MAT 280: Harmonic Analysis on Graphs & Networks
Lecture 16: Wavelets on Graphs I

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Outline

1. Brief Introduction to Wavelets

2. Wavelets using Graph Laplacians

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2 Wavelets using Graph Laplacians

What are Wavelets?

- For usual signals and images, *wavelet transforms* have a proven track record of success: an excellent tool to analyze complicated signals; the backbone of the JPEG *2000* Image Compression Standard; ...
- A wavelet transform *decomposes* a given signal into a *multiresolution* representation (i.e., *analysis*) whereas the inverse wavelet transform *reconstructs* the original signal from such a representation (i.e., *synthesis*).
- The key idea of wavelet transform is to use *translations* and *dilations* of a *single* function, say, $\psi(x) \in L^2(\mathbb{R})$ in the case of 1D signals.

Definition (Mother Wavelet Function)

A *mother wavelet function* is a function $\psi \in L^2(\mathbb{R})$ with

- $\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 0$;
- $\|\psi\|_2 = 1$;
- $\int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\,\mathrm{d}\xi < \infty$ (admissibility condition).

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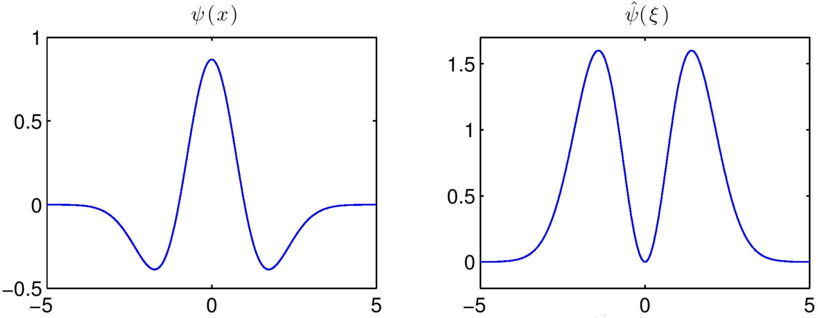
An Example of Mother Wavelet Function

The *Laplacian of Gaussian* (a.k.a. *Mexican hat*) function:

$$\begin{cases} \psi(x) = \frac{2}{\sqrt[4]{\pi}\sqrt{3\sigma}}\left(1 - \frac{x^2}{\sigma^2}\right)\exp\left(-x^2/(2\sigma^2)\right); \\ \widehat{\psi}(\xi) = \mathscr{F}\psi(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x = 8\sqrt{\frac{2}{3}}\pi^{9/4}\sigma^{5/2}\xi^2\exp\left(-2\pi^2\sigma^2\xi^2\right). \end{cases}$$



- $\widehat{\psi}(0) = 0$; $\widehat{\psi}(\xi) \sim \xi^2$ near $x = 0$, i.e., it approximates $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, so it's a *pseudo differential operator*!
- Note: $\exists$ many different mother wavelet functions!

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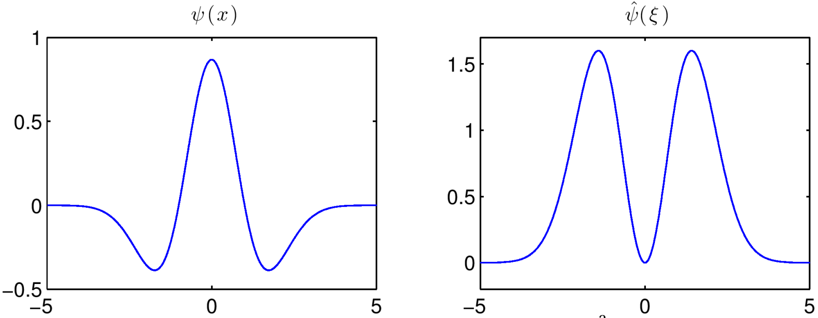

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- Note: ∃ many different mother wavelet functions!

The Continuous Wavelet Transform

- Given a mother wavelet function, let's generate its *family* via

$$\psi_{a,b}(x) := \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right),$$

  where $a > 0$, $b \in \mathbb{R}$ are the *dilation (or scale)* and *translation (or shift)* parameters, respectively. Note $\|\psi_{a,b}\|_2 = 1$.

- Then, the *continuous wavelet transform* of an input function $f \in L^2(\mathbb{R})$ is defined as:

$$Wf(a,b) = W_\psi f(a,b) := \int_{-\infty}^{\infty} f(x)\overline{\psi_{a,b}(x)}\,\mathrm{d}x = \langle f, \psi_{a,b}\rangle.$$

- This can be viewed as a *linear filtering*: $Wf(a,b) = f * \overline{\widetilde{\psi}_a}(b)$ where

$$\widetilde{\psi}_a(x) := \frac{1}{\sqrt{a}}\psi(-x/a) \xrightarrow{\mathscr{F}} \widehat{\widetilde{\psi}}_a(\xi) = \sqrt{a}\overline{\widehat{\psi}(a\xi)}$$

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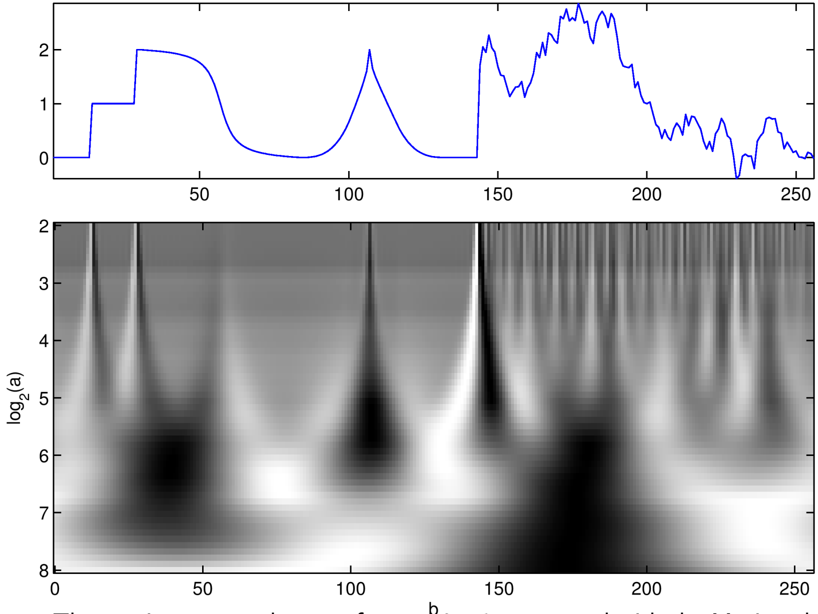

Figure: The continuous wavelet transform $Wf(a,b)$ computed with the Mexican hat mother wavelet. The vertical axis is in the logarithmic scale $\log a$.

Motivation of Lifting Wavelets to Graphs

- Classical harmonic analysis tools such as Fourier and wavelet transforms have been the 'crown jewels' for analyzing regularly-sampled data or functions defined on simple Euclidean domains (e.g., a rectangle).
- They have a proven track record of success in a variety of applications, e.g., data compression, image analysis, and statistical signal processing, ...
- However, these classical harmonic analysis tools *cannot* directly handle datasets recorded on general graphs and networks.
- Hence, the community of applied and computational harmonic analysts including my group, has recognized the importance of transferring these tools to the graph setting, and in fact, efforts have been made to extend classical wavelets and their relatives to the ever-expanding realm of data on graphs.
- This is not an easy endeavor mainly because of the *lack* of the notion of proper *"frequency"* or the *"dual"* domain for graphs unlike the Euclidean domains.

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Work of Hammond-Vandergheynst-Gribonval

- By now, there are many different methods to construct wavelet-like transforms on graphs. Today, we'll focus on the *spectral graph wavelet transform* (SGWT) of D. K. Hammond, P. Vandergheynst, R. Gribonval: "Wavelets on graphs via spectral graph theory," *Applied and Computational Harmonic Analysis*, vol. 30, no. 2, pp. 129-150, 2011.
- Conceptually, it is an adaptation of the *continuous wavelet transform* for graphs.
- Let $G(V,E)$ be a weighted graph with $|V| = n$.
- The *graph Fourier transform* of $f \in \mathscr{L}^2(V)$ is defined as

$$\widehat{f}(\ell) = \langle f, \phi_\ell \rangle = \sum_{k=1}^{n} f(k)\phi_\ell^*(k), \quad \ell = 0, 1, \ldots, n-1$$

where $\phi_\ell := (\phi_\ell(1), \ldots, \phi_\ell(n))^{\mathsf{T}} \in \mathbb{R}^n$ is the ℓth graph Laplacian eigenvector corresponding to the eigenvalue λ_ℓ, and ϕ_ℓ^* is its hermitian transpose. The original vector f can be reconstructed by

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- Let $T_g = g(L) : \mathscr{L}^2(V) \to \mathscr{L}^2(V)$ be defined as a *Fourier multiplier* as

$$\widehat{T_g f}(\ell) = g(\lambda_\ell)\widehat{f}(\ell),$$

where g is a wavelet generating kernel, also called the *spectral graph wavelet kernel* (SGWT kernel). We now have:

$$(T_g f)(k) = \sum_{\ell=0}^{n-1} g(\lambda_\ell)\widehat{f}(\ell)\phi_\ell(k).$$

- The *spectral graph wavelet operator* at scale $s > 0$ is defined by $T_g^s = g(sL)$. Hence, the *spectral wavelet function* at scale s and vertex v_m is realized as $\psi_{s,m}(k) := T_g^s \delta_m(k)$ where δ_m is an impulse located at v_m. Because $\widehat{\delta}_m(\ell) = \phi_\ell^*(m)$, we have

$$\psi_{s,m}(k) = \sum_{\ell=0}^{n-1} g(s\lambda_\ell)\phi_\ell^*(m)\phi_\ell(k).$$

Work of Hammond-Vandergheynst-Gribonval . . .

- The *wavelet coefficient* of a given function $f \in \mathscr{L}^2(V)$ is computed by

$$W_f(s,m) := \langle f, \psi_{s,m} \rangle = \left(T_g^s f\right)(m) = \sum_{\ell=0}^{n-1} g(s\lambda_\ell)\widehat{f}(\ell)\phi_\ell(m).$$

Lemma (H-V-G)

If the SGWT kernel g satisfies the admissibility condition:

$$\int_0^\infty \frac{g^2(x)}{x}\,\mathrm{d}x =: C_g < \infty, \text{ and } g(0) = 0, \text{ then}$$

$$\frac{1}{C_g}\sum_{m=1}^{n}\int_0^\infty W_f(s,m)\psi_{s,m}(k)\frac{\mathrm{d}s}{s} =: f^\sharp(k)$$

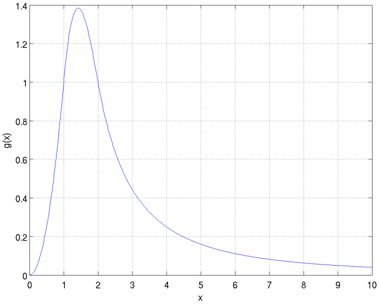
where $f^\sharp = f - \langle f, \phi_0 \rangle \phi_0$. In other words, f can be reconstructed from the information $\{W_f(s,m)\}$ and the DC component.

Work of Hammond-Vandergheynst-Gribonval . . .

An example of $g(x)$:

$$g(x) = \begin{cases} (x/x_1)^\alpha & \text{for } 0 \le x < x_1; \\ s(x) & \text{for } x_1 \le x \le x_2; \\ (x_2/x)^\beta & \text{for } x > x_2. \end{cases}$$

H-V-G used $x_1 = 1$; $x_2 = 2$; $\alpha = \beta = 2$; and $s(x) = -5 + 11x - 6x^2 + x^3$.



Work of Hammond-Vandergheynst-Gribonval . . .

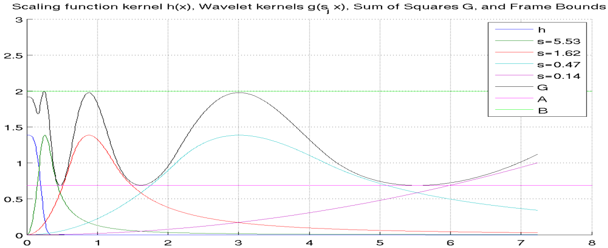
- The *scaling* (a.k.a. *father wavelet*) function φ can be defined as:

$$\varphi_m = \varphi_{1,m} := T_h \delta_m = h(L)\delta_m,$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ acts as a low pass filter with $h(0) > 0$ and $h(x) \to 0$ as $x \to \infty$. An example is: $h(x) = \gamma \exp\left(-(x/0.6\lambda_{\min})^4\right)$ where γ is set such that $h(0) = \max_{x \ge 0} g(x)$.

- The scaling coefficient of a given function $f \in \mathscr{L}^2(V)$ is computed by

$$S_f(m) := \langle f, \varphi_m \rangle.$$



Work of Hammond-Vandergheynst-Gribonval ...

- For a discrete transform, sample the scale parameter s in a logarithmically equispaced manner between $s_J = x_2/\lambda_{\max}$ and $s_1 = x_2/\lambda_{\min}$ where $\lambda_{\max} \geq \lambda_{n-1}$, $\lambda_{\min} = \lambda_{\max}/K$ for some $K > 0$.

Theorem (H-V-G)

Given a set of scales $\{s_j\}_{1\leq j\leq J}$, the set $\mathcal{F} := \{\varphi_m\}_{1\leq m\leq n} \cup \{\psi_{s_j,m}\}_{1\leq j\leq J; 1\leq m\leq n}$ constitutes a frame with bounds A, B given by

$$A = \min_{\lambda\in[0,\lambda_{n-1}]} G(\lambda); \quad B = \max_{\lambda\in[0,\lambda_{n-1}]} G(\lambda)$$

where $G(\lambda) := h^2(\lambda) + \sum_{j=1}^{J} g^2(s_j\lambda)$.

Hence, for any $f \in \mathscr{L}^2(V)$, we have:

$$A\|f\|_2^2 \leq \sum_{m=1}^{n} \left(|S_f(m)|^2 + \sum_{j=1}^{J} |W_f(s_j, m)|^2 \right) \leq B\|f\|_2^2.$$

Work of Hammond-Vandergheynst-Gribonval ...

- H-V-G proposed a fast transform instead of computing all of the graph Laplacian eigenvalues and eigenvectors that would require $O(n^3)$ operations.
- The fast algorithm requires $O(C \cdot |E| + C' \cdot Jn)$, where $C, C' > 0$ are some constants.
- The fast algorithm is based on the *Chebyshev polynomial* approximation $p(s_j x)$ to the function $g(s_j x)$ and fully utilizes the Chebyshev recurrence relation.
- Hence $W_f(s_j, m) \approx \delta_m^* p(s_j L) f$, i.e., done by matrix-vector products.
- It is especially effective if the graph (hence L) is *sparse*.
- As for the inverse transform, a stable algorithm exists because $\mathcal{F}$ forms a frame. As the frame theory indicates, it involves the pseudo inverse. Hence, it is not super fast even if one uses the conjugate gradient method.
- Software demo now!

My Reaction to SGWT

- The eigenvalue axis is not the same as the frequency axes particularly if a given graph comes from data from a topologically skewed shape or a narrow strip. In those cases, the eigenvalue orders are not intuitive. $\Longrightarrow$ See later lectures as well as Lecture 3.
- The inverse transform is still slow even if one uses the Conjugate Gradient (CG) method.