Numerical Analysis of the Non-uniform Sampling Problem

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Abstract

We give an overview of recent developments in the problem of reconstructing a band-limited signal from non-uniform sampling from a numerical analysis view point. It is shown that the appropriate design of the finite-dimensional model plays a key role in the numerical solution of the non-uniform sampling problem. In the one approach (often proposed in the literature) the finite-dimensional model leads to an ill-posed problem even in very simple situations. The other approach that we consider leads to a well-posed problem that preserves important structural properties of the original infinite-dimensional problem and gives rise to efficient numerical algorithms. Furthermore a fast multilevel algorithm is presented that can reconstruct signals of unknown bandwidth from noisy non-uniformly spaced samples. We also discuss the design of efficient regularization methods for ill-conditioned reconstruction problems. Numerical examples from spectroscopy and exploration geophysics demonstrate the performance of the proposed methods.

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1 Introduction

The problem of reconstructing a signal f from non-uniformly spaced measurements $f(t_j)$ arises in areas as diverse as geophysics, medical imaging, communication engineering, and astronomy. A successful reconstruction of f from its samples $f(t_j)$ requires a priori

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information about the signal, otherwise the reconstruction problem is ill-posed. This a priori information can often be obtained from physical properties of the process generating the signal. In many of the aforementioned applications the signal can be assumed to be (essentially) band-limited.

Recall that a signal (function) is band-limited with bandwidth Ω if it belongs to the space $\boldsymbol{B}_\Omega$, given by

$$\boldsymbol{B}_\Omega = \left\{ f \in \boldsymbol{L}^2(\mathbb{R}) : \hat{f}(\omega) = 0 \text{ for } |\omega| > \Omega \right\}, \tag{1}$$

where $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i \omega t}\, dt\,.$$

For convenience and without loss of generality we restrict our attention to the case $\Omega = \frac{1}{2}$, since any other bandwidth can be reduced to this case by a simple dilation. Therefore we will henceforth use the symbol $\boldsymbol{B}$ for the space of band-limited signals.

It is now more than 50 years ago that Shannon published his celebrated sampling theorem [35]. His theorem implies that any signal $f \in \boldsymbol{B}$ can be reconstructed from its regularly spaced samples $\{f(n)\}_{n\in\mathbb{Z}}$ by

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}\,. \tag{2}$$

In practice however we seldom enjoy the luxury of equally spaced samples. The solution of the nonuniform sampling problem poses much more difficulties, the crucial questions being:

- Under which conditions is a signal $f \in \boldsymbol{B}$ uniquely defined by its samples $\{f(t_j)\}_{j\in\mathbb{Z}}$?
- How can f be stably reconstructed from its samples $f(t_j)$?

These questions have led to a vast literature on nonuniform sampling theory with deep mathematical contributions see [11, 25, 3, 6, 15] to mention only a few. There is also no lack of methods claiming to efficiently reconstruct a function from its samples [42, 41, 1, 14, 40, 26, 15]. These numerical methods naturally have to operate in a finite-dimensional model, whereas theoretical results are usually derived for the infinite-dimensional space $\boldsymbol{B}$. From a numerical point of view the “reconstruction” of a bandlimited signal f from a finite number of samples $\{f(t_j)\}_{j=1}^r$ amounts to computing an approximation to f (or $\hat{f}$) at sufficiently dense (regularly) spaced grid points in an interval (t_1, t_r).

Hence in order to obtain a "complete" solution of the sampling problem following questions have to be answered:

- Does the approximation computed within the finite-dimensional model actually converge to the original signal f, when the dimension of the model approaches infinity?
- Does the finite-dimensional model give rise to fast and stable numerical algorithms?

These are the questions that we have in mind, when presenting an overview on recent advances and new results in the nonuniform sampling problem from a numerical analysis view point.

In Section 2 it is demonstrated that the celebrated frame approach does only lead to fast and stable numerical methods when the finite-dimensional model is carefully designed. The approach usually proposed in the literature leads to an ill-posed problem even in very simple situations. We discuss several methods to stabilize the reconstruction algorithm in this case. In Section 3 we derive an alternative finite-dimensional model, based on trigonometric polynomials. This approach leads to a well-posed problem that preserves important structural properties of the original infinite-dimensional problem and gives rise to efficient numerical algorithms. Section 4 describes how this approach can be modified in order to reconstruct band-limited signals for the in practice very important case when the bandwidth of the signal is not known. Furthermore we present regularization techniques for ill-conditioned sampling problems. Finally Section 5 contains numerical experiments from spectroscopy and geophysics.

Before we proceed we introduce some notation that will be used throughout the paper. If not otherwise mentioned $\|h\|$ always denotes the $\boldsymbol{L}^2(\mathbb{R})$-norm ($\boldsymbol{\ell}^2(\mathbb{Z})$-norm) of a function (vector). For operators (matrices) $\|T\|$ is the standard operator (matrix) norm. The condition number of an invertible operator T is defined by $\kappa(A) = \|A\|\|A^{-1}\|$ and the spectrum of T is $\sigma(T)$. I denotes the identity operator.

1.1 Nonuniform sampling, frames, and numerical algorithms

The concept of frames is an excellent tool to study nonuniform sampling problems [13, 2, 1, 24, 15, 44]. The frame approach has the advantage that it gives rise to deep theoretical results and also to the construction of efficient numerical algorithms – *if* (and this point is often ignored in the literature) the finite-dimensional model is properly designed.

Following Duffin and Schaeffer [11], a family $\{f_j\}_{j\in\mathbb{Z}}$ in a separable Hilbert space $\boldsymbol{H}$ is said to be a frame for $\boldsymbol{H}$, if there exist constants (the *frame bounds*) $A, B > 0$ such that

$$A\|f\|^2 \le \sum_j |\langle f, f_j\rangle|^2 \le B\|f\|^2\,, \qquad \forall f \in \boldsymbol{H}. \tag{3}$$

We define the *analysis operator* T by

$$T : f \in \boldsymbol{H} \to Ff = \{\langle f, f_j\rangle\}_{j\in\mathbb{Z}}\,, \tag{4}$$

and the *synthesis operator*, which is just the adjoint operator of T, by

$$T^* : c \in \boldsymbol{\ell}^2(\mathbb{Z}) \to T^*c = \sum_j c_j f_j\,. \tag{5}$$

The *frame operator* S is defined by $S = T^*T$, hence $Sf = \sum_j \langle f, f_j\rangle f_j$. S is bounded by $AI \le S \le BI$ and hence invertible on $\boldsymbol{H}$.

We will also make use of the operator TT^* in form of its Gram matrix representation $R : \boldsymbol{\ell}^2(\mathbb{Z}) \to \boldsymbol{\ell}^2(\mathbb{Z})$ with entries $R_{j,l} = \langle f_j, f_l\rangle$. On $\mathcal{R}(T) = \mathcal{R}(R)$ the matrix R is bounded by $AI \le R \le BI$ and invertible. On $\boldsymbol{\ell}^2(\mathbb{Z})$ this inverse extends to the *Moore-Penrose inverse* or pseudo-inverse R^+ (cf. [12]).

Given a frame $\{f_j\}_{j\in\mathbb{Z}}$ for $\boldsymbol{H}$, any $f \in \boldsymbol{H}$ can be expressed as

$$f = \sum_{j\in\mathbb{Z}} \langle f, f_j\rangle \gamma_j = \sum_{j\in\mathbb{Z}} \langle f, \gamma_j\rangle f_j\,, \tag{6}$$

where the elements $\gamma_j := S^{-1} f_j$ form the so-called dual frame and the frame operator induced by γ_j coincides with S^{-1}. Hence if a set $\{f_j\}_{j\in\mathbb{Z}}$ establishes a frame for $\boldsymbol{H}$, we can reconstruct any function $f \in \boldsymbol{H}$ from its moments $\langle f, f_j\rangle$.

One possibility to connect sampling theory to frame theory is by means of the *sinc*-function

$$\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}\,. \tag{7}$$

Its translates give rise to a *reproducing kernel* for $\boldsymbol{B}$ via

$$f(t) = \langle f, \mathrm{sinc}(\cdot - t)\rangle \quad \forall t, f \in \boldsymbol{B}\,. \tag{8}$$

Combining (8) with formulas (3) and (6) we obtain following well-known result [13, 2].

Theorem 1.1 *If the set $\{sinc(\cdot - t_j)\}_{j\in\mathbb{Z}}$ is a frame for $\boldsymbol{B}$, then the function $f \in \boldsymbol{B}$ is uniquely defined by the sampling set $\{f(t_j)\}_{j\in\mathbb{Z}}$. In this case we can recover f from its samples by*

$$f(t) = \sum_{j\in\mathbb{Z}} f(t_j)\gamma_j\,, \qquad \textit{where } \gamma_j = S^{-1} sinc(\cdot - t_j)\,, \tag{9}$$

or equivalently by

$$f(t) = \sum_{j\in\mathbb{Z}} c_j sinc(t - t_j)\,, \qquad \textit{where } Rc = b\,, \tag{10}$$

with R being the frame Gram matrix with entries $R_{j,l} = sinc(t_j - t_l)$ and $b = \{b_j\} = \{f(t_j)\}$.

The challenge is now to find easy-to-verify conditions for the sampling points t_j such that $\{\mathrm{sinc}(\cdot - t_j)\}_{j\in\mathbb{Z}}$ (or equivalently the exponential system $\{e^{2\pi i t_j \omega}\}_{j\in\mathbb{Z}}$) is a frame for $\boldsymbol{B}$. This is a well-traversed area (at least for one-dimensional signals), and the reader should consult [1, 15, 24] for further details and references. If not otherwise mentioned from now on we will assume that $\{\mathrm{sinc}(\cdot - t_j)\}_{j\in\mathbb{Z}}$ is a frame for $\boldsymbol{B}$.

Of course, neither of the formulas (9) and (10) can be actually implemented on a computer, because both involve the solution of an infinite-dimensional operator equation, whereas in practice we can only compute a finite-dimensional approximation. Although the design of a valid finite-dimensional model poses severe mathematical challenges, this step is often neglected in theoretical but also in numerical treatments of the nonuniform sampling problem. We will see in the sequel that the way we design our finite-dimensional model is crucial for the stability and efficiency of the resulting numerical reconstruction algorithms.

In the next two sections we describe two different approaches for obtaining finite-dimensional approximations to the formulas (9) and (10). The first and more traditional approach, discussed in Section 2, applies a finite section method to equation (10). This approach leads to an ill-posed problem involving the solution of a large unstructured linear system of equations. The second approach, outlined in Section 3, constructs a finite model for the operator equation in (9) by means of trigonometric polynomials. This technique leads to a well-posed problem that is tied to efficient numerical algorithms.

2 Truncated frames lead to ill-posed problems

According to equation (10) we can reconstruct f from its sampling values $f(t_j)$ via $f(t) = \sum_{j\in\mathbb{Z}} c_j \,\mathrm{sinc}(t - t_j)$, where $c = R^{+}b$ with $b_j = f(t_j), j \in \mathbb{Z}$. In order to compute a finite-dimensional approximation to $c = \{c_j\}_{j\in\mathbb{Z}}$ we use the finite section method [17]. For $x \in \boldsymbol{\ell}^2(\mathbb{Z})$ and $n \in \mathbb{N}$ we define the orthogonal projection P_n by

$$P_n x = (\dots, 0, 0, x_{-n}, x_{-n+1}, \dots, x_{n-1}, x_n, 0, 0, \dots) \tag{11}$$

and identify the image of P_n with the space $\mathbb{C}^{2n+1}$. Setting $R_n = P_n R P_n$ and $b^{(n)} = P_n b$, we obtain the n-th approximation $c^{(n)}$ to c by solving

$$R_n c^{(n)} = b^{(n)} . \tag{12}$$

It is clear that using the truncated frame $\{\mathrm{sinc}(\cdot - t_j)\}_{j=-n}^{n}$ in (10) for an approximate reconstruction of f leads to the same system of equations.

If $\{\mathrm{sinc}(\cdot - t_j)\}_{j\in\mathbb{Z}}$ is an exact frame (i.e., a Riesz basis) for $\boldsymbol{B}$ then we have following well-known result.

Lemma 2.1 *Let $\{sinc(\cdot - t_j)\}_{j\in\mathbb{Z}}$ be an exact frame for $\boldsymbol{B}$ with frame bounds A, B and $Rc = b$ and $R_n c^{(n)} = b^{(n)}$ as defined above. Then R_n^{-1} converges strongly to R^{-1} and hence $c^{(n)} \to c$ for $n \to \infty$.*

Since the proof of this result given in [9] is somewhat lengthy we include a rather short proof here.

Proof: Note that R is invertible on $\boldsymbol{\ell}^2(\mathbb{Z})$ and $A \le R \le B$. Let $x \in \mathbb{C}^{2n+1}$ with $\|x\| = 1$, then $\langle R_n x, x\rangle = \langle P_n R P_n x, x\rangle = \langle Rx, x\rangle \ge A$. In the same way we get $\|R_n\| \le B$, hence the matrices R_n are invertible and uniformly bounded by $A \le R_n \le B$ and

$$\frac{1}{B} \le R_n^{-1} \le \frac{1}{A} \qquad \text{for all } n \in \mathbb{N}.$$

The Lemma of Kantorovich [32] yields that $R_n^{-1} \to R^{-1}$ strongly. ❏

If $\{\mathrm{sinc}(\cdot - t_j)\}_{j\in\mathbb{Z}}$ is a non-exact frame for $\boldsymbol{B}$ the situation is more delicate. Let us consider following situation.

Example 1: Let $f \in \boldsymbol{B}$ and let the sampling points be given by $t_j = \frac{j}{m}, j \in \mathbb{Z}, 1 < m \in \mathbb{N}$, i.e., the signal is regularly oversampled at m times the Nyquist rate. In this case the reconstruction of f is trivial, since the set $\{\mathrm{sinc}(\cdot - t_j)\}_{j\in\mathbb{Z}}$ is a tight frame with frame bounds $A = B = m$. Shannon's Sampling Theorem implies that f can be expressed as $f(t) = \sum_{j\in\mathbb{Z}} c_j \,\mathrm{sinc}(t - t_j)$ where $c_j = \frac{f(t_j)}{m}$ and the numerical approximation is obtained by truncating the summation, i.e.,

$$f_n(t) = \sum_{j=-n}^{n} \frac{f(t_j)}{m} \,\mathrm{sinc}(t - t_j) .$$

Using the truncated frame approach one finds that R is a Toeplitz matrix with entries

$$R_{j,l} = \frac{\sin \frac{\pi}{m}(j-l)}{\frac{\pi}{m}(j-l)}, \qquad j,l \in \mathbb{Z},$$

in other words, R_n coincides with the prolate matrix [36, 39]. The unpleasant numerical properties of the prolate matrix are well-documented. In particular we know that the singular values λ_n of R_n cluster around 0 and 1 with $\log n$ singular values in the transition region. Since the singular values of R_n decay exponentially to zero the finite-dimensional reconstruction problem has become *severely ill-posed* [12], although the infinite-dimensional problem is "perfectly posed" since the frame operator satisfies $S = mI$, where I is the identity operator.

Of course the situation does not improve when we consider non-uniformly spaced samples. In this case it follows from standard linear algebra that $\sigma(R) \subseteq \{0 \cup [A, B]\}$, or expressed in words, the singular values of R are bounded away from zero. However for the truncated matrices R_n we have

$$\sigma(R_n) \subseteq \{(0, B]\}$$

and the smallest of the singular values of R_n will go to zero for $n \to \infty$, see [23].

Let $A = U\Sigma V^*$ be the singular value decomposition of a matrix A with $\Sigma = \mathrm{diag}(\{\lambda_k\})$. Then the Moore-Penrose inverse of A is $A^+ = V\Sigma^+U^*$, where (e.g., see [18])

$$\Sigma^+ = \mathrm{diag}(\{\lambda_k^+\}), \quad \lambda_k^+ = \begin{cases} 1/\lambda_k & \text{if } \lambda_k \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

For $R_n = U_n\Sigma_n V_n$ this means that the singular values close to zero will give rise to extremely large coefficients in R_n^+. In fact $\|R_n^+\| \to \infty$ for $n \to \infty$ and consequently $c^{(n)}$ does not converge to c.

Practically $\|R_n^+\|$ is always bounded due to finite precision arithmetics, but it is clear that it will lead to meaningless results for large n. If the sampling values are perturbed due to round-off error or data error, then those error components which correspond to small singular values λ_k are amplified by the (then large) factors $1/\lambda_k$. Although for a given R_n these amplifications are theoretically bounded, they may be practically unacceptable large.

Such phenomena are well-known in regularization theory [12]. A standard technique to compute a stable solution for an ill-conditioned system is to use a truncated singular value decomposition (TSVD) [12]. This means in our case we compute a regularized pseudo-inverse $R_n^{+,\tau} = V_n \Sigma_n^{+,\tau} U_n^*$ where

$$\Sigma^{+,\tau} = \mathrm{diag}(\{d_k^+\}), \quad d_k^+ = \begin{cases} 1/\lambda_k & \text{if } \lambda_k \geq \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

In [23] it is shown that for each n we can choose an appropriate truncation level τ such that the regularized inverses $R_n^{+,\tau}$ converge strongly to R^+ for $n \to \infty$ and consequently $\lim\limits_{n\to\infty} \|f - f^{(n)}\| = 0$, where

$$f^{(n)}(t) = \sum_{j=-n}^{n} c_j^{(n,\tau)} \mathrm{sinc}(t - t_j)$$

with

$$c^{(n,\tau)} = R_n^{+,\tau} b^{(n)} .$$

The optimal truncation level τ depends on the dimension n, the sampling geometry, and the noise level. Thus it is not known a priori and has in principle to be determined for each n independently.

Since τ is of vital importance for the quality of the reconstruction, but no theoretical explanations for the choice of τ are given in the sampling literature, we briefly discuss this issue. For this purpose we need some results from regularization theory.

2.1 Estimation of regularization parameter

Let $Ax = y^\delta$ be given where A is ill-conditioned or singular and y^δ is a perturbed right-hand side with $\|y - y^\delta\| \leq \delta\|y\|$. Since in our sampling problem the matrix under consideration is symmetric, we assume for convenience that A is symmetric. From a numerical point of view ill-conditioned systems behave like singular systems and additional information is needed to obtain a satisfactory solution to $Ax = y$. This information is usually stated in terms of "smoothness" of the solution x. A standard approach to qualitatively describe smoothness of x is to require that x can be represented in the form $x = Sz$ with some vector z of reasonable norm, and a "smoothing" matrix S, cf. [12, 29]. Often it is useful to construct S directly from A by setting

$$S = A^p , \qquad p \in \mathbb{N}_0 . \tag{15}$$

Usually, p is assumed to be fixed, typically at $p = 1$ or $p = 2$.

We compute a regularized solution to $Ax = y^\delta$ via a truncated SVD and want to determine the optimal regularization parameter (i.e., truncation level) τ.

Under the assumption that

$$x = Sz\,, \quad \|Ax - y^\delta\| \leq \Delta \|z\| \tag{16}$$

it follows from Theorem 4.1 in [29] that the optimal regularization parameter τ for the TSVD is

$$\hat{\tau} = \left(\frac{\gamma_1 \delta}{\gamma_2 p}\right)^{\frac{1}{p+1}}, \tag{17}$$

where $\gamma_1 = \gamma_2 = 1$ (see Section 6 in [29]).

However z and Δ are in general not known. Using $\|Ax - y^\delta\| \leq \delta\|y\|$ and $\|y\| = \|Ax\| = \|ASz\| = \|A^{p+1}z\|$ we obtain $\|y\| \leq \|A\|^{p+1}\|z\|$. Furthermore, setting $\delta\|y\| = \Delta\|z\|$ implies

$$\Delta \leq \delta \|A\|^{p+1}\,. \tag{18}$$

Hence combining (17) and (18) we get

$$\hat{\tau} \leq \left(\frac{\delta \|A\|^{p+1}}{p}\right)^{\frac{1}{p+1}} = \|A\| \left(\frac{\delta}{p}\right)^{\frac{1}{p+1}}. \tag{19}$$

Applying these results to solving $R_n c^{(n)} = b^{(n)}$ via TSVD as described in the previous section, we get

$$\hat{\tau} \leq \|R_n\| \left(\frac{\delta}{p}\right)^{\frac{1}{p+1}} \leq \|R\| \left(\frac{\delta}{p}\right)^{\frac{1}{p+1}} = B \left(\frac{\delta}{p}\right)^{\frac{1}{p+1}}, \tag{20}$$

where B is the upper frame bound. Fortunately estimates for the upper frame bound are much easier to obtain than estimates for the lower frame bound.

Thus using the standard setting $p = 1$ or $p = 2$ a good choice for the regularization parameter τ is

$$\tau \subseteq [B(\delta/2)^{1/3}, B(\delta)^{1/2}]\,. \tag{21}$$

Extensive numerical simulations confirm this choice, see also Section 5.

For instance for the reconstruction problem of Example 1 with noise-free data and machine precision $\varepsilon = \delta = 10^{-16}$, formula (21) implies $\tau \subseteq [10^{-6}, 10^{-8}]$. This coincides very well with numerical experiments.

If the noise level δ is not known, it has to be estimated. This difficult problem will not be discussed here. The reader is referred to [29] for more details.

Although we have arrived now at an implementable algorithm for the nonuniform sampling problem, the disadvantages of the approach described in the previous section are obvious. In general the matrix R_n does not have any particular structure, thus the computational costs for the singular value decomposition are $\mathcal{O}(n^3)$ which is prohibitive large in many applications. It is definitely not a good approach to transform a well-posed infinite-dimensional problem into an ill-posed finite-dimensional problem for which a stable solution can only be computed by using a "heavy regularization machinery".

The methods in [42, 41, 40, 33, 2] coincide with or are essentially equivalent to the truncated frame approach, therefore they suffer from the same instability problems and the same numerical inefficiency.

2.2 CG and regularization of the truncated frame method

As mentioned above one way to stabilize the solution of $R_n c^{(n)} = b^{(n)}$ is a truncated singular value decomposition, where the truncation level serves as regularization parameter. For large n the costs of the singular value decomposition become prohibitive for practical purposes.

We propose the conjugate gradient method [18] to solve $R_n c^{(n)} = b^{(n)}$. It is in general much more efficient than a TSVD (or Tikhonov regularization as suggested in [40]), and at the same time it can also be used as a regularization method.

The standard error analysis for CG cannot be used in our case, since the matrix is ill-conditioned. Rather we have to resort to the error analysis developed in [28, 22].

When solving a linear system $Ax = y$ by CG for noisy data y^δ following happens. The iterates x_k of CG may diverge for $k \to \infty$, however the error propagation remains limited in the beginning of the iteration. The quality of the approximation therefore depends on how many iterative steps can be performed until the iterates turn to diverge. The idea is now to stop the iteration at about the point where divergence sets in. In other words the iterations count is the regularization parameter which remains to be controlled by an appropriate stopping rule [27, 22].

In our case assume $\|b^{(n,\delta)} - b^{(n)}\| \le \delta \|b^{(n)}\|$, where $b_j^{(n,\delta)}$ denotes a noisy sample. We terminate the CG iterations when the iterates $(c^{(n,\delta)})_k$ satisfy for the first time [22]

$$\|b^{(n)} - (c^{(n,\delta)})_k\| \leq \tau\delta\|b^{(n)}\| \tag{22}$$

for some fixed $\tau > 1$.

It should be noted that one can construct "academic" examples where this stopping rule does not prevent CG from diverging, see [22], "most of the time" however it gives satisfactory results. We refer the reader to [27, 22] for a detailed discussion of various stopping criteria.

There is a variety of reasons, besides the ones we have already mentioned, that make the conjugate gradient method and the nonuniform sampling problem a "perfect couple". See Sections 3, 4.1, and 4.2 for more details.

By combining the truncated frame approach with the conjugate gradient method (with appropriate stopping rule) we finally arrive at a reconstruction method that is of some practical relevance. However the only existing method at the moment that can handle large scale reconstruction problems seems to be the one proposed in the next section.

3 Trigonometric polynomials and efficient signal reconstruction

In the previous section we have seen that the naive finite-dimensional approach via truncated frames is not satisfactory, it already leads to severe stability problems in the ideal case of regular oversampling. In this section we propose a different finite-dimensional model, which resembles much better the structural properties of the sampling problem, as can be seen below.

The idea is simple. In practice only a finite number of samples $\{f(t_j)\}_{j=1}^r$ is given, where without loss of generality we assume $-M \leq t_1 < \cdots < t_r \leq M$ (otherwise we can always re-normalize the data). Since no data of f are available from outside this region we focus on a local approximation of f on $[-M, M]$. We extend the sampling set periodically across the boundaries, and identify this interval with the (properly normalized) torus $\mathbb{T}$. To avoid technical problems at the boundaries in the sequel we will choose the interval somewhat larger and consider either $[-M-1/2, M+1/2]$ or $[-N, N]$ with $N = M + \frac{M}{r-1}$. For theoretical considerations the choice $[-M-1/2, M+1/2]$ is more convenient.

Since the dual group of the torus $\mathbb{T}$ is $\mathbb{Z}$, periodic band-limited functions on $\mathbb{T}$ reduce to trigonometric polynomials (of course technically f does then no longer belong to $\boldsymbol{B}$ since it is no longer in $\boldsymbol{L}^2(\mathbb{R})$). This suggests to use trigonometric polynomials as a realistic finite-dimensional model for a numerical solution of the nonuniform sampling problem. We consider the space $\boldsymbol{P}_M$ of trigonometric polynomials of degree M of the form

$$p(t) = (2M+1)^{-1} \sum_{k=-M}^{M} a_k e^{2\pi i k t/(2M+1)} . \tag{23}$$

The norm of $p \in \boldsymbol{P}_M$ is

$$\|p\|^2 = \int_{-N}^{N} |p(t)|^2 \, dt = \sum_{k=-M}^{M} |a_k|^2 .$$

Since the distributional Fourier transform of p is $\hat{p} = (2M+1)^{-1} \sum_{k=-M}^{M} a_k \delta_{k/(2M+1)}$ we have $\operatorname{supp} \hat{p} \subseteq \{k/(2M+1), |k| \le M\} \subseteq [-1/2, 1/2]$. Hence $\boldsymbol{P}_M$ is indeed a natural finite-dimensional model for $\boldsymbol{B}$.

In general the $f(t_j)$ are not the samples of a trigonometric polynomial in $\boldsymbol{P}_M$, moreover the samples are usually perturbed by noise, hence we may not find a $p \in \boldsymbol{P}_M$ such that $p(t_j) = b_j = f(t_j)$. We therefore consider the least squares problem

$$\min_{p \in \boldsymbol{P}_M} \sum_{j=1}^{r} |p(t_j) - b_j|^2 w_j . \tag{24}$$

Here the $w_j > 0$ are user-defined weights, which can be chosen for instance to compensate for irregularities in the sampling geometry [14].

By increasing M so that $r \le 2M+1$ we can certainly find a trigonometric polynomial that interpolates the given data exactly. However in the presence of noise, such a solution is usually rough and highly oscillating and may poorly resemble the original signal. We will discuss the question of the optimal choice of M if the original bandwidth is not known and in presence of noisy data in Section 4.2.

The following theorem provides an efficient numerical reconstruction algorithm. It is also the key for the analysis of the relation between the finite-dimensional approximation in $\boldsymbol{P}_M$ and the solution of the original infinite-dimensional sampling problem in $\boldsymbol{B}$.

Theorem 3.1 (and Algorithm) *[19, 14] Given the sampling points $-M \le t_1 < \ldots, t_r \le M$, samples $\{b_j\}_{j=1}^r$, positive weights $\{w_j\}_{j=1}^r$ with $2M+1 \le r$.*
Step 1: Compute the $(2M+1) \times (2M+1)$ Toeplitz matrix T_M with entries

$$(T_M)_{k,l} = \frac{1}{2M+1} \sum_{j=1}^{r} w_j e^{-2\pi i (k-l) t_j/(2M+1)} \qquad \textit{for } |k|, |l| \le M \tag{25}$$

and $y_M \in \mathbb{C}^{(2M+1)}$ *by*

$$(y_M)_k = \frac{1}{\sqrt{2M+1}} \sum_{j=1}^{r} b_j w_j e^{-2\pi i k t_j/(2M+1)} \qquad \textit{for } |k| \le M\,. \tag{26}$$

Step 2: Solve the system

$$T_M a_M = y_M\,. \tag{27}$$

Step 3: Then the polynomial $p_M \in \boldsymbol{P}_M$ *that solves* (24) *is given by*

$$p_M(t) = \frac{1}{\sqrt{2M+1}} \sum_{k=-M}^{M} (a_M)_k e^{2\pi i k t/(2M+1)}\,. \tag{28}$$

Numerical Implementation of Theorem/Algorithm 3.1:
Step 1: The entries of T_M and y_M of equations (25) and (26) can be computed in $\mathcal{O}(M \log M + r \log(1/\varepsilon))$ operations (where ε is the required accuracy) using Beylkin's unequally spaced FFT algorithm [4].
Step 2: We solve $T_M a_M = y_M$ by the conjugate gradient (CG) algorithm [18]. The matrix-vector multiplication in each iteration of CG can be carried out in $\mathcal{O}(M \log M)$ operations via FFT [8]. Thus the solution of (27) takes $\mathcal{O}(kM \log M)$ operations, where k is the number of iterations.
Step 3: Usually the signal is reconstructed on regularly space nodes $\{u_i\}_{i=1}^N$. In this case $p_M(u_i)$ in (28) can be computed by FFT. For non-uniformly spaced nodes u_i we can again resort to Beylkin's USFFT algorithm.

There exists a large number of fast algorithms for the solution of Toeplitz systems. Probably the most efficient algorithm in our case is CG. We have already mentioned that the Toeplitz system (27) can be solved in $\mathcal{O}(kM \log M)$ via CG. The number of iterations k depends essentially on the clustering of the eigenvalues of T_M, cf. [8]. It follows from equation (31) below and perturbation theory [10] that, if the sampling points stem from a perturbed regular sampling set, the eigenvalues of T_M will be clustered around β, where β is the oversampling rate. In such cases we can expect a very fast rate of convergence. The simple frame iteration [26, 1] is not able to take advantage of such a situation.

For the analysis of the relation between the solution p_M of Theorem 3.1 and the solution f of the original infinite-dimensional problem we follow Gröchenig [20]. Assume that the samples $\{f(t_j)\}_{j\in\mathbb{Z}}$ of $f \in \boldsymbol{B}$ are given. For the finite-dimensional approximation we consider only those samples $f(t_j)$ for which t_j is contained in the interval $[-M - \frac{1}{2}, M + \frac{1}{2}]$

and compute the least squares approximation p_M with degree M and period $2M+1$ as in Theorem 3.1. It is shown in [20] that if $\sigma(T_M) \subseteq [\alpha, \beta]$ for all M with $\alpha > 0$ then

$$\lim_{M\to\infty} \int_{[-M,M]} |f(t) - p_M(t)|^2 \, dt = 0, \tag{29}$$

and also $\lim p_M(t) = f(t)$ uniformly on compact sets.

Under the Nyquist condition $\sup(t_{j+1} - tj) := \gamma < 1$ and using weights $w_j = (t_{j+1} - t_{j-1})/2$ Gröchenig has shown that

$$\sigma(T_M) \subseteq \left[(1-\gamma)^2, 6\right], \tag{30}$$

independently of M, see [20]. These results validate the usage of trigonometric polynomials as finite-dimensional model for nonuniform sampling.

Example 1 – reconsidered: Recall that in Example 1 of Section 2 we have considered the reconstruction of a regularly oversampled signal $f \in \boldsymbol{B}$. What does the reconstruction method of Theorem 3.1 yield in this case? Let us check the entries of the matrix T_M when we take only those samples in the interval $[-n, n]$. The period of the polynomial becomes $2N$ with $N = n + \frac{n}{r-1}$ where r is the number of given samples. Then

$$(T_M)_{k,l} = \frac{1}{2N} \sum_{j=1}^{r} e^{2\pi i (k-l) t_j/(2N)} = \sum_{j=-nm}^{nm} e^{2\pi i (k-l) \frac{j}{2nm+1}} = m\delta_{k,l} \tag{31}$$

for $k, l = -M, \dots, M$, where $\delta_{k,l}$ is Kronecker's symbol with the usual meaning $\delta_{k,l} = 1$ if $k = l$ and 0 else. Hence we get

$$T_M = mI\,,$$

where I is the identity matrix on $\mathbb{C}^{2M+1}$, thus T_M resembles the structure of the infinite-dimensional frame operator S in this case (including exact approximation of the frame bounds). Recall that the truncated frame approach leads to an "artificial" ill-posed problem even in such a simple situation.

The advantages of the trigonometric polynomial approach compared to the truncated frame approach are manifold. In the one case we have to deal with an ill-posed problem which has no specific structure, hence its solution is numerically very expensive. In the other case we have to solve a problem with rich mathematical structure, whose stability depends only on the sampling density, a situation that resembles the original infinite-dimensional sampling problem.

In principle the coefficients $a_M = \{(a_M)_k\}_{k=-M}^{M}$ of the polynomial p_M that minimizes (24) could also be computed by directly solving the Vandermonde type system

$$W V a_M = W b\,, \tag{32}$$

where $V_{j,k} = \frac{1}{\sqrt{2M+1}} e^{-2\pi i k t_j/(2M+1)}$ for $j = 1, \ldots, r$, $k = -M, \ldots, M$ and W is a diagonal matrix with entries $W_{j,j} = \sqrt{w_j}$, cf. [31]. Several algorithms are known for a relatively efficient solution of Vandermonde systems [5, 31]. However this is one of the rare cases, where, instead of directly solving (32), it is advisable to explicitly establish the system of normal equations

$$T_M a_M = y_M\,, \tag{33}$$

where $T = V^* W^2 V$ and $y = V^* W^2 b$.

The advantages of considering the system $T_M a_M = y_M$ instead of the Vandermonde system (32) are manifold:

- The matrix T_M plays a key role in the analysis of the relation of the solution of (24) and the solution of the infinite-dimensional sampling problem (9), see (29) and (30) above.
- T_M is of size $(2M+1) \times (2M+1)$, independently of the number of sampling points. Moreover, since $(T_M)_{k,l} = \sum_{j=1}^{r} w_j e^{2\pi i (k-l) t_j}$, it is of Toeplitz type. These facts give rise to fast and robust reconstruction algorithms.
- The resulting reconstruction algorithms can be easily generalized to higher dimensions, see Section 3.1. Such a generalization to higher dimensions seems not to be straightforward for fast solvers of Vandermonde systems such as the algorithm proposed in [31].

We point out that other finite-dimensional approaches are proposed in [16, 7]. These approaches may provide interesting alternatives in the few cases where the algorithm outlined in Section 3 does not lead to good results. These cases occur when only a few samples of the signal f are given in an interval $[a, b]$ say, and at the same time we have $|f(a) - f(b)| \gg 0$ and $|f'(a) - f'(b)| \gg 0$, i.e., if f is "strongly non-periodic" on $[a, b]$. However the computational complexity of the methods in [16, 7] is significantly larger.

3.1 Multi-dimensional nonuniform sampling

The approach presented above can be easily generalized to higher dimensions by a diligent book-keeping of the notation. We consider the space of d-dimensional trigonometric polynomials $\boldsymbol{P}_M^d$ as finite-dimensional model for $\boldsymbol{B}^d$. For given samples $f(t_j)$ of $f \in \boldsymbol{B}^d$, where $t_j \in \mathbb{R}^d$, we compute the least squares approximation p_M similar to Theorem 3.1 by solving the corresponding system of equations $T_M a_M = y_M$.

In 2-D for instance the matrix T_M becomes a block Toeplitz matrix with Toeplitz blocks [37]. For a fast computation of the entries of T we can again make use of Beylkin's USFFT algorithm [4]. And similar to 1-D, multiplication of a vector by T_M can be carried out by 2-D FFT.

Also the relation between the finite-dimensional approximation in $\boldsymbol{P}_M^d$ and the infinite-dimensional solution in $\boldsymbol{B}^d$ is similar as in 1-D. The only mathematical difficulty is to give conditions under which the matrix T_M is invertible. Since the fundamental theorem of algebra does not hold in dimensions larger than one, the condition $(2M+1)^d \leq r$ is necessary but no longer sufficient for the invertibility of T_M. Sufficient conditions for the invertibility, depending on the sampling density, are presented in [21].

4 Bandwidth estimation and regularization

In this section we discuss several numerical aspects of nonuniform sampling that are very important from a practical viewpoint, however only few answers to these problems can be found in the literature.

4.1 A multilevel signal reconstruction algorithm

In almost all theoretical results and numerical algorithms for reconstructing a band-limited signal from nonuniform samples it is assumed that the bandwidth is known a priori. This information however is often not available in practice.

A good choice of the bandwidth for the reconstruction algorithm becomes crucial in case of noisy data. It is intuitively clear that choosing a too large bandwidth leads to over-fit of the noise in the data, while a too small bandwidth yields a smooth solution but also to under-fit of the data. And of course we want to avoid the determination of the "correct" Ω by trial-and-error methods. Hence the problem is to design a method that can reconstruct a signal from non-uniformly spaced, noisy samples without requiring a priori information about the bandwidth of the signal.

The multilevel approach derived in [34] provides an answer to this problem. The approach applies to an infinite-dimensional as well as to a finite-dimensional setting. We describe the method directly for the trigonometric polynomial model, where the determination of the bandwidth Ω translates into the determination of the polynomial degree M of the reconstruction. The idea of the multilevel algorithm is as follows.

Let the noisy samples $\{b_j^\delta\}_{j=1}^r = \{f^\delta(t_j)\}_{j=1}^r$ of $f \in \boldsymbol{B}$ be given with $\sum_{j=1}^r |f(t_j) - b^\delta(t_j)|^2 \le \delta^2 \|b^\delta\|^2$ and let Q_M denote the orthogonal projection from $\boldsymbol{B}$ into $\boldsymbol{P}_M$. We start with initial degree $M = 1$ and run Algorithm 3.1 until the iterates $p_{0,k}$ satisfy for the first time the *inner* stopping criterion

$$\sum_{j=1}^r |p_{1,k}(t_j) - b_j^\delta|^2 \le 2\tau(\delta\|b^\delta\| + \|Q_0 f - f\|)\|b^\delta\|$$

for some fixed $\tau > 1$. Denote this approximation (at iteration k_*) by p_{1,k_*}. If p_{1,k_*} satisfies the *outer* stopping criterion

$$\sum_{j=1}^r |p_{1,k}(t_j) - b_j^\delta|^2 \le 2\tau\delta\|b^\delta\|^2 \tag{34}$$

we take p_{1,k_*} as final approximation. Otherwise we proceed to the next level $M = 2$ and run Algorithm 3.1 again, using p_{1,k_*} as initial approximation by setting $p_{2,0} = p_{1,k_*}$.

At level $M = N$ the inner level-dependent stopping criterion becomes

$$\sum_{j=1}^r |p_{N,k}(t_j) - b_j^\delta|^2 \le 2\tau(\delta\|b^\delta\| + \|Q_N f - f\|)\|b^\delta\|, \tag{35}$$

while the outer stopping criterion does not change since it is level-independent.

Stopping rule (35) guarantees that the iterates of CG do not diverge. It also ensures that CG does not iterate too long at a certain level, since if M is too small further iterations at this level will not lead to a significant improvement. Therefore we switch to the next level. The outer stopping criterion (34) controls over-fit and under-fit of the data, since in presence of noisy data is does not make sense to ask for a solution p_M that satisfies $\sum_{j=1}^r |p_M(t_j) - b_j^\delta|^2 = 0$.

Since the original signal f is not known, the expression $\|f - Q_N f\|$ in (35) cannot be computed. In [34] the reader can find an approach to estimate $\|f - Q_N f\|$ recursively.

4.2 Solution of ill-conditioned sampling problems

A variety of conditions on the sampling points $\{t_j\}_{j\in\mathbb{Z}}$ are known under which the set $\{\mathrm{sinc}(\cdot - t_j)\}_{j\in\mathbb{Z}}$ is a frame for $\boldsymbol{B}$, which in turn implies (at least theoretically) perfect reconstruction of a signal f from its samples $f(t_j)$. This does however not guarantee a stable reconstruction from a numerical viewpoint, since the ratio of the frame bounds

B/A can still be extremely large and therefore the frame operator S can be ill-conditioned. This may happen for instance if γ in (30) goes to 1, in which case cond(T) may become large. The sampling problem may also become numerically unstable or even ill-posed, if the sampling set has large gaps, which is very common in astronomy and geophysics. Note that in this case the instability of the system $T_M a_M = y_M$ does *not* result from an inadequate discretization of the infinite-dimensional problem.

There exists a large number of (circulant) Toeplitz preconditioners that could be applied to the system $T_M a_M = y_M$, however it turns out that they do not improve the stability of the problem in this case. The reason lies in the distribution of the eigenvalues of T_M, as we will see below.

Following [38], we call two sequences of real numbers $\{\lambda^{(n)}\}_{k=1}^{n}$ and $\{\nu^{(n)}\}_{k=1}^{n}$ *equally distributed*, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [F(\lambda_k^{(n)}) - F(\nu_k^{(n)})] = 0 \tag{36}$$

for any continuous function F with compact support[1].

Let C be a $(n \times n)$ circulant matrix with first column $(c_0, \dots, c_{n-1})$, we write $C = \mathrm{circ}(c_0, \dots, c_{n-1})$. The eigenvalues of C are distributed as $\lambda_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} c_l e^{2\pi i k l/n}$. Observe that the Toeplitz matrix A_n with first column $(a_0, a_1, \dots, a_n)$ can be embedded in the circulant matrix

$$C_n = \mathrm{circ}(a_0, a_1, \dots, a_n, \bar{a_n}, \dots, \bar{a_1}) \,. \tag{37}$$

Thms 4.1 and 4.2 in [38] state that the eigenvalues of A_n and C_n are equally distributed as $f(x)$ where

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k x} \,. \tag{38}$$

The partial sum of the series (38) is

$$f_n(x) = \sum_{k=-n}^{n} a_k e^{2\pi i k x} \,. \tag{39}$$

To understand the clustering behavior of the eigenvalues of T_M in case of sampling sets with large gaps, we consider a sampling set in $[-M, M)$, that consists of one large

[1] In H.Weyl's definition $\lambda_k^{(n)}$ and $\nu_k^{(n)}$ are required to belong to a common interval.

block of samples and one large gap, i.e., $t_j = \frac{j}{Lm}$ for $j = -mM, \dots mM$ for $m, L \in \mathbb{N}$. (Recall that we identify the interval with the torus). Then the entries z_k of the Toeplitz matrix T_M of (25) (with $w_j = 1$) are

$$z_k = \frac{1}{2M+1} \sum_{j=-mM}^{mM} e^{-2\pi i k \frac{j}{Lm}/(2M+1)}, \quad k = 0, \dots, 2M\,.$$

To investigate the clustering behavior of the eigenvalues of T_M for $M \to \infty$, we embed T_M in a circulant matrix C_M as in (37). Then (39) becomes

$$f_{mM}(x) = \frac{1}{Lm(2M+1)} \sum_{l=-mM}^{mM} \sum_{j=-mM}^{mM} e^{2\pi i l[k/(4M+1) - j/((2M+1)mL)]} \tag{40}$$

whence $f_{mM} \to \mathbf{1}_{[-1/(2L),1/(2L)]}$ for $M \to \infty$, where $\mathbf{1}_{[-a,a]}(x) = 1$, if $-a < x < a$ and 0 else.

Thus the eigenvalues of T_M are asymptotically clustered around zero and one. For general nonuniform sampling sets with large gaps the clustering at 1 will disappear, but of course the spectral cluster at 0 will remain. In this case it is known that the preconditioned problem will still have a spectral cluster at the origin [43] and preconditioning will not be efficient.

Fortunately there are other possibilities to obtain a stabilized solution of $T_M a_M = y_M$. The condition number of T_M essentially depends on the ratio of the maximal gap in the sampling set to the Nyquist rate, which in turn depends on the bandwidth of the signal. We can improve the stability of the system by adapting the degree M of the approximation accordingly. Thus the parameter M serves as a regularization parameter that balances stability and accuracy of the solution. This technique can be seen as a specific realization of *regularization by projection*, see Chapter 3 in [12]. In addition, as described in Section 4.2, we can utilize CG as regularization method for the solution of the Toeplitz system in order to balance approximation error and propagated error. The multilevel method introduced in Section 4.1 combines both features. By optimizing the level (bandwidth) and the number of iterations in each level it provides an efficient and robust regularization technique for ill-conditioned sampling problems. See Section 5 for numerical examples.

5 Applications

We present two numerical examples to demonstrate the performance of the described methods. The first one concerns a 1-D reconstruction problem arising in spectroscopy. In the second example we approximate the Earth's magnetic field from noisy scattered data.

5.1 An example from spectroscopy

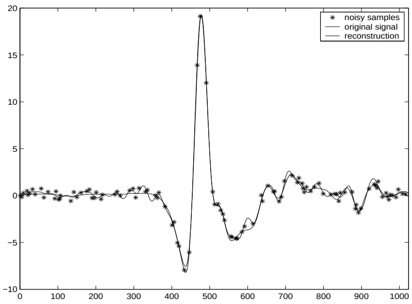
The original spectroscopy signal f is known at 1024 regularly spaced points t_j. This discrete sampling sequence will play the role of the original continuous signal. To simulate the situation of a typical experiment in spectroscopy we consider only 107 randomly chosen sampling values of the given sampling set. Furthermore we add noise to the samples with noise level (normalized by division by $\sum_{k=1}^{1024} |f(t_j)|^2$) of $\delta = 0.1$. Since the samples are contaminated by noise, we cannot expect to recover the (discrete) signal f completely. The bandwidth is approximately $\Omega = 5$ which translates into a polynomial degree of $M \approx 30$. Note that in general Ω and (hence M) may not be available. We will also consider this situation, but in the first experiments we assume that we know Ω. The error between the original signal f and an approximation f_n is measured by computing $\|f - f_n\|^2/\|f\|^2$.

First we apply the truncated frame method with regularized SVD as described in Section 2. We choose the truncation level for the SVD via formula (21). This is the optimal truncation level in this case, providing an approximation with least squares error 0.0944. Figure 1(a) shows the reconstructed signal together with the original signal and the noisy samples. Without regularization we get a much worse "reconstruction" (which is not displayed).

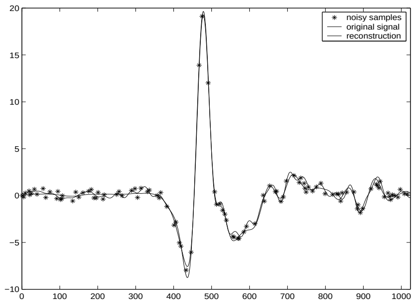
We apply CG to the truncated frame method, as proposed in Section 2.2 with stopping criterion (22) (for $\tau = 1$). The algorithm terminates already after 3 iterations. The reconstruction error is with 0.1097 slightly higher than for truncated SVD (see also Figure 1(b)), but the computational effort is much smaller.

Also Algorithm 3.1 (with $M = 30$) terminates after 3 iterations. The reconstruction is shown in Figure 1(c), the least squares error (0.0876) is slightly smaller than for the truncated frame method, the computational effort is significantly smaller.

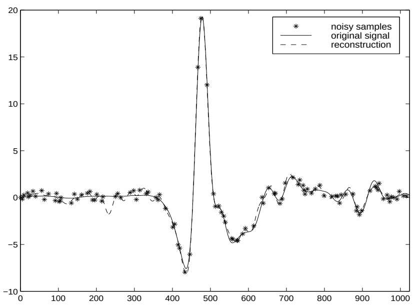
We also simulate the situation where the bandwidth is not known a priori and demonstrate the importance of a good estimate of the bandwidth. We apply Algorithm 3.1 using a too small degree ($M = 11$) and a too high degree ($M = 40$). (We get qualitatively the same results using the truncated frame method when using a too small or too large bandwidth). The approximations are shown in Figs. 1(d) and (e), The approximation errors are 0.4648 and 0.2805, respectively. Now we apply the multilevel algorithm of Section 4.1 which does not require any initial choice of the degree M. The algorithm terminates at "level" $M = 22$, the approximation is displayed in Fig. 1(f), the error is 0.0959, thus within the error bound δ, as desired. Hence without requiring explicit information about the bandwidth, we are able to obtain the same accuracy as for the methods above.



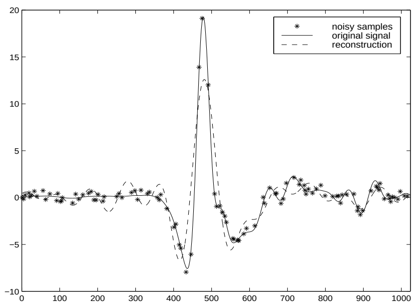
(a) Truncated frame method with TSVD, error=0.0944.



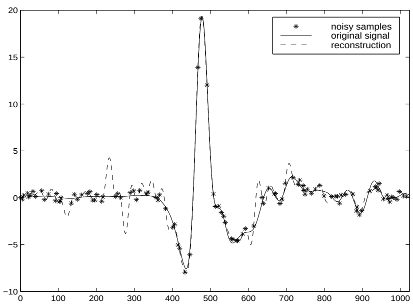
(b) Truncated frame method with CG, error=0.1097.



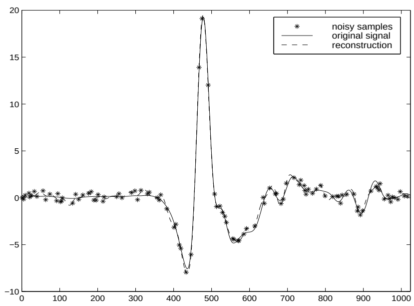
(c) Algorithm 3.1 with "correct" bandwidth, error=0.0876



(d) Using a too small bandwidth, error=0.4645.



(e) Using a too large bandwidth, error = 0.2412.



(f) Multilevel algorithm, error=0.0959.

Figure 1: Example from spectroscopy – comparison of reconstruction methods.

5.2 Approximation of geophysical potential fields

Exploration geophysics relies on surveys of the Earth's magnetic field for the detection of anomalies which reveal underlying geological features. Geophysical potential field-data are generally observed at scattered sampling points. Geoscientists, used to looking at their measurements on maps or profiles and aiming at further processing, therefore need a representation of the originally irregularly spaced data at a regular grid.

The reconstruction of a 2-D signal from its scattered data is thus one of the first and crucial steps in geophysical data analysis, and a number of practical constraints such as measurement errors and the huge amount of data make the development of reliable reconstruction methods a difficult task.

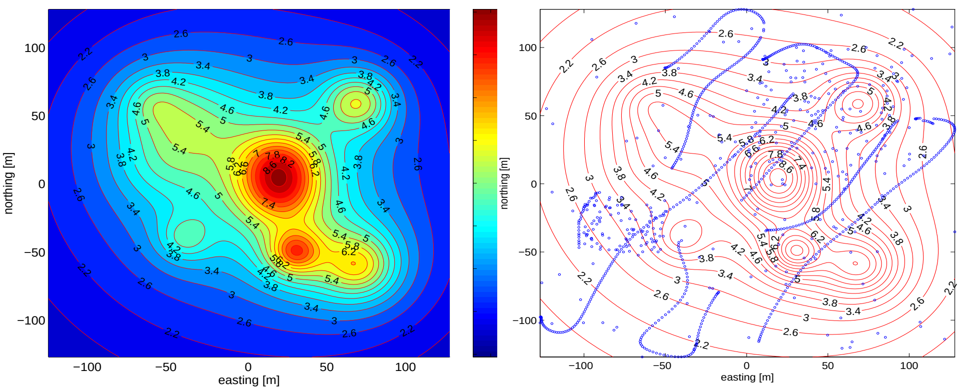
It is known that the Fourier transform of a geophysical potential field f has decay $|\hat{f}(\omega)| = \mathcal{O}(e^{-|\omega|})$. This rapid decay implies that f can be very well approximated by band-limited functions [30]. Since in general we may not know the (essential) bandwidth of f, we can use the multilevel algorithm proposed in Section 4.1 to reconstruct f.

The multilevel algorithm also takes care of following problem. Geophysical sampling sets are often highly anisotropic and large gaps in the sampling geometry are very common. The large gaps in the sampling set can make the reconstruction problem ill-conditioned or even ill-posed. As outlined in Section 4.2 the multilevel algorithm iteratively determines the optimal bandwidth that balances the stability and accuracy of the solution.

Figure 5.2(a) shows a synthetic gravitational anomaly f. The spectrum of f decays exponentially, thus the anomaly can be well represented by a band-limited function, using a "cut-off-level" of $|f(\omega)| \leq 0.01$ for the essential bandwidth of f.

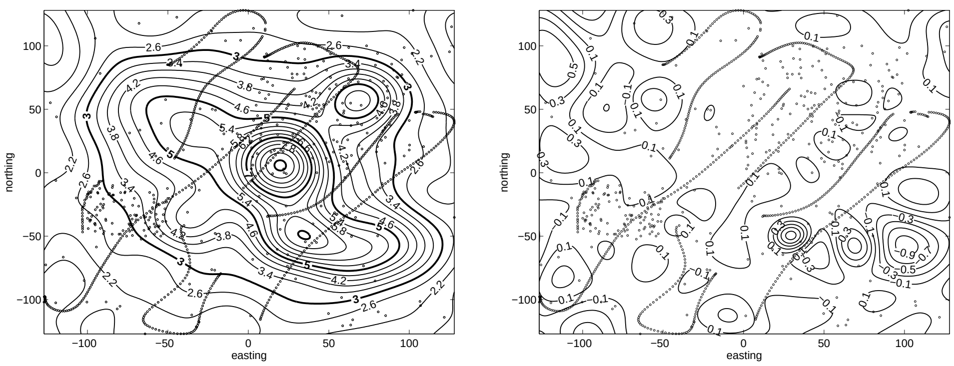
We have sampled the signal at 1000 points (u_j, v_j) and added 5% random noise to the sampling values $f(u_j, v_j)$. The sampling geometry – shown in Figure 5.2 as black dots – exhibits several features one encounters frequently in exploration geophysics [30]. The essential bandwidth of f would imply to choose a polynomial degree of $M = 12$ (i.e., $(2M+1)^2 = 625$ spectral coefficients). With this choice of M the corresponding block Toeplitz matrix T_M would become ill-conditioned, making the reconstruction problem unstable. As mentioned above, in practice we usually do not know the essential bandwidth of f. Hence we will not make use of this knowledge in order to approximate f.

We apply the multilevel method to reconstruct the signal, using only the sampling points $\{(u_j, v_j)\}$, the samples $\{f^\delta(u_j, v_j)\}$ and the noise level $\delta = 0.05$ as a priori information. The algorithm terminates at level $M = 7$. The reconstruction is displayed in Figure 5.2(c), the error between the true signal and the approximation is shown in Figure 5.2(d). The reconstruction error is 0.0517 (or 0.193 mGal), thus of the same order as the data error, as desired.



(a) Contour map of synthetic gravity anomaly, gravity is in mGal.

(b) Sampling set and synthetic gravity anomaly.



(c) Approximation by multi-level algorithm

(d) Error between approximation and actual anomaly.

Figure 2: Approximation of synthetic gravity anomaly from 1000 non-uniformly spaced noisy samples by the multilevel algorithm of Section 4.1. The algorithm iteratively determines the optimal bandwidth (i.e. level) for the approximation.

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