Localization of Matrix Factorizations

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Abstract

Matrices with off-diagonal decay appear in a variety of fields in mathematics and in numerous applications, such as signal processing, statistics, communications engineering, condensed matter physics, and quantum chemistry. Numerical algorithms dealing with such matrices often take advantage (implicitly or explicitly) of the empirical observation that this off-diagonal decay property seems to be preserved when computing various useful matrix factorizations, such as the Cholesky factorization or the QR-factorization. There is a fairly extensive theory describing when the *inverse* of a matrix inherits the localization properties of the original matrix. Yet, except for the special case of band matrices, surprisingly very little theory exists that would establish similar results for matrix factorizations. We will derive a comprehensive framework to rigorously answer the question when and under which conditions the matrix factors inherit the localization of the original matrix for such fundamental matrix factorizations as the LU-, QR-, Cholesky, and Polar factorization.

1 Introduction

Matrices with off-diagonal decay appear in numerous areas of mathematics including PDEs, numerical analysis, pseudo-differential operator theory, and applied harmonic analysis. They also play a prominent role in applications, including signal processing, statistics, condensed matter physics, quantum chemistry, and communications engineering. For instance, such matrices arise naturally from the fact that, in many systems, perturbations are localized. That is, local disturbances are not felt globally. One way to study such matrices is via various non-commutative generalizations of Wiener's Lemma, of which [25] provides an overview. However in a number of applications one is not interested in just the decay properties of the matrix describing the system and its inverse, but also in the decay properties of various fundamental factorizations of that matrix. We give a few such examples below.

In quantum chemistry, the so-called *density matrix* is a vital tool to determine the electronic structure of (possibly large) molecules [41, 38]. Many of the important values in electronic structure theory can be obtained as functionals of the density matrix. The typical basis vectors used to discretize the Hamiltonian give rise to a matrix which, while not sparse, exhibits off-diagonal decay. For non-metallic systems the off-diagonal decay of the density matrix is exponential, while for metallic systems the decay is algebraic [21, 10]. This decay is exploited to devise fast numerical algorithms for electronic structure calculations. Many of these algorithms implicitly assume that the (inverses of the) Cholesky factors of the density matrix inherit the decay properties of the density matrix. Benzi, Boito, and Razouk recently showed that this assumption is indeed justified in case of exponential decay [10]. However, for other types of decay, such as algebraic decay, it has been an open problem whether the Cholesky factors do indeed possess the same off-diagonal decay as the density matrix.

In astronomy, the detection and identification of signals must be done in the presence of cosmic microwave background radiation. This is a specific instance of detecting sparse, and possibly faint, signals in the

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presence of noise. In [31], methods are developed to detect signals in the case that the noise is correlated. Such methods work within bounds which depend on the sparsity and strength of the signal, and require off-diagonal decay of the correlation matrix. Essential to this method is that the Cholesky factors of the correlation matrix exhibit the same form of decay.

In many applications one is interested in constructing an orthonormal system consisting of vectors that are well localized, e.g. see [16, 47, 20] and [40, Chapter 9] for some examples. A standard approach is to apply the Gram-Schmidt algorithm to a set of vectors, which is of course nothing else than computing the QR-factorization of a matrix $\mathbf{A}$, where $\mathbf{A}$ contains the initial set of vectors as its columns. When the matrix $\mathbf{A}$ is localized, then it would be desirable to know (and often it is tacitly assumed, without any proof) that the orthonormal vectors obtained via Gram-Schmidt inherit these localization properties. In numerical analysis, the off-diagonal decay of the Cholesky factors or of matrix functions such as the matrix exponential, can be exploited to construct efficient preconditioners [11, 32, 12]. Current theory only covers the case where the matrix is banded or has exponential off-diagonal decay.

Furthermore, various signal processing algorithms for wireless communications involve the QR-factorization of a large matrix that represents the wireless communication channel [43] as an efficient form of "precoding". Moreover, the LU-factorization and the Cholesky factorization plays a prominent role in signal processing, filter design, and digital communications [35], often in connection with the concept of causality. In all these applications, knowledge about the off-diagonal decay behavior of the matrix factors can greatly help to reduce computational complexity and mitigate truncation errors, as well as simplify hardware design.

The results presented in this paper can be seen as fundamental extensions of noncommutative versions of the famous Wiener's Lemma. Recall that these noncommutative generalizations of Wiener's Lemma state that under certain conditions the inverse $\mathbf{A}^{-1}$ of a matrix $\mathbf{A}$ will indeed inherit the off-diagonal decay properties of $\mathbf{A}$, see e.g. [34, 4, 13, 37, 28, 48, 25]. Informally, this *Wiener property* can be stated as "*if* $\mathbf{A}$ *is localized, then so is* $\mathbf{A}^{-1}$". It has been an open problem whether and under which conditions the Wiener property extends to matrix factorizations. The results in this paper provide affirmative answers.

We will show that for a wide range of off-diagonal decay, localized matrices give rise to LU-, Cholesky, QR-, and polar factorizations, whose factors inherit this localization. Below we state two examples of the type of results we prove in this paper. In both theorems the algebra $\mathscr{A}$ represents a Banach algebra that describes localized matrices in the form of some off-diagonal decay. To give a concrete example, we might assume that $\mathbf{A} \in \mathscr{A}$ satisfies polynomial off-diagonal decay, i.e., there exists a parameter $s > 1$ and a constant $C > 0$ such that the entries a_{jk} of $\mathbf{A}$ obey

$$|a_{jk}| \leq C(1 + |j - k|)^{-s}, \qquad \forall j, k \in \mathbb{Z}.$$

The precise statements of the theorems with the accompanying definitions are given in the following sections.

Theorem 1.1. *Let $\mathscr{B}_c$ be the closure of the algebra of band matrices with respect to the operator norm. Let $\mathscr{A} \subset \mathscr{B}_c$ be an inverse-closed sub-algebra satisfying certain technical conditions, given in Section 3. Suppose $\mathbf{A} \in \mathscr{A}$. Then, if $\mathbf{A} = \mathbf{LU}$ in $\mathscr{B}_c$, we have $\mathbf{L}, \mathbf{U} \in \mathscr{A}$.*

Theorem 1.2. *Let $\mathscr{A}$ be one of the decay algebras specified in Section 4, such as the algebra with polynomial off-diagonal decay discussed above. Assume that $\mathbf{A} \in \mathscr{A}$. Then, if $\mathbf{A} = \mathbf{QR}$, we have $\mathbf{Q}, \mathbf{R} \in \mathscr{A}$.*

The LU-factorization turns out to be the most challenging one for which to prove that the matrix factors exhibit the same localization properties as the original matrix. We present two quite different methods for obtaining such localization results for the LU-factorization, see Sections 3 and 4. The first method is more abstract and algebraic, while the second is more concrete and computational. The first method has the advantage of being applicable to a broad array of types of decay. To prove our theorem, we only need to assume that the matrix belongs to some inverse closed Banach algebra, and that it possesses an LU-factorization in a very weak decay algebra. The second method requires us to consider each form of decay separately, but it gives a better quantitative understanding of the decay properties of the factors. Moreover, we need to assume only that the matrix possesses an LU-factorization, but nothing about the factors themselves. In Section 5 we prove similar localization results for the QR-factorization, the Cholesky factorization, and the polar factorization. In fact, for the latter the result is quickly obtained from basic facts

and does not actually require the main machinery developed in this paper. Finally, localization of matrix functions, such as the matrix exponential, is the topic of Section 6.

Prior work on the topic of localized matrix factorizations starts with the seminal work of Wiener on spectral factorization, see e.g. [49]. His research has led to a plethora of extensions, generalizations, and refinements, which we cannot possibly review here. This work, however, is firmly rooted in the commutative setting, as the operators under consideration correspond to convolution operators (including even those papers dealing with matrix-valued convolution operators). To our knowledge the first result that truly addresses localization of matrix factorizations in the noncommutative setting is due to Gohberg, Kasshoek, and Woerdeman. In [23], the authors show if a positive definite matrix is in the so-called non-stationary Wiener algebra $\mathcal{W}$ (in current terminology, $\mathcal{W}$ is the unweighted Baskakov-Gohberg-Sjöstrand algebra, see Definition 2.2, item 4), then the Cholesky factors belong to the same algebra. Our research was definitely influenced by their paper as well as by its notable precursors [24, 22]. We also gladly acknowledge inspiration by the work of Benzi and coauthors, who have analyzed the localization of (inverses of) Cholesky factors for the case of band matrices [10]. Finally, Baskakov's work on the spectral theory of Banach modules [6] and especially causal operators [9, 7] (in collaboration with one of the authors of this paper), has paved the way for one of the approaches presented in this paper.

The reader may wonder why we have not yet mentioned the eigenvalue decomposition and the singular value decomposition. It is easy to see that localization of the matrix is in general not sufficient to ensure localization of its eigenvectors or singular vectors. Take for example a bi-infinite Laurent matrix (also called Toeplitz matrix), then we know that its (generalized) eigenvectors are given by the complex exponentials, which have no decay whatsoever, regardless of how strong the off-diagonal decay of the Laurent matrix may be. Nevertheless, under certain conditions the eigenvectors or singular vectors of a matrix *do inherit* the localization properties of a matrix. We will report on these results in a forthcoming paper.

2 Preliminaries

2.1 Notation

For a vector $\mathbf{y} = \{y_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we define its projection $P_n\mathbf{y}$ to be

$$P_n\mathbf{y} = (\dots, 0, y_{-n}, y_{-n+1}, \dots, y_{n-1}, y_n, 0, \dots). \tag{1}$$

The range of P_n is written as $\operatorname{Im} P_n$. Since $\operatorname{Im} P_n$ is a subspace of $\ell^2(\mathbb{Z})$ of dimension $2n+1$, it can be identified with $\mathbb{C}^{2n+1}$. Given a matrix $\mathbf{A} = (a_{ij}), i, j \in \mathbb{Z}$ we denote $\mathbf{A}^{(n)} = P_n\mathbf{A}P_n$ restricted to $\operatorname{Im} P_n$, that is

$$\mathbf{A}^{(n)} : \operatorname{Im} P_n \to \operatorname{Im} P_n \subset \ell^2(\mathbb{Z}). \tag{2}$$

By definition $\mathbf{A}^{(n)}$ is an operator acting on $\operatorname{Im} P_n$, and we can interpret $\mathbf{A}^{(n)}$ as a finite $(2n+1)\times(2n+1)$ matrix with entries $\{a_{ij}\}_{|i|,|j|\le n}$, acting on $\mathbb{C}^{2n+1}$.

Recall that a matrix $\mathbf{A}$ is lower triangular if $a_{ij} = 0$ for $i > j$, and $\mathbf{A}$ is upper triangular if $a_{ij} = 0$ for $i < j$. We shall use the notations $\mathscr{L}, \mathscr{L}^*, \mathscr{D}$ to refer to the sub-algebras of $\mathscr{B}(\ell^2)$ consisting of lower-triangular, upper-triangular, and diagonal matrices, respectively. We shall denote by $\mathscr{L}_0$ and $\mathscr{L}_0^*$ the sub-algebras of strictly lower-triangular and strictly upper-triangular matrices, respectively.

2.2 Matrix Factorizations

For the convenience of the reader we briefly review the definitions of the matrix factorizations under consideration.

Definition 2.1. Let $\mathscr{A}$ be a Banach algebra of matrices in $\mathscr{B}(\ell^2)$ and assume that $\mathbf{A} \in \mathscr{A}$.
1) We say that $\mathbf{A} \in \mathscr{A}$ admits an *LU-factorization* in $\mathscr{A}$ if $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}, \mathbf{L}^{-1} \in \mathscr{L} \cap \mathscr{A}$ and $\mathbf{U}, \mathbf{U}^{-1} \in \mathscr{L}^* \cap \mathscr{A}$.

2) $\mathbf{A} \in \mathscr{A}$ admits a *QR-factorization* in $\mathscr{A}$ if $\mathbf{A} = \mathbf{QR}$, where $\mathbf{R}, \mathbf{R}^{-1} \in \mathscr{L}^* \cap \mathscr{A}$ and $\mathbf{Q} \in \mathscr{A}$ is a unitary matrix.
3) Assume $\mathbf{A}$ is hermitian positive definite. We say that $\mathbf{A} \in \mathscr{A}$ admits a *Cholesky factorization* in $\mathscr{A}$ if $\mathbf{A} = \mathbf{CC}^*$, where $\mathbf{C}, \mathbf{C}^{-1} \in \mathscr{L} \cap \mathscr{A}$ and the diagonal entries of $\mathbf{C}$ are positive.
4) The *polar factorization* of $\mathbf{A} \in \mathscr{A}$ is given by $\mathbf{A} = \mathbf{UP}$, where $\mathbf{U}$ is a unitary matrix and $\mathbf{P}$ is a positive-semidefinite Hermitian matrix, and $\mathbf{U}, \mathbf{P} \in \mathscr{A}$.
Remark 1. For bi-infinite matrices the existence and uniqueness of some of these matrix factorizations is non-trivial, see [2, 15, 42, 46] for more detailed discussions[1]. The existence of the QR-factorization and the Cholesky factorization follows for instance from the results of Section 3 in Arveson's seminal paper [2]. The existence of the LU-factorization is less clear (unless we consider finite matrices), the interested reader may want to consult [1] and the references therein. A necessary condition that a matrix $\mathbf{A}$ has an LU-factorization on $\ell^p, 1 \leq p \leq \infty$, is that $\mathbf{A}$ and all $\mathbf{A}^{(n)}$ are uniformly invertible, i.e., $\sup_n\{\|(\mathbf{A}^{(n)})^{-1}\|_{\mathscr{B}(\ell^2)}, \|\mathbf{A}^{-1}\|_{\mathscr{B}(\ell^2)}\} < \infty$, see [3, 1]. However, this condition is not sufficient as a counter example in [1] shows. We observe that if $\mathbf{LU} = \bar{\mathbf{L}}\bar{\mathbf{U}}$ are two different LU-factorizations then $\bar{\mathbf{L}}^{-1}\mathbf{L} = \bar{\mathbf{U}}\mathbf{U}^{-1}$ is a diagonal matrix, i.e. the LU-factorization is unique up to multiplication by a diagonal matrix. In Section 4 we shall assume that the diagonal entries of $\mathbf{L}$ are all equal to 1 to avoid ambiguity. For positive definite matrices the existence of such an LU-factorization follows immediately from the existence of the Cholesky factorization. Indeed, one simply rescales $\mathbf{C}$ via multiplication by a diagonal matrix, i.e., if $\mathbf{A} = \mathbf{CC}^*$ then $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L} = \mathbf{CD}^{-1}$ and the main diagonal of $\mathbf{D}$ coincides with the main diagonal of $\mathbf{C}$.

2.3 Decay Algebras

We first consider some typical matrix norms that express various forms of off-diagonal decay. In applications, one might encounter such forms of decay in signal and image processing, digital communication, quantum chemistry and quantum physics. Off-diagonal decay is quantified by means of weight functions.
Definition 2.2. A non-negative function v on $\mathbb{Z}$ is called an *admissible weight* if it satisfies the following properties:

1. v is even and normalized so that $v(0) = 1$;
2. v is sub-multiplicative, i.e. $v(j+k) \leq v(j)v(k)$ for all $j, k \in \mathbb{Z}$;
3. v satisfies the *Gelfand-Raikov-Shilov (GRS)* condition [19]: $\lim_{n\to\infty} v(nk)^{\frac{1}{n}} = 1$ for all $k \in \mathbb{Z}$.

The assumption that v is even assures that the corresponding Banach algebra is closed under taking the adjoint. The GRS property is crucial for the inverse-closedness of the Banach algebra, as we will see below. The standard weight functions on $\mathbb{Z}$ are of the form

$$v(k) = e^{a\cdot d(k)^b}(1 + d(k))^s,$$

where $d(k)$ is a norm on $\mathbb{Z}$. Such a weight is sub-multiplicative when $a, s \geq 0$ and $0 \leq b \leq 1$, and satisfies the GRS condition if and only if $0 \leq b < 1$.
Definition 2.3. We consider the following types of off-diagonal decay.

1. The *Jaffard Class* [34], denoted $\mathscr{A}_s$, is the collection of matrices $\mathbf{A} = (a_{jk}), j, k \in \mathbb{Z}$ such that

$$|a_{jk}| \leq C(1 + |j-k|)^{-s}, \tag{3}$$

endowed with the norm

$$\|\mathbf{A}\|_{\mathscr{A}_s} := \sup_{j,k\in\mathbb{Z}} |a_{jk}(1 + |j-k|)^s.$$

[1]The definitions of the factorizations in Definition 2.1 imply that $\mathbf{A}$ is invertible in $\mathscr{B}(\ell^2)$. Some of the factorizations could be defined slightly more generally than we have don here. For instance in the QR-factorization and the polar factorization we could replace the unitary matrix by a partial isometry. Many of the results in our paper can be extended to hold for these more general factorizations, but for clarity of presentation we prefer to work with the factorizations as defined in Definition 2.1.

2. More generally, let v be an admissible weight such that $v^{-1} \in \ell^1(\mathbb{Z})$ and $v^{-1} * v^{-1} \leq Cv^{-1}$. Then we denote by $\mathscr{A}_v$ the collection of matrices satisfying

$$|a_{jk}| \leq Cv^{-1}(j-k), \tag{4}$$

endowed with the norm

$$\|\mathbf{A}\|_{\mathscr{A}_v} := \sup_{j,k\in\mathbb{Z}} |a_{jk}|v(j-k).$$

3. *Schur-type algebras:* Let v be an admissible weight. Then we denote by $\mathscr{A}_v^1$ the collection of matrices $\mathbf{A} = (a_{jk}), j, k \in \mathbb{Z}$ such that

$$\sup_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |a_{jk}|v(j-k) < \infty \qquad \text{and} \qquad \sup_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |a_{jk}|v(j-k) < \infty, \tag{5}$$

endowed with the norm

$$\|\mathbf{A}\|_{\mathscr{A}_v^1} := \max\left\{ \sup_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |a_{jk}|v(j-k) < \infty,\ \sup_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |a_{jk}|v(j-k) < \infty \right\}.$$

4. Let v be an admissible weight. Then, the *Gohberg-Baskakov-Sjöstrand class*, denoted by $\mathscr{C}_v$, is the collection of matrices such that the norm

$$\|\mathbf{A}\|_{\mathscr{C}_v} := \sum_{j\in\mathbb{Z}} \sup_{k\in\mathbb{Z}} |a_{k,k-j}|v(j) = \inf_{\alpha\in\ell^1_v} \left\{ \|\alpha\|_{\ell^1_v} : |a_{jk}| \leq \alpha(j-k) \right\}$$

is finite.

If $\mathscr{A}$ is one of the Banach algebras defined in 2.3, then any matrix $\mathbf{A} \in \mathscr{A}$ is bounded on $\ell^p, 1 \leq p \leq \infty$, see [30]. The results derived in this paper hold for a variety of other Banach algebras that describe off-diagonal decay, such as the ones in [48]. But for clarity of presentation we mainly focus on the Banach algebras introduced in Definition 2.3.

We recall that a Banach algebra $\mathscr{A}$ is *inverse-closed* in $\mathscr{B}(\ell^2(\mathbb{Z}))$ if for every $\mathbf{A} \in \mathscr{A}$ that is invertible on $\ell^2(\mathbb{Z})$ we have that $\mathbf{A}^{-1} \in \mathscr{A}$. The matrix algebras above are inverse-closed essentially when v is an admissible weight. The precise statement is slightly more involved, because we need to be a bit meticulous about the weights.

Theorem 2.4. *Let v be an admissible weight.*

1. *Assume that $v^{-1} \in \ell^1(\mathbb{Z})$ and $v^{-1} * v^{-1} \leq Cv^{-1}$. Then $\mathscr{A}_v$ is inverse-closed in $\mathscr{B}(\ell^2(\mathbb{Z}))$. In particular, $\mathscr{A}_s$ possesses this property if $s > 1$.*
2. *If $v(k) \geq C(1+|k|)^\delta$ for some $\delta > 0$, then $\mathscr{A}_v^1$ is inverse-closed in $\mathscr{B}(\ell^2(\mathbb{Z}))$.*
3. *$\mathscr{C}_v$ is inverse closed in $\mathscr{B}(\ell^2(\mathbb{Z}))$ for arbitrary admissible weights.*

While for C^*-(sub)algebras inverse-closedness is easy to prove and always true, it is highly non-trivial to establish inverse-closedness of a Banach algebra. Inverse-closedness for $\mathscr{A}_s$ is due to Jaffard [34] and Baskakov [4, 5]. For $\mathscr{A}_v$ it was proved by Baskakov [5], and a different proof is given in [29]. The result for $\mathscr{A}_v^1$ is proven in [28]. Inverse-closedness for $\mathscr{C}_v$ with $v \equiv 1$ is due to Gohberg, Kaashoek, and Woerdeman [23], and was rediscovered by Sjöstrand [44]. The case of arbitrary weights is due to Baskakov [4, 5], but see also Kurbatov [37] and Blatov [13].

Remark 2. We note that for a singly-infinite matrix $\mathbf{A} \in \mathscr{A}_v$ with admissible weight v, the existence of the LU-factorization of $\mathbf{A}$ can be derived from Theorem 2 in [1]. This follows essentially from the fact that condition (5) together with the inverse-closedness of $\mathscr{A}_v$ imply the decay condition in Theorem 2 of [1].

Furthermore we will make use of the following two classes of matrices. The class of *band* matrices, denoted $\mathscr{B}_b$, is the collection of matrices $\mathbf{A} = (a_{jk}), j, k \in \mathbb{Z}$, such that there is some natural number $N = N(\mathbf{A})$

with the property $a_{jk} = 0$ if $|j-k| > N$. The class of matrices with *exponentially decaying* diagonals, denoted $\mathscr{B}_\gamma$, is the collection of matrices $\mathbf{A} = (a_{jk}), j, k \in \mathbb{Z}$, such that $|a_{jk}| \leq C\gamma^{|j-k|}$ for some constants $C = C(\mathbf{A}) > 0$ and $\gamma = \gamma(\mathbf{A}) \in (0,1)$.

All of the Banach algebras described in Definition 2.3 are contained in a larger Banach algebra, which we will denote $\mathscr{B}_c$. In some sense, $\mathscr{B}_c$ defines the weakest sort of off-diagonal decay, but we will need to lay a little more groundwork before we are ready to give it a formal definition.

For $\theta \in \mathbb{T}$ and $x \in \ell^2(\mathbb{Z})$, the modulation representation $M : \mathbb{T} \to \ell^2(\mathbb{Z})$ is defined by

$$M(\theta)x(n) = \theta^n x(n).$$

Given any $\mathbf{A} \in \mathscr{B}(\ell^2)$ we shall denote by $f_{\mathbf{A}} \in L^\infty(\mathbb{T}, \mathscr{B}(\ell^2))$ the function

$$f_{\mathbf{A}}(\theta) = M(\theta)\mathbf{A}M(\theta^{-1}),\ \theta \in \mathbb{T}. \tag{6}$$

The Fourier series of the operator $\mathbf{A}$ is defined [17, 5] as the Fourier series of the function $f_{\mathbf{A}}(\theta) \sim \sum_k \theta^k \mathbf{A}_k$. An easy computation shows that the operators $\mathbf{A}_k$ are the diagonals of the matrix $\mathbf{A} \in \mathscr{B}(\ell^2)$.

Definition 2.5. The algebra of *M-continuous* matrices, denoted $\mathscr{B}_c$, is the collection of all matrices such that the function $f_{\mathbf{A}}$ defined above is a continuous map from $\mathbb{T}$ to $\mathscr{B}(\ell^2)$, i.e. $f_{\mathbf{A}} \in C(\mathbb{T}, \mathscr{B}(\ell^2))$.

In the following proposition we collect some useful known relationships between the above subalgebras, see [8, 9] and references therein, as well as [26, 27].

Proposition 2.6. The following properties hold:

1. $\mathscr{B}_c(\ell^2) = \overline{\mathscr{B}_b(\ell^2)}$, (because of this property $\mathscr{B}_c(\ell^2)$ is sometimes also referred to as the algebra of band-dominated operators, see e.g. [39]);
2. $\mathscr{B}_b(\ell^2) \subseteq \{\mathbf{A} \in \mathscr{B}(\ell^2)$: $f_{\mathbf{A}}$ admits an extension to an entire function$\}$;
3. $\mathscr{B}_\gamma(\ell^2) = \{\mathbf{A} \in \mathscr{B}(\ell^2)$: $f_{\mathbf{A}}$ admits a holomorphic extension to an annulus $\{1-\varepsilon < |z| < 1+\varepsilon\}$ for some $\varepsilon > 0$ depending on $\gamma\}$.
4. $\mathscr{B}_b(\ell^2) \subset \mathscr{B}_\gamma(\ell^2) \subset \mathscr{C}_1 \subset \mathscr{B}_c(\ell^2)$;
5. $\mathscr{L} \cap \mathscr{B}_c(\ell^2) = \{\mathbf{A} \in \mathscr{B}(\ell^2)$: $f_{\mathbf{A}}$ admits a holomorphic extension into the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that is continuous in $\overline{\mathbb{D}}\}$;
6. $\mathscr{L}^* \cap \mathscr{B}_c(\ell^2) = \{\mathbf{A} \in \mathscr{B}(\ell^2)$: $f_{\mathbf{A}}$ admits a bounded holomorphic extension outside the unit disc $\mathbb{D}$ that is continuous in $\mathbb{C}\backslash\mathbb{D}\}$;
7. $\mathscr{L}_0$ and $\mathscr{L}_0^*$ are two-sided ideals in $\mathscr{L}$ and $\mathscr{L}^*$ respectively.

By $\mathscr{A}$ we shall denote some Banach algebra of matrices in $\mathscr{B}(\ell^2)$. Typically we shall assume that

$$\mathscr{B}_\gamma(\ell^2) \subset \mathscr{A} \subseteq \mathscr{B}_c(\ell^2) \subset \mathscr{B}(\ell^2) \tag{7}$$

and

$$\|\mathbf{A}\|_{\mathscr{B}(\ell^2)} \leq c\|\mathbf{A}\|_{\mathscr{A}}, \tag{8}$$

where the constant c is independent of $\mathbf{A}$. These conditions are satisfied for the Banach Algebras of Definition 2.3. Indeed, for condition (7) this is obvious and for condition (8) see [30].

Definition 2.7. We say that an algebra $\mathscr{A}$ is *strongly decomposable* if there exists a bounded projection $\mathcal{P} : \mathscr{A} \to \mathscr{A}$ which maps $\mathscr{A}$ onto $\mathscr{L} \cap \mathscr{A}$ parallel to $\mathscr{L}_0^*$. In this case, we let $\mathcal{Q} = I - \mathcal{P} \in B(\mathscr{A})$ be the projection onto $\mathscr{L}_0^* \cap \mathscr{A}$ parallel to $\mathscr{L}$.

Remark 3. It is easy to check that the Banach algebras of Definition 2.3 are strongly decomposable. In most cases, $\mathscr{B} \subseteq \mathscr{C}_1$ will imply strong decomposability.

In order to prove the main result in Section 3, we need to embed the algebra $\mathscr{A}$ into the algebra of (uniformly) continuous operator valued functions $C(\mathbb{T}, \mathscr{B}(\ell^2))$. The above definitions transfer to the realm of such functions in the following way.

Definition 2.8. [24]. Let $\mathfrak{A}$ be a subalgebra of $C(\mathbb{T}, \mathscr{B}(\ell^2))$. Let $\mathfrak{A}^+$ be the set of all functions in $\mathfrak{A}$ that admit holomorphic extensions into $\mathbb{D}$ that are continuous in $\overline{\mathbb{D}}$. Similarly, let $\mathfrak{A}^-$ be the set of all functions in $\mathfrak{A}$ that admit bounded holomorphic extensions into $\mathbb{C}\backslash\overline{\mathbb{D}}$ that are continuous in $\mathbb{C}\backslash\mathbb{D}$. Let also $\mathfrak{A}_0^- = \{f \in \mathfrak{A}^- : \ f(\infty) = 0\}$. The algebra $\mathfrak{A}$ is *decomposing* if $\mathfrak{A} = \mathfrak{A}^+ \oplus \mathfrak{A}_0^-$. We shall denote by $\mathfrak{P}$ and $\mathfrak{Q}$ the projections in $B(\mathfrak{A})$ associated with this decomposition.

Definition 2.9. [24]. A function $f \in \mathfrak{A}$ admits a *canonical factorization* if $f = f_\ell f_u$, where $f_\ell, f_\ell^{-1} \in \mathfrak{A}^+$ and $f_u, f_u^{-1} \in \mathfrak{A}^-$.

Remark 4. We observe that an algebra $\mathfrak{A}$ is inverse closed in $C(\mathbb{T}, \mathscr{B}(\ell^2))$ if and only if the function f^{-1} defined by $f^{-1}(\theta) = [f(\theta)]^{-1}$ is in $\mathfrak{A}$ whenever $f \in \mathfrak{A}$ and $[f(\theta)]^{-1} \in \mathscr{B}(\ell^2)$ for all $\theta \in \mathbb{T}$. In particular, a subalgebra $\mathfrak{A}_{\mathscr{A}} = \{f_{\mathbf{A}}\colon \ \mathbf{A} \in \mathscr{A}\}$ is inverse closed in $C(\mathbb{T}, \mathscr{B}(\ell^2))$ if $f_{\mathbf{A}}$ is defined by (6) and $\mathscr{A}$ is inverse closed in $\mathscr{B}_c(\ell^2)$. We also note that the algebra $\mathfrak{A}_{\mathscr{A}}$ is decomposing if $\mathscr{A}$ is strongly decomposable.

3 Abstract Harmonic Analysis Approach

In this section we present our first theorem concerning the off-diagonal decay of the LU-factors. Our abstract approach, which is based on some advanced harmonic analysis results, leads to a fairly short proof, albeit at the cost of more conceptual effort.

Theorem 3.1. *Let $\mathscr{A}$ be a strongly decomposable inverse closed subalgebra of $\mathscr{B}(\ell^2)$ that satisfies conditions* (7) *and* (8)*. Assume also that $\mathbf{A} \in \mathscr{A}$ admits an LU-factorization $\mathbf{A} = \mathbf{LU}$ in $\mathscr{B}_c(\ell^2)$. Then $\mathbf{A}$ admits an LU-factorization in $\mathscr{A}$.*

The proof of this result relies heavily on the following two abstract results.

Theorem 3.2. *[9, Theorem 8.14]. Let $\mathbf{A} \in \mathscr{L} \cap \mathscr{B}_c(\ell^2)$. The following are equivalent:*

1. $\mathbf{A}^{-1} \in \mathscr{L}$*;*
2. $f_{\mathbf{A}}(z)$ *is invertible in $\mathscr{B}(\ell^2)$ for all $z \in \overline{\mathbb{D}}$.*

Theorem 3.3. *[24, Theorem 1.1]. Assume that $\mathfrak{A}$ is a decomposing inverse closed subalgebra of $C(\mathbb{T}, \mathscr{B}(\ell^2))$ that satisfies the following two conditions:*

1. *For all $f \in \mathfrak{A}$*
$$\max_{\theta\in\mathbb{T}} \|f(\theta)\|_{\mathscr{B}(\ell^2)} \le c\|f\|_{\mathfrak{A}},$$
where $c > 0$ is a constant independent of f;
2. *If $f \in C(\mathbb{T}, \mathscr{B}(\ell^2))$ admits a holomorphic extension to an annulus $\{1-\varepsilon < |z| < 1+\varepsilon\}$ for some $\varepsilon > 0$ then $f \in \mathfrak{A}$ and the set of all such operator-valued functions is dense in $\mathfrak{A}$.*

Then any function $f \in \mathfrak{A}$ that satisfies

$$\max_{\theta\in\mathbb{T}} \|f(\theta) - I\|_{\mathscr{B}(\ell^2)} < 1 \tag{9}$$

admits a canonical factorization $f = f_\ell f_u$ and

$$f_\ell^{-1}(\theta) = f_\ell(\theta)^{-1} = I - \mathfrak{P}g(\theta) + \mathfrak{P}[g(\theta)\mathfrak{P}g(\theta)] - \dots, \tag{10}$$

$$f_u^{-1}(\theta) = f_u(\theta)^{-1} = I - \mathfrak{Q}g(\theta) + \mathfrak{Q}[(\mathfrak{Q}g(\theta))g(\theta)] - \dots, \tag{11}$$

where $g(\theta) = f(\theta) - I$, and the series converge in $\mathfrak{A}$.

The relationships between various subalgebras mentioned in Proposition 2.6 and Remark 4 allow us to apply the above result to the algebras of the form

$$\mathfrak{A}_{\mathscr{A}} = \{f \in C(\mathbb{T}, \mathscr{B}(\ell^2)) : f = f_{\mathbf{A}}, \mathbf{A} \in \mathscr{A}\}.$$

As a result, we obtain the following special case of Theorem 3.3.

Theorem 3.4. *Assume that $\mathscr{A}$ is a strongly decomposable inverse closed subalgebra of $\mathscr{B}(\ell^2)$ that satisfies conditions* (7) *and* (8)*. Then any matrix $\mathbf{A} \in \mathscr{A}$ that satisfies $\|\mathbf{A}-\mathbf{I}\|_{\mathscr{B}(\ell^2)} < 1$ admits an LU-factorization $\mathbf{A} = \mathbf{LU}$ in $\mathscr{A}$ such that*

$$\mathbf{L}^{-1} = \mathbf{I} - \mathcal{P}\mathbf{M} + \mathcal{P}[\mathbf{M}\mathcal{P}\mathbf{M}] - \mathcal{P}[\mathbf{M}\mathcal{P}[\mathbf{M}\mathcal{P}\mathbf{M}]] + \dots, \tag{12}$$

$$\mathbf{U}^{-1} = \mathbf{I} - \mathcal{Q}\mathbf{M} + \mathcal{Q}[[\mathcal{Q}\mathbf{M}]\mathbf{M}] - \mathcal{Q}[\mathcal{Q}[[\mathcal{Q}\mathbf{M}]\mathbf{M}]\mathbf{M}] + \dots, \tag{13}$$

where $\mathbf{M} = \mathbf{A} - \mathbf{I}$ and the series converge in $\mathscr{A}$.

Proof of Theorem 3.1. Let $\mathbf{A} = \mathbf{LU}$ and consider the holomorphic extensions $f_{\mathbf{L}}(z) = \sum_k z^k \mathbf{L}_k$, $z \in \mathbb{D}$, and $f_{\mathbf{U}}(z) = \sum_k z^k \mathbf{U}_k$, $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$. Pick $\varepsilon \in (0,1)$ such that

$$\|[f_{\mathbf{L}}(\varepsilon)]^{-1}\mathbf{LU}[f_{\mathbf{U}}(1/\varepsilon)]^{-1} - \mathbf{I}\| < 1.$$

Such an ε exists because $\mathbf{L}, \mathbf{U} \in \mathscr{B}_c(\ell^2)$ and invertibility is stable under small perturbations. Hence, Theorem 3.4 implies that $\mathbf{A}' := [f_{\mathbf{L}}(\varepsilon)]^{-1}\mathbf{LU}[f_{\mathbf{U}}(1/\varepsilon)]^{-1}$ admits an LU-factorization $\mathbf{A}' = \mathbf{L}'\mathbf{U}'$ in $\mathscr{A}$. On the other hand, since $\mathbf{L}^{-1} \in \mathscr{L}$ and $\mathbf{U}^{-1} \in \mathscr{L}^*$, Theorem 3.2 implies $[f_{\mathbf{L}}(\varepsilon)]^{-1} \in \mathscr{L}$ and $[f_{\mathbf{U}}(1/\varepsilon)]^{-1} \in \mathscr{L}^*$. Hence, $\mathbf{A}'$ admits two LU-factorizations in $\mathscr{B}_c(\ell^2)$ and Remark 1 implies that there exists an invertible diagonal matrix $\mathbf{D} \in \mathscr{D}$ such that

$$(\mathbf{L}')^{-1}[f_{\mathbf{L}}(\varepsilon)]^{-1}\mathbf{L} = \mathbf{U}' f_{\mathbf{U}}(1/\varepsilon)\mathbf{U}^{-1} = \mathbf{D}.$$

Moreover, since $f_{\mathbf{L}}(\varepsilon)$, $f_{\mathbf{U}}(1/\varepsilon) \in \mathscr{B}_\gamma(\ell^2)$, we have $\mathbf{L} = f_{\mathbf{L}}(\varepsilon)\mathbf{L}'\mathbf{D} \in \mathscr{A}$ and $\mathbf{U} = \mathbf{D}^{-1}\mathbf{U}' f_{\mathbf{U}}(1/\varepsilon) \in \mathscr{A}$. □

Remark 5. From the proofs above it is not hard to see that an analogous result holds in a much more general setting than just for matrix algebras. However, this general result is beyond the scope of this paper as it would require introduction of too much heavy machinery. We cite [9, 8] which describe the setup for the general result.

Remark 6. If in Theorem 3.1, we additionally required that $\|\mathbf{I}-\mathbf{A}\|_{\mathscr{A}} < 1$, then the result would follow immediately from the observation that the series in (12) and (13) would converge in $\mathscr{A}$ under this additional assumption. However, the condition $\|\mathbf{I}-\mathbf{A}\|_{\mathscr{A}} < 1$ is rather restrictive, as it *cannot* be enforced by a simple rescaling of $\mathbf{A}$, even not if we assumed $\mathbf{A}$ to be positive definite. Indeed, it is highly non-trivial that we can switch from the $\mathscr{A}$-norm to the operator norm in the condition $\|\mathbf{I}-\mathbf{A}\| < 1$ in Theorem 3.4. Note furthermore that for a non-positive definite matrix $\mathbf{A}$ even the condition $\|\mathbf{I}-\mathbf{A}\|_{\mathscr{B}(\ell^2)} < 1$ cannot be enforced by simple rescaling. Instead we need to resort to a smart choice of "preconditioning" by $[f_{\mathbf{L}}(\varepsilon)]^{-1}$. Establishing the existence of such a preconditioner with all the right properties involves some advanced results from abstract harmonic analysis.

4 Linear Algebra Approach

As pointed out in Section 2, if the LU-factorization of a matrix exists, then it is unique up to multiplication by an invertible diagonal matrix. In this section we assume that $\mathbf{A}$ has an LU-factorization $\mathbf{A} = \mathbf{LU}$, and to avoid ambiguity we stipulate that the diagonal entries of $\mathbf{L}$ are equal to 1.

In what follows it will be useful to write our matrices in block form, i.e.

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_{11} & 0 \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ 0 & \mathbf{U}_{22} \end{pmatrix},$$

with $\mathbf{A}_{11} = (a_{ij})_{i<0,j<0}$, $\mathbf{A}_{12} = (a_{ij})_{i<0,j\geq 0}$, $\mathbf{A}_{21} = (a_{ij})_{i\geq 0,j<0}$, and $\mathbf{A}_{22} = (a_{ij})_{i\geq 0,j\geq 0}$. Analogously,

$$\mathbf{A}^{-1} = \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{\Omega}_{11} & \mathbf{\Omega}_{12} \\ 0 & \mathbf{\Omega}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda}_{11} & 0 \\ \mathbf{\Lambda}_{21} & \mathbf{\Lambda}_{22} \end{pmatrix},$$

where each of the blocks is only singly infinite. It is sometimes convenient to consider the individual blocks of $\mathbf{A}$ and $\mathbf{A}^{-1}$ as operators acting on $\ell^2(\mathbb{Z})$. Thus (with slight abuse of notation) we can think of, say, $\mathbf{A}_{11}$ as

$$\begin{bmatrix} \mathbf{A}_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

We will also make use of the relations $\mathbf{\Lambda}_{22} = \mathbf{U}_{22}\mathbf{B}_{22}$, $\mathbf{L}_{11} = \mathbf{A}_{11}\mathbf{\Omega}_{11}$, $\mathbf{\Omega}_{22}^{-1} = \mathbf{U}_{22}$ and $\mathbf{\Lambda}_{22}^{-1} = \mathbf{L}_{22}$, and note that these relations imply invertibility of $\mathbf{B}_{22}$ and $\mathbf{A}_{11}$. Furthermore, for a singly-infinite matrix $\mathbf{M} = (m_{jk})_{j,k=-1}^{-\infty}$ and $n \in \mathbb{N}$, we use the notation $\mathbf{M}^{(n)}$ to refer to the sub-matrix consisting of those entries m_{jk} such that $j, k = -1, \ldots, -n$. We will apply this notation to "upper-left" block matrices such as $\mathbf{A}_{11}$ and $\mathbf{B}_{11}$. Similarly, for a singly-infinite matrix $\mathbf{M} = (m_{jk})_{j,k=0}^{\infty}$ and $n \in \mathbb{N}$, we use the notation $\mathbf{M}^{(n)}$ to refer to the sub-matrix consisting of those entries m_{jk} such that $j, k = 0, \ldots, n$. We will apply this notation to "lower-right' block matrices such as $\mathbf{A}_{22}$ and $\mathbf{B}_{22}$. We use the notation $\mathbf{m}_{n-1}$ to refer to the column vector consisting of the entries $m_{k,-n}$ such that $-n < k \leq 0$ and $\mathbf{m}_{n-1}^*$ to refer to the row vector consisting of the entries $m_{n,k}$ such that $0 \leq k < n$.

Theorem 4.1. *Let $\mathscr{A}$ be one of the algebras $\mathscr{A}_v, \mathscr{A}_v^1, \mathscr{C}_v$, introduced in Definition 2.3, and let $\mathbf{A} \in \mathscr{A}$. If $\mathbf{A} = \mathbf{LU}$ is the LU-factorization of $\mathbf{A}$ in $\mathscr{B}(\ell^2)$, then $\mathbf{L}, \mathbf{U} \in \mathscr{A}$.*

The proof of Theorem 4.1 is quite different from the approach in the previous section, it is more concrete (and longer) and implicitly utilizes the Schur complement. Before we proceed to the proof of this theorem we need some preparation.

We first prove some estimates on the entries. We shall deal with the lower-right blocks (e.g. $\mathbf{\Lambda}_{22}$) and the upper-left blocks (e.g. $\mathbf{L}_{11}$) separately. The following lemma is adapted and expanded from [31].

Lemma 4.2. *1) Let $\mathbf{B}_{22}^{-1} = (\beta_{jk})$. Then*

$$|\lambda_{kj}| \leq \left| \boldsymbol{\beta}_{k-1}^* \mathbf{B}_{22}^{(k-1)}(j) \right| \tag{14}$$

for $0 \leq j \leq k-1$.
2) Let $\mathbf{A}_{11}^{-1} = (\alpha_{jk})$. Then

$$|\ell_{jk}| \leq \left| \mathbf{A}_{11}^{(k-1)} \boldsymbol{\alpha}_{k-1}^*(j) \right| \tag{15}$$

for $1-k \leq j \leq 0$.

Proof. We first write

$$\mathbf{L}_{22} = \begin{pmatrix} \mathbf{L}_{22}^{(k-1)} & 0 & 0 \\ \boldsymbol{\ell}_{k-1}^* & \ell_{kk} & 0 \\ \mathbf{M} & \mathbf{v} & \mathbf{D} \end{pmatrix},$$

from which we see

$$\mathbf{L}_{22}^{-1} = \mathbf{\Lambda}_{22} = \begin{pmatrix} \mathbf{\Lambda}_{22}^{(k-1)} & 0 & 0 \\ \boldsymbol{\lambda}_{k-1}^* & \lambda_{kk} & 0 \\ \mathbf{N} & \mathbf{w} & \mathbf{D}^{-1} \end{pmatrix}.$$

By observing that

$$\begin{bmatrix} \boldsymbol{\ell}_{k-1}^* & \ell_{kk} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{22}^{(k-1)} \\ \boldsymbol{\lambda}_{k-1}^* \\ \mathbf{N} \end{bmatrix} = 0,$$

we obtain

$$\boldsymbol{\lambda}_{k-1}^* = -\ell_{kk}^{-1} \boldsymbol{\ell}_{k-1}^* \mathbf{\Lambda}_{22}^{(k-1)}. \tag{16}$$

Since $\mathbf{\Lambda}_{22} = \mathbf{U}_{22}\mathbf{B}_{22}$, we see that $\mathbf{L}_{22} = \mathbf{B}_{22}^{-1}\mathbf{\Omega}_{22}$. Hence,

$$\begin{aligned}
\boldsymbol{\ell}_{k-1}^{*}\mathbf{\Lambda}_{22}^{(k-1)} &= (\ell_{k1},\dots,\ell_{k,k-1})\mathbf{\Lambda}_{22}^{(k-1)} = \left(\sum_{i=1}^{k-1}\beta_{ki}\omega_{i1},\dots,\sum_{i=1}^{k-1}\beta_{ki}\omega_{i,k-1}\right)\mathbf{\Lambda}_{22}^{(k-1)} \\
&= \left(\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\beta_{ki}\omega_{ij}\lambda_{j1},\dots,\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\beta_{ki}\omega_{ij}\lambda_{j,k-1}\right) \\
&= \left(\sum_{i=1}^{k-1}\beta_{ki}\sum_{j=1}^{k-1}\omega_{ij}\lambda_{j1},\dots,\sum_{i=1}^{k-1}\beta_{ki}\sum_{j=1}^{k-1}\omega_{ij}\lambda_{j,k-1}\right) \\
&= \left(\sum_{i=1}^{k-1}\beta_{ki}b_{i1},\dots,\sum_{i=1}^{k-1}\beta_{ki}b_{i,k-1}\right) = \boldsymbol{\beta}_{k-1}^{*}\mathbf{B}_{22}^{(k-1)}. \qquad (17)
\end{aligned}$$

Combining (17) with (16) and noting that $\lambda_{kk} = 1$ (since the diagonal entries of $\mathbf{L}$ are 1), we arrive at (14).

The proof for (15) is similar. □

Lemma 4.3. *Assume* $\mathbf{A}$ *has the LU-factorization* $\mathbf{A} = \mathbf{LU}$ *in* $\mathscr{B}(\ell^2)$ *and let* $\mathscr{A}$ *be one of the algebras in Definition 2.3. Suppose* $\mathbf{A} \in \mathscr{A}$ *and the elements of* $\mathbf{L}$ *in the blocks* $\mathbf{L}_{11}$ *satisfy the appropriate decay condition. Then the elements in the block* $\mathbf{L}_{21}$ *also satisfy that decay condition.*

Proof. It is convenient for the proof of this lemma to consider the individual blocks of $\mathbf{A}$ and $\mathbf{A}^{-1}$ as operators acting on $\ell^2(\mathbb{Z})$ as indicated in the beginning of this section. Clearly, $\mathbf{L}_{21} = \mathbf{A}_{21}\mathbf{\Omega}_{11}$, and due to the upper-triangular structure of $\mathbf{U}$ we also have that $\mathbf{\Omega}_{11} = \mathbf{U}_{11}^{-1}$. Moreover $\mathbf{A}_{11} = \mathbf{L}_{11}\mathbf{U}_{11}$. Hence, since the algebra $\mathscr{A}$ to which $\mathbf{A}_{11}$ and $\mathbf{L}_{11}$ belong is inverse-closed, we have that $\mathbf{U}_{11} \in \mathscr{A}$. Thus $\mathbf{\Omega}_{11} \in \mathscr{A}$ and therefore $\mathbf{L}_{21} \in \mathscr{A}$. □

Now, to prove Theorem 4.1, we only need to show that if $\mathbf{A} \in \mathscr{A}$, then $\mathbf{L}_{11}$ and $\mathbf{L}_{22}$ are also in $\mathscr{A}$.

Proof of Theorem 4.1. 1. Suppose that $\mathbf{A} \in \mathscr{A}_v$. Then, by (14) of Lemma 4.2, we have

$$|\lambda_{kj}| \le \left|\boldsymbol{\beta}_{k-1}^{*}\mathbf{B}_{22}^{(k-1)}(j)\right|,$$

for $k, j \ge 0$. Since both $\mathbf{B}_{22}$ and $\mathbf{B}_{22}^{-1}$ have the appropriate decay property, we have

$$|\lambda_{kj}| \le \left|\boldsymbol{\beta}_{k-1}^{*}\mathbf{B}_{22}^{(k-1)}(j)\right| \le \sum_{i=1}^{k-1}|\beta_{ki}||b_{ij}| \le \sum_{i=1}^{k-1}\frac{1}{v(k-i)}\frac{1}{v(i-j)} \le \frac{1}{v(k-j)},$$

where the last inequality holds by the fact that v is sub-convolutive. Thus we have shown that the entries of $\mathbf{\Lambda}_{22}$ satisfy the appropriate decay estimate. Then, since $\mathscr{A}_v$ is inverse closed, we have that $\mathbf{L}_{22}$ satisfies the decay estimate.

By (15) of Lemma 4.2 we have

$$|\ell_{jk}| \le \left|\mathbf{A}_{11}^{(k-1)}\boldsymbol{\alpha}_{k-1}(j)\right|,$$

for $j, k \le -1$. Since both $\mathbf{A}_{11}$ and $\mathbf{A}_{11}^{-1}$ have the appropriate decay, we have

$$|\ell_{jk}| \le \left|\mathbf{A}_{11}^{(k-1)}\boldsymbol{\alpha}_{k-1}(j)\right| \le \sum_{i=-1}^{1-k}|a_{ji}||\alpha_{ik}| \le \sum_{i=1}^{k-1}\frac{1}{v(j-i)}\frac{1}{v(i-k)} \le \frac{1}{v(j-k)}.$$

Hence $\mathbf{L}_{11}$ satisfies the decay estimate. Then, by Lemma 4.3, $\mathbf{L}_{21}$ satisfies the decay condition. So all three blocks of $\mathbf{L}$ have the appropriate decay. Therefore, $\mathbf{L} \in \mathscr{A}_v$. Then, since $\mathscr{A}_v$ is inverse-closed, $\mathbf{\Lambda} \in \mathscr{A}_v$. Thus, since $\mathbf{U} = \mathbf{\Lambda}\mathbf{A}$, we have that $\mathbf{U} \in \mathscr{A}_v$.

2. Suppose that $\mathbf{A} \in \mathscr{A}_v^1$. Then, by (14), we have

$$|\lambda_{kj}| \leq \left| \boldsymbol{\beta}_{k-1}^* \mathbf{B}_{22}^{(k-1)}(j) \right|,$$

for $k, j \geq 0$. Since both $\mathbf{B}_{22}$ and $\mathbf{B}_{22}^{-1}$ have the appropriate decay property, we have

$$\begin{aligned}
\sup_{k\geq 0} \sum_{j\geq 0} |\lambda_{kj}| v(k-j) &\leq \sup_{k\geq 0} \sum_{j\geq 0} \left| \boldsymbol{\beta}_{k-1}^* \mathbf{B}_{22}^{(k-1)}(j) \right| v(k-j) \\
&\leq \sup_{k\geq 0} \sum_{j\geq 0} \sum_{i=1}^{k-1} |\beta_{ki}||b_{ij}| v(k-j) \\
&\leq \sup_{k\geq 0} \sum_{j\geq 0} \sum_{i\geq 0} |\beta_{ki}||b_{ij}| v(k-i) v(i-j) \\
&\leq \sup_{k\geq 0} \sum_{i\geq 0} |\beta_{ki}| v(k-i) \sum_{j\geq 0} |b_{ij}| v(i-j) \\
&\leq \sup_{k\geq 0} \sum_{i\geq 0} |\beta_{ki}| v(k-i) \sup_{i\geq 0} \sum_{j\geq 0} |b_{ij}| v(i-j) < \infty.
\end{aligned}$$

Note that the same argument applies if we take the supremum with respect to j and the sum with respect to k instead. Hence we have shown that the entries of $\boldsymbol{\Lambda}_{22}$ satisfy the appropriate decay estimate. Then, since $\mathscr{A}_v^1$ is inverse closed, we have that $\mathbf{L}_{22}$ satisfies the decay estimate.

Now, by (15) we have

$$|\ell_{jk}| \leq \left| \mathbf{A}_{11}^{(k-1)} \boldsymbol{\alpha}_{k-1}(j) \right|,$$

for $j, k \leq -1$. Since both $\mathbf{A}_{11}$ and $\mathbf{A}_{11}^{-1}$ have the appropriate decay, we have

$$\begin{aligned}
\sup_{k\leq -1} \sum_{j\leq -1} |\ell_{jk}| v(j-k) &\leq \sup_{k\leq -1} \sum_{j\leq -1} \left| \mathbf{A}_{11}^{(k-1)} \boldsymbol{\alpha}_{k-1}(j) \right| v(j-k) \\
&\leq \sup_{k\leq -1} \sum_{j\leq -1} \sum_{i=-1}^{1-k} |a_{ji}||\alpha_{ik}| v(j-k) \\
&\leq \sup_{k\leq -1} \sum_{j\leq -1} \sum_{i\leq -1} |a_{ji}||\alpha_{ik}| v(j-i) v(i-k) \\
&\leq \sup_{k\leq -1} \sum_{i\leq -1} |\alpha_{ik}| v(i-k) \sum_{j\leq -1} |a_{ji}| v(j-i) \\
&\leq \sup_{k\leq -1} \sum_{i\leq -1} |\alpha_{ik}| v(i-k) \sup_{i\leq -1} \sum_{j\leq -1} |a_{ji}| v(j-i) < \infty.
\end{aligned}$$

Again, the same argument applies if we take the supremum with respect to j and the sum with respect to k instead. Hence $\mathbf{L}_{11}$ satisfies the decay estimate. Then, by Lemma 4.3, $\mathbf{L}_{21}$ satisfies the decay condition. So all three blocks of $\mathbf{L}$ have the appropriate decay. Therefore, $\mathbf{L} \in \mathscr{A}_v^1$. Then, since $\mathscr{A}_v^1$ is inverse-closed, $\boldsymbol{\Lambda} \in \mathscr{A}_v$. Hence, since $\mathbf{U} = \boldsymbol{\Lambda}\mathbf{A}$, we have that $\mathbf{U} \in \mathscr{A}_v^1$.

3. Suppose that $\mathbf{A} \in \mathscr{C}_v$. By [27], we know that there is some $\gamma_{\mathbf{B}} \in \ell_v^1$ such that the entries of $\mathbf{B}_{22}$ and $\mathbf{B}_{22}^{-1}$ are bounded by $\gamma_{\mathbf{B}}$ for all n. Then, by (17), we have

$$|\lambda_{kj}| \leq \sum_{i=1}^{k-1} |\beta_{ki} b_{ij}| \leq \sum_{i\in\mathbb{Z}} \gamma_{\mathbf{B}}(k-i) \gamma_{\mathbf{B}}(i-j) = [\gamma_{\mathbf{B}} * \gamma_{\mathbf{B}}](k-j).$$

Since ℓ_v^1 is a convolution algebra, we have that $\boldsymbol{\Lambda}$ satisfies the appropriate decay estimate, and so does $\mathbf{L}$ by inverse-closedness.

Now consider $\mathbf{L}_{11}$. Then, there is some sequence $\gamma_{\mathbf{A}} \in \ell^1_v$ such that the entries of $\mathbf{A}_{11}$ and $\mathbf{A}_{11}^{-1}$ are bounded by $\gamma_{\mathbf{A}}$ for all n. Then, by (15), we have

$$|\ell_{jk}| \leq \sum_{i=-1}^{1-k} |a_{ji}\alpha_{ik}| \leq \sum_{i\in\mathbb{Z}} \gamma_{\mathbf{A}}(j-i)\gamma_{\mathbf{A}}(i-k) = [\gamma_{\mathbf{A}} * \gamma_{\mathbf{A}}](j-k).$$

Again, the result follows from the fact that ℓ^1_v is a convolution algebra.

□

5 Localization of the Cholesky, QR-, and Polar factorization

5.1 The Cholesky factorization

Corollary 5.1. *Assume that $\mathscr{A}$ is a strongly decomposable inverse closed subalgebra of $\mathscr{B}(\ell^2)$ that satisfies* (7) *and* (8) *(for example we can choose $\mathscr{A}$ to be one of the algebras in Definition 2.3). Then any positive definite matrix $\mathbf{A} \in \mathscr{A}$ admits a Cholesky factorization in $\mathscr{A}$.*

Proof. Since $\mathbf{A}$ is positive definite we can always rescale $\mathbf{A}$ so that $\|\mathbf{I} - \alpha\mathbf{A}\|_{\mathscr{B}(\ell^2)} < 1$ for some $\alpha > 0$ (note that we only claim $\|\mathbf{I} - \alpha\mathbf{A}\|_{\mathscr{B}(\ell^2)} < 1$ and not $\|\mathbf{I} - \alpha\mathbf{A}\|_{\mathscr{A}} < 1$) . Hence, by Theorem 3.4 we know that $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}, \mathbf{U} \in \mathscr{A}$ and $\mathbf{L}_{ii} = 1$, $i \in \mathbb{Z}$. Since $\mathbf{A} = \mathbf{A}^* = \mathbf{LU} = \mathbf{U}^*\mathbf{L}^*$, there holds $\mathbf{A} = \mathbf{LDL}^*$, where $\mathbf{D}$ has to be positive definite, because $\mathbf{A}$ is. Then $\mathbf{A} = (\mathbf{LD}^{\frac{1}{2}})(\mathbf{LD}^{\frac{1}{2}})^*$ is the Cholesky factorization of $\mathbf{A}$, and obviously $\mathbf{LD}^{\frac{1}{2}} \in \mathscr{A}$. □

We note that for the special case when $\mathscr{A}$ is the Jaffard algebra and $\mathbf{A}$ is finite-dimensional, Corollary 5.1 recovers Lemma A.1 in [31]. Furthermore, Corollary 5.1 contains as special case Lemma II.3.2. of [23], by setting $\mathscr{A} = \mathcal{C}_v$ with $v \equiv 1$.

5.2 The QR-factorization

Corollary 5.2. *Assume that $\mathscr{A}$ is a strongly decomposable inverse closed subalgebra of $\mathscr{B}(\ell^2)$ that satisfies* (7) *and* (8) *(for example we can choose $\mathscr{A}$ to be one of the algebras in Definition 2.3). Let $\mathbf{A} \in \mathscr{A}$ be such that $\mathbf{A}$ is invertible and suppose that its QR-decomposition $\mathbf{A} = \mathbf{QR}$ exists. Then $\mathbf{Q}, \mathbf{R} \in \mathscr{A}$.*

Proof. Consider the matrix $\mathbf{V} = \mathbf{A}^*\mathbf{A} = \mathbf{RQ}^*\mathbf{QR} = \mathbf{R}^*\mathbf{R}$. Note that $\mathbf{V} \in \mathscr{A}$ and it is invertible, positive definite, and hermitian. Let $\mathbf{V} = \mathbf{C}^*\mathbf{C}$ be its Cholesky factorization. Then $\mathbf{R} = \mathbf{DC}$, where $\mathbf{D}$ is a unitary diagonal matrix. So $\mathbf{R} \in \mathscr{A}$, which implies $\mathbf{R}^{-1} \in \mathscr{A}$. Since $\mathscr{A}$ is an algebra, it follows that $\mathbf{Q} \in \mathscr{A}$. □

It is claimed (without proof) in Section II.B of [36] that the Gram-Schmidt orthogonalization would destroy the localization properties of a given set of vectors, while applying the Löwdin orthogonalization procedure (which essentially amounts to applying the inverse square root of a matrix) preserves localization. Corollary 5.2 shows that this claim is incorrect, since Gram-Schmidt does preserve localization as well.

5.3 The Polar factorization

Theorem 5.3. *Let $\mathscr{A}$ be one of the decay algebras in Definition 2.3. Let $\mathbf{A} \in \mathscr{A}$ be invertible and suppose its polar decomposition $\mathbf{A} = \mathbf{PU}$ exists, where $\mathbf{P}$ is positive definite. Then $\mathbf{P}, \mathbf{U} \in \mathscr{A}$.*

Proof. Recall that $\mathbf{P} = (\mathbf{A}^*\mathbf{A})^{-\frac{1}{2}}$. Then, the Banach square root theorem [18] implies that $\mathbf{P} \in \mathscr{A}$. So, since $\mathscr{A}$ is inverse-closed, we have $\mathbf{P}^{-1} \in \mathscr{A}$, which in turn implies $\mathbf{U} \in \mathscr{A}$. □

A close relative of the polar factorization is the singular value decomposition $\mathbf{A} = \mathbf{V}\mathbf{\Sigma}\mathbf{W}^*$. Since $\mathbf{V}\mathbf{W}^* = \mathbf{U}$, where $\mathbf{U}$ is the partial isometry from the polar decomposition of $\mathbf{A}$, we can say that $\mathbf{V}\mathbf{W}^*$ inherits the decay of $\mathbf{A}$. Of course, this says nothing about the localization of $\mathbf{V}$ and $\mathbf{W}$. Indeed, as already mentioned in the introduction, it is clear that without any additional assumptions the individual factors $\mathbf{V}$ and $\mathbf{W}$ cannot inherit the decay properties of $\mathbf{A}$. For example, note that the identity matrix $\mathbf{I}$ can be written $\mathbf{I} = \mathbf{V}\mathbf{I}\mathbf{V}^*$ for any arbitrary unitary matrix $\mathbf{V}$. Eigenvector localization is a heavily studied subject, in particular in connection with the famous phenomenon of Anderson localization (see e.g. [45] and its many references). We will report on eigenvector localization in connection with Banach algebras with off-diagonal decay in a forthcoming paper.

6 Factorizations, Localization, and Functional Calculi

The question whether matrix functions, such as the matrix exponential, inherit the localization properties of a matrix has been investigated for the cases of band matrices in [11, 32, 12]. This issue is of relevance in various areas of numerical mathematics [11, 32]. What can we say about functions of a matrix if the matrix is not banded, but has some form of off-diagonal decay?

In the previous sections we were able to prove that localization of various matrix factors is preserved in general under reasonably mild conditions and regardless of how these factors were obtained. In practice, the factors are often obtained via some kind of functional calculus. Since many useful functional calculi restrict naturally to the inverse closed subalgebras, the factors will automatically inherit the property.

For example, in case of Laurent matrices a necessary and sufficient condition for factorization of a matrix $\mathbf{A} = (a_{jk}) \in \mathscr{C}_1$ is given in terms of its symbol $\sigma_{\mathbf{A}} \in C(\mathbb{T})$, $\sigma_{\mathbf{A}}(\theta) = \sum_{n\in\mathbb{Z}} c_n\theta^n$, $\theta \in \mathbb{T}$, $c_n = a_{j,j-n}$, $j,k,n \in \mathbb{Z}$. Recall that in this case $\mathbf{A}$ is the matrix of an operator of multiplication by $\sigma_{\mathbf{A}}$ in $L^2(\mathbb{T})$ in the standard Fourier basis. Thus, the LU-factorization of $\mathbf{A}$ is equivalent to the *spectral* factorization $\sigma_{\mathbf{A}} = \sigma_{\mathbf{L}}\sigma_{\mathbf{U}}$, where $\sigma_{\mathbf{L}}$ admits a non-vanishing continuous extension that is holomorphic in $\mathbb{D}$ and $\sigma_{\mathbf{U}}$ admits a non-vanishing continuous extension that is holomorphic in $\mathbb{C}\backslash\overline{\mathbb{D}}$. The factorization condition then is the well-known Paley-Wiener condition [33]

$$\int_{-\pi}^{\pi} \ln \sigma_{\mathbf{A}}(e^{it})dt > -\infty,$$

and the (extension of the) symbol of one of the factors, $\sigma_{\mathbf{L}}$, is then computed via

$$\sigma_{\mathbf{L}}(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \sigma_{\mathbf{A}}(e^{it})\frac{e^{it}+z}{e^{it}-z}dt\right), \quad z \in \mathbb{D}.$$

Since the series defining the matrix exponential will converge in any Banach algebra of matrices, the factors will belong to a subalgebra $\mathscr{A}$ as long as the logarithm of the symbol of the matrix stays in the isomorphic subalgebra of $C(\mathbb{T})$. We cite [33, 14] for more information and detailed estimates on the coefficients of the spectral factors.

Riesz-Dunford functional calculus is another natural example of a calculus that restricts to a Banach subalgebra. Indeed, if $\mathbf{A} \in \mathscr{A}$ and $\mathscr{A}$ is inverse closed then

$$\mathbf{B} = \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda\mathbf{I} - \mathbf{A})^{-1}d\lambda \in \mathscr{A},$$

where f is holomorphic inside a positively oriented contour Γ surrounding the spectrum or a spectral component of $\mathbf{A}$. To give a concrete example, it now follows easily that the matrix exponential $\exp(\mathbf{A}) \in \mathscr{A}$ if $\mathbf{A} \in \mathscr{A}$, whenever $\mathscr{A}$ is an inverse-closed Banach algebra, such as e.g. one of the Banach algebras in Definition 2.3.

Yet another example of a useful functional calculus is in some sense a generalization of the approach in the Laurent case. Given $\mathbf{A} \in \mathscr{B}_c$ and $h \in L^1(\mathbb{R})$, we can define a Banach $L^1(\mathbb{R})$-module structure [9] via

$$h\mathbf{A} = \int_{\mathbb{T}} h(\theta)f_{\mathbf{A}}(\theta^{-1})d\theta = \int_{\mathbb{T}} h(\theta)M(\theta^{-1})\mathbf{A}M(\theta)d\theta.$$

This structure extends to a closed operator calculus as in [8]. Again, we have $h\mathbf{A} \in \mathscr{A}$ as long as $\mathbf{A}$ is in a Banach algebra $\mathscr{A}$ that is invariant under the modulation representation.

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References

[1] K.T. Andrews, P. W. Smith, and J. D. Ward. LU factorization of operators on l_1. *Proceedings of the AMS*, 98(2):247–252, 1986.

[2] W. Arveson. Interpolation problems in nest algebras. *J. Functional Analysis*, 20(3):208–233, 1975.

[3] M.A. Barkar and I. Gohberg. On factorization of operators relative to a discrete chain of projectors in Banach space. *Amer. Math. Soc. Transl. (2)*, 90:81–103, 1970.

[4] A. G. Baskakov. Wiener's theorem and asymptotic estimates for elements of inverse matrices. *Funktsional. Anal. i Prilozhen.*, 24(3):64–65, 1990.

[5] A. G. Baskakov. Estimates for the elements of inverse matrices, and the spectral analysis of linear operators. (Russian). *Izv. Ross. Akad. Nauk Ser. Mat.*, 61(6):3–26, 1997. Translation in Izv. Math. 61 (1997), no. 6, pp. 1113–1135.

[6] A. G. Baskakov. Theory of representations of Banach algebras, and abelian groups and semigroups in the spectral analysis of linear operators. *Sovrem. Mat. Fundam. Napravl.*, 9:3–151 (electronic), 2004. English translation: J. Math. Sci. (N. Y.) 137 (2006), no. 4, 48855036.

[7] A. G. Baskakov and I. A. Krishtal. Harmonic analysis of causal operators. *Funktsional. Anal. i Prilozhen.*, 40(1):65–68, 2006. English translation: Funct. Anal. Appl. 40 (2006), no. 1, 5255.

[8] A. G. Baskakov and I. A. Krishtal. Memory Estimation of Inverse Operators. *ArXiv e-prints*, March 2011.

[9] A.G. Baskakov and I.A. Krishtal. Harmonic analysis of causal operators and their spectral properties. (Russian). *Izv. Ross. Akad. Nauk Ser. Mat.*, 69(3):3–54, 2005. Translation in Izv. Math., 69(3):439–486, 2005.

[10] M. Benzi, P. Boito, and N. Razouk. Decay properties of spectral projectors with applications to electronic structure. *ArXiv e-prints*, 2012.

[11] M. Benzi and G. Golub. Bounds for the entries of matric functions with applications to preconditioning. *BIT*, 39(3):417–438, 1999.

[12] M. Benzi and N. Razouk. Decay bounds and $\mathcal{O}(n)$ algorithms for approximating functions of sparse matrices. *Electr. Trans. Numer. Anal.*, 28:16–39, 2007.

[13] I. A. Blatov. On operator algebras with pseudo-sparse matrices, and their applications. (Russian). *Sibirsk. Mat. Zh.*, 37(1):36–59, i, 1996. Translation in Siberian Math. J. 37 (1996), no. 1, 32–52.

[14] H. Boche and V. Pohl. On the boundedness behavior of the spectral factorization in the Wiener algebra for FIR data. In *15th European Signal Processing Conference (EUSIPCO 2007)*, Poznan, Poland, 2007.

[15] C. K. Chui, J. D. Ward, and P. W. Smith. Cholesky factorization of positive definite bi-infinite matrices. *Numer. Funct. Anal. Optim.*, 5(1):1–20, 1982.

[16] R.R. Coifman and M. Maggioni. Diffusion wavelets. *Appl. Comp. Harm. Anal.*, 21(1):53–94, 2006.

[17] K. de Leeuw. Fourier series of operators and an extension of the F. and M. Riesz theorem. *Bull. Amer. Math. Soc.*, 79(2):342–344, 1973.

[18] L.T. Gardner. Square roots in Banach algebras. *Proc. Amer. Math. Soc.*, 17:132–134, 1966.

[19] I. Gel'fand, D. Raikov, and G. Shilov. *Commutative normed rings*. Chelsea Publishing Co., New York, 1964.

[20] L. Genovese and et al. Daubechies wavelets as a basis set for density functional pseudopotential calculations. *J. Chem. Phys.*, 129(1):014109, 2008.

[21] S. Goedecker. Decay properties of the finite-temperature density matrix in metals. *Phys. Rev. B*, 58:3501–3502, 1998.

[22] I. C. Gohberg and I. A. Fel′dman. *Proektsionnye metody resheniya uravneniiBinera-Khopfa*. With an appendix by A. A. Semencul. Academy of Sciences of the Moldavian SSR. Institute of Mathematics with Computing Center. Akad. Nauk Moldav. SSR, Kishinev, 1967.

[23] I. Gohberg, M. A. Kaashoek, and H. J. Woerdeman. The band method for positive and strictly contractive extension problems: an alternative version and new applications. *Integral Equations Operator Theory*, 12(3):343–382, 1989.

[24] I. Gohberg and J. Laĭterer. The factorization of operator functions relative to a contour. II. the canonical factorization of operator functions that are close to the identity operator. *Math. Nachr.*, 54:41–74, 1972. (Russian).

[25] K. Gröchenig. Wiener's lemma: Theme and variations. An introduction to spectral invariance and its applications. In P. Massopust and B. Forster, editors, *Four Short Coures on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*. Birkhäuser, 2010.

[26] K. Gröchenig and A. Klotz. Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices. *Constr. Approx.*, 32(3):429–466, 2010.

[27] K. Gröchenig and A. Klotz. Norm-Controlled Inversion in Smooth Banach Algebras, I. *ArXiv e-prints*, July 2012.

[28] K. Gröchenig and M. Leinert. Wiener's lemma for twisted convolution and Gabor frames. *J. Amer. Math. Soc.*, 17:1–18, 2004.

[29] K. Gröchenig and M. Leinert. Symmetry of matrix algebras and symbolic calculus for infinite matrices. *Trans. Amer. Math. Soc.*, 358:2695–2711, 2006.

[30] K. Gröchenig, Z. Rzeszotnik, and T. Strohmer. Convergence analysis of the finite section method and banach algebras of matrices. *Integral Equations and Operator Theory*, 67(2):183–202, 2010.

[31] P. Hall and J. Jin. Innovated higher criticism for detecting sparse signals in correlated noise. *Ann. Statist.*, 38(3):1686–1732, 2010.

[32] A. Iserles. How large is the exponential of a banded matrix? *New Zealand J. Math.*, 29:177–192, 2000.

[33] B. Jacob and J. R. Partington. On the boundedness and continuity of the spectral factorization mapping. *SIAM J. Control Optim.*, 40(1):88–106, 2001.

[34] S. Jaffard. Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.

[35] T. Kailath, A. H. Sayed, and B. Hassibi. *Linear Estimation*. Prentice Hall, 2000.

[36] W. Kozek and A. Molisch. Nonorthogonal pulseshapes for multicarrier communications in doubly dispersive channels. *IEEE J. Sel. Areas Comm.*, 16(8):1579–1589, 1998.

[37] V.G. Kurbatov. *Functional Differential Operators and Equations*, volume 473 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Dotrecht, Boston, London, 1999.

[38] C. Le Bris. Computational chemistry from the perspective of numerical analysis. *Acta Numerica*, 14:363–444, 2005.

[39] M. Lindner. *Infinite Matrices and their finite sections*. Frontiers in Mathematics. Birkäuser, 2006.

[40] S. Mallat. *A wavelet tour of signal processing*. Academic Press, San Diego, CA, 1998.

[41] R.M. Martin. *Electronic Structure. Basic Theory and Practical Methods*. Cambridge University Press, Cambridge, UK, 2004.

[42] C. Mee, G. Rodriguez, and S. Seatzu. LDU factorization results for bi-infinite and semi-infinite scalar and block Toeplitz matrices. *CALCOLO*, 33:307–335, 1996.

[43] P. Schniter, S.-J. Hwang, S. Das, and A. P. Kannu. Equalization of time-varying channels. In F. Hlawatsch and G. Matz, editors, *Wireless Communications over Rapidly Time-Varying Channels*, chapter 6. Elsevier Academic Press, 2011.

[44] J. Sjöstrand. Wiener type algebras of pseudodifferential operators. In *Séminaire sur les Équations aux Dérivées Partielles, 1994–1995*, pages Exp. No. IV, 21. École Polytech., Palaiseau, 1995.

[45] G. Stolz. An introduction to the mathematics of Anderson localization. *Preprint, arXiv:1104.2317*, 2011.

[46] G. Strang. Triangular factorizations: The algebra of elimination, 2011. Preprint.

[47] T. Strohmer. Pseudodifferential operators and Banach algebras in mobile communications. *Applied and Computational Harmonic Analysis*, 20(2):237–249, 2006.

[48] Q. Sun. Wiener's lemma for infinite matrices. *Trans. Amer. Math. Soc.*, 359(7):3099–3123, 2007.

[49] N. Wiener. Tauberian theorems. *Ann. of Math. (2)*, 33(1):1–100, 1932.