'$ obtained by restricting $\boldsymbol{\phi}$ to $V(G) \setminus \{v\}$ is an eigenfunction of G' affording λ.

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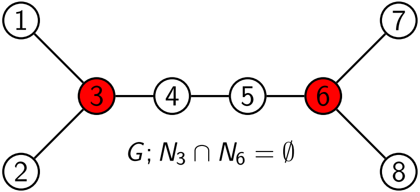
A Simple Example



$\lambda_2(G) = 1; \boldsymbol{\phi}_2(G) = [-0.0261, -0.0261, \mathit{0}, 0.0523, 0.0523, \mathit{0}, -0.7303, 0.6781]^{\mathsf{T}}$

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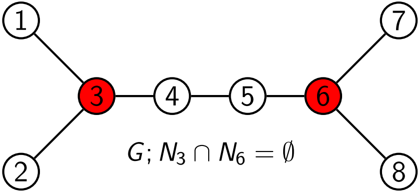

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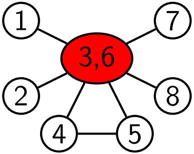

G'

$$\lambda_2(G') = 1; \boldsymbol{\phi}_2(G') \propto [-0.0261, -0.0261, 0, 0.0523, 0.0523, -0.7303, 0.6781]^{\mathsf{T}}$$

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The Perron-Frobenius Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a rather general symmetric matrix associated with a graph G such that $A_{uv} \neq 0$ iff $e = (u, v) \in E(G)$. Then, A is called *irreducible* if its underlying graph is *connected*.

Theorem (Perron-Frobenius Theorem)

Let A, B be real symmetric irreducible nonnegative $n \times n$ matrices. Then,

(i) *the spectral radius $\rho(A)$ is a simple eigenvalue of A. If ϕ is an eigenfunction for $\rho(A)$, then no entries of ϕ are zero, and all have the same sign.*

(ii) *Furthermore, if $A - B$ is nonnegative, then $\rho(B) \leq \rho(A)$, with equality iff $B = A$.*

Corollary

Let G be a connected graph. Then, the smallest eigenvalue of $L(G)$, $L_{\mathrm{rw}}(G)$, $L_{\mathrm{sym}}(G)$, i.e., $\lambda_0 = 0$, is simple, and ϕ_0 can be taken to have all entries positive. ϕ_0 is often called the Perron vector of G.

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My Comments on the Perron-Frobenius Theorem

- If $G = P_n$, then $\boldsymbol{\phi}_j$ of $L(G)$ is jth DCT-II basis vector, as I discussed in Lecture 3. Hence, the Perron vector of P_n is the constant vector for the *DC component* in the signal processing terminology.
- For the continuous case, I talked about the integral operator $\mathscr{K}$ that commutes with the Laplace operator in Lecture 2. In particular, I showed the 1D example where the domain is the unit interval $\Omega = (0, 1)$. In that case, the smallest eigenvalue is $\lambda_0 \approx -5.756915$, and $\phi_0(x) \propto \cosh \sqrt{-\lambda_0}\left(x - \frac{1}{2}\right)$. This function also does not change its sign, hence it can be viewed as the Perron vector of $\mathscr{K}$.

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- Does there exist the P-F theory for *compact operators*? $\Longrightarrow$ YES!

Theorem (Krein & Rutman (1948))

Let X be a Banach space, and let $K \subset X$ be a convex cone such that the set $K - K = \{f - g \mid f, g \in K\}$ is dense in X. Let $T : X \to X$ be a non-zero compact operator which is positive, meaning that $T(K) \subset K$, and assume that its spectral radius $\rho(T)$ is strictly positive. Then $\rho(T)$ is an eigenvalue of T with positive eigenfunction, meaning that there exists $\phi \in K \setminus \{0\}$ such that $T(\phi) = \rho(T)\phi$.

- Generally, one of my research goals is to consider *the graph version of the integral operator commuting with a given graph Laplacian*, and analyze its properties!

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- Generally, one of my research goals is to consider *the graph version of the integral operator commuting with a given graph Laplacian*, and analyze its properties!

My Comments on the Perron-Frobenius Theorem . . .

- Does there exist the P-F theory for *compact operators*? $\Longrightarrow$ YES!

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Outline

Perron-Frobenius/Fiedler $\Longrightarrow$ Courant

- From the Perron-Frobenius theorem, for a connected graph G, we know that $\lambda_0 = 0$ and $\boldsymbol{\phi}_0$ is all positive.
- By Fielder, we also know that the algebraic connectivity $a(G) = \lambda_1(G) > 0$, $\boldsymbol{\phi}_1$ (called the *Fiedler vector* of G) splits V into three subsets $V = V_+ \cup V_- \cup V_0$ where the values of $\boldsymbol{\phi}_1$ on V_+, V_-, V_0 are positive, negative, and zero (note that V_0 could be $\emptyset$).
- These reminds us of Courant's celebrated *Nodal Domain Theorem* for elliptic operatros on manifolds.

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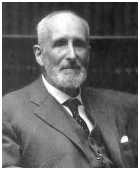
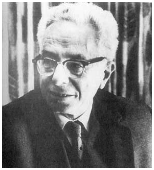
(a) Ferdinand Georg Frobenius (1849–1917)



(b) Oskar Perron (1880–1975)



(c) Richard Courant (1888–1972)

(d) Miroslav Fiedler (1926–2015)

Courant's Nodal Domain Theorem

Theorem (Courant (1923))

Let L be a self-adjoint second order differential operator, and consider the following elliptic eigenvalue problem on a domain $\Omega \subset \mathbb{R}^d$:

$$Lu + \lambda \rho u = 0, \quad \rho > 0,$$

with arbitrary homogeneous boundary conditions. If its eigenfunctions are ordered according to increasing eigenvalues, then the nodes (a.k.a. nodal sets or nodal lines) of the kth eigenfunction ϕ_k ($k = 0, 1, \ldots$) divide Ω into no more than $k+1$ subdomains.

Of course, the nodal sets of a function $f(x)$ in Ω is defined as

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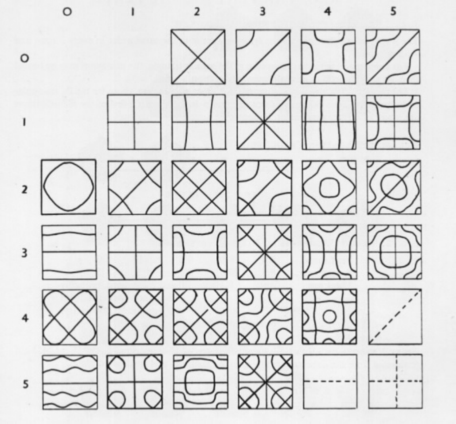
A Famous Example of Nodal Domain Theorem

Courtesy: `http://www.cymascope.com/cyma_research/history.html`



(a) Chladni Plates

(b) Ernst Chladini (1756–1827)

Discrete Nodal Domains

- In the context of manifolds, the *nodal domains* of f refers to the connected components of the complement of the nodal set $\mathscr{N}[f]$, i.e., to the components of $\{\boldsymbol{x} \in \Omega \mid f(\boldsymbol{x}) \neq 0\}$, which are bounded by the nodal sets.
- The discrete analog of a "nodal domain" is a maximal connected induced subgraph consisting entirely of positive and negative vertices w.r.t. a given function f defined over $V(G)$.
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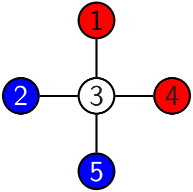

$K_{1,4}$

$\lambda_1 = 1$; $m_{K_{1,4}}(1) = 3$; $\boldsymbol{\phi}_1 \propto [1, -1, 0, 1, -1]^{\mathsf{T}}$.

Discrete Nodal Domains . . .

- A *positive* (or *negative*) *strong nodal domain* of f on $V(G)$ is a maximal connected induced subgraph of G on vertices $v \in V$ with $f(v) > 0$ (or $f(v) < 0$). The number of strong nodal domains of f is denoted by $\mathfrak{S}(f)$.
- In contrast, a *positive* (or *negative*) *weak nodal domain* of f on $V(G)$ is a maximal connected induced subgraph of G on vertices $v \in V$ with $f(v) \geq 0$ (or $f(v) \leq 0$) that contains at least one nonzero vertex. The number of weak nodal domains of f is denoted by $\mathfrak{W}(f)$.
- In the above example of $K_{1,4}$, $\mathfrak{S}(\phi_1) = 4$ and $\mathfrak{W}(\phi_1) = 2$ because the strong nodal domains are $\{\{1\},\{2\},\{4\},\{5\}\}$ while the weak nodal domains are $\{\{1,3,4\},\{2,3,5\}\}$.
- Obviously, we always have $\mathfrak{W}(f) \leq \mathfrak{S}(f)$.
- The *zero vertices* separate positive (or negative) strong nodal domains while they join weak nodal domains. In fact, each zero vertex simultaneously belongs to exactly one weak positive nodal domain and exactly one weak negative nodal domain.

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We focus our attention on the kth eigenvalue λ_k with multiplicity r of a graph Laplacian (L, L_{rw}, L_{sym}).

$$\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r-1} < \lambda_{k+r} \le \cdots \le \lambda_{n-1}.$$

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Let G be a connected graph with n vertices. Then, any graph Laplacian eigenfunction $\boldsymbol{\phi}_k$ corresponding to λ_k with multiplicity r has at most $k+1$ weak nodal domains and $k+r$ strong nodal domains, i.e.,

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In the example of $K_{1,4}$, $\lambda_1 = 1$ has multiplicity $r = 3$. Hence, $\mathfrak{W}(\boldsymbol{\phi}_1) = 2 \le 1+1$ and $\mathfrak{S}(\boldsymbol{\phi}_1) = 4 \le 1+3$ are satisfied!

Discrete Nodal Domains . . .

Corollary (Fiedler (1975))

If G is connected, then $\mathfrak{W}(\boldsymbol{\phi}_1) = 2$.

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The eigenfunction $\boldsymbol{\phi}_k$ affording λ_k has at most k positive weak nodal domains for $k \geq 1$. Consequently, $\mathfrak{W}(\boldsymbol{\phi}_k) \leq 2k$.

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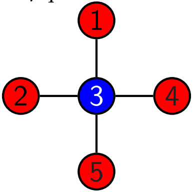
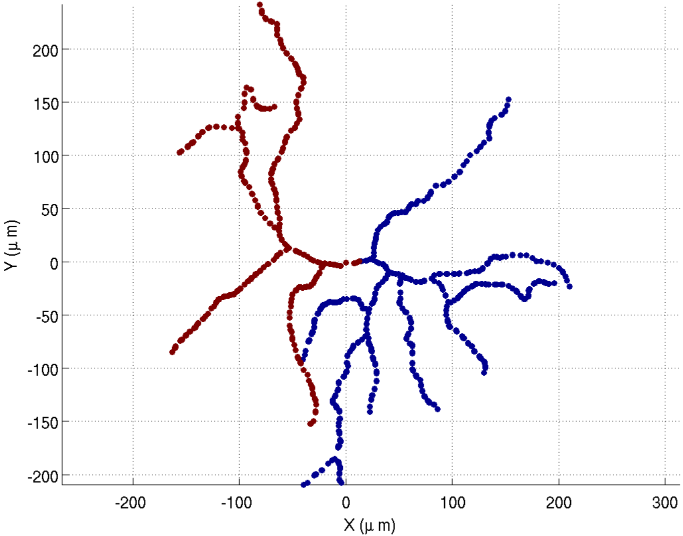
In the previous example of $K_{1,4}$, we have $\lambda_{\max} = \lambda_4 = 5$, and $\boldsymbol{\phi}_4 \propto [1, 1, -4, 1, 1]^{\mathsf{T}}$. Hence, $\mathfrak{W}(\boldsymbol{\phi}_4) = 5 \leq 2 \cdot 4 = 8$, satisfying the corollary.

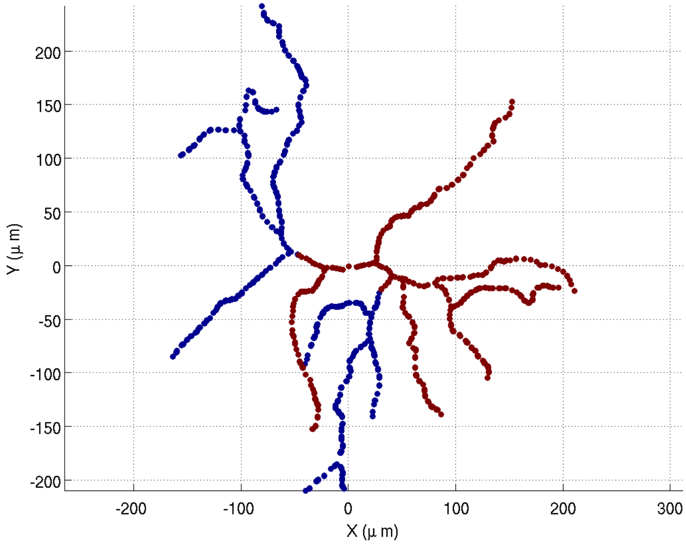


$K_{1,4}$

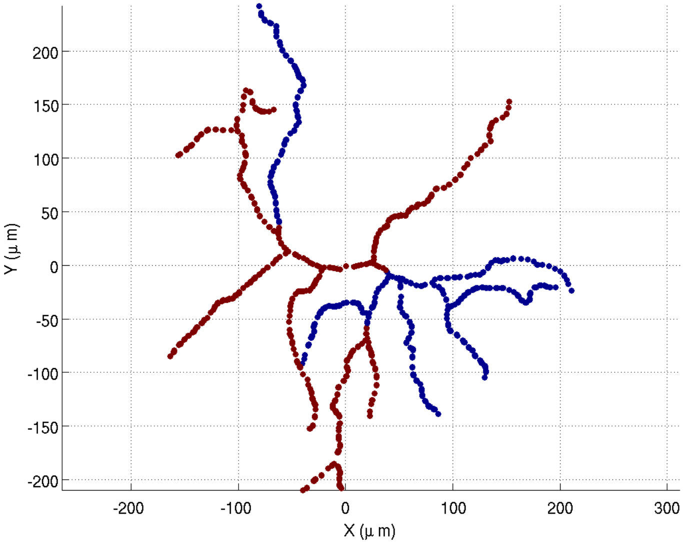
Discrete Nodal Domains of a Dendritic Tree: $\mathrm{sign}(\phi_1)$



Discrete Nodal Domains of a Dendritic Tree: $\mathrm{sign}(\boldsymbol{\phi}_2)$



Discrete Nodal Domains of a Dendritic Tree: $\mathrm{sign}(\boldsymbol{\phi}_3)$



Discrete Nodal Domains of a Dendritic Tree: $\mathrm{sign}(\boldsymbol{\phi}_4)$