Stability Results in the Theory of Relativistic Stars

Asad Lodhia

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Abstract

In this article, we discuss, at an accessible level, the relativistic theory of stars. We overview the history of the problem, paying particular attention to the monumental results that lay at the foundations of the theory. Our primary attention will be directed towards the Buchdahl Stability Theorem, as well as the work of Chandrasekhar on relativistic stability. We assume a basic knowledge of General Relativity – up to the Schwarzschild solution, along with an understanding of the techniques of perturbation theory.

1 Introduction – The History of the Problem

Einstein's General Theory of Relativity profoundly impacted our perception of the Universe. General Relativistic corrections helped us understand the precession of the perihelion of Mercury, as well as the deflection of light passing by a massive body. Since the corrections of General Relativity tended to be extremely small, it was not clear whether or not Relativity had any bearing on general astrophysical phenomenon. Indeed, Chandrasekhar's famous work on the mass limit of white dwarf stars restricted itself to the study of Newtonian Stars.

Another curiousity was the mysterious mass limit that creates a singularity in the Schwarzschild solution of the Einstein equations. One may ask, if it is possible that there are celestial objects whose radii are less than the Schwarzschild limit? If so, do the objects form stable planets? Any attempt to explore such a "planet" would be futile, since its escape speed would exceed the speed of light – an unbreakable barrier. Can one at least then assess what kind of objects could be allowed to live "inside" the Schwarzschild radius?

Buchdahl's theorem indicates that if we consider a stable star (composed of a perfect fluid in thermodynamic equilibrium), then its radius *necessarily* exceeds the Schwarszchild radius. That is to say, a star never has the property that its escape speed is larger than light as long as the assumptions that we took are fulfilled. We will begin by deriving the equations of hydrodynamic stability of a star (the Tolman-Oppenheimer-Volkoff equations) and then derive the Buchdahl Result.

2 The Tolmann-Oppenheimer-Volkoff Equations

We will begin with a spherically symmetric system, i.e. a metric of the form:

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

or in matrix form:

$$(g_{\alpha\beta}) = \begin{bmatrix} e^\nu & 0 & 0 & 0 \\ 0 & -e^\lambda & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}.$$

Recall, the Einstein field equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{-8\pi G}{c^4}T_{\alpha\beta},$$

where

$$R_{\alpha\beta} = \Gamma^\rho_{\alpha\rho,\beta} - \Gamma^\rho_{\alpha\beta,\rho} + \Gamma^\rho_{\alpha\mu}\Gamma^\mu_{\beta\rho} - \Gamma^\mu_{\alpha\beta}\Gamma^\rho_{\mu\rho},$$

and $\Gamma^\rho_{\alpha\beta}$ are the Christoffel connection symbols given by

$$\Gamma^\rho_{\alpha\beta} = \frac{1}{2}g^{\rho\mu}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})$$

(commas represent derivatives, and $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$), and

$$R = g^{\mu\nu}R_{\nu\mu}.$$

The value of $T_{\alpha\beta}$ is the stress-energy tensor of a perfect fluid

$$T_{\alpha\beta} = (\epsilon + p)u_\alpha u_\beta + pg_{\alpha\beta},$$

where u_α is the four-velocity, ϵ is the energy density, and p is the pressure of the fluid. When the fluid is static, there are no other contributions to the fluid velocity except the time-like component (u_0); the velocity is normalized so that $u^\alpha u_\alpha = -1$. This gives us:

$$(T_{\alpha\beta}) = \begin{bmatrix} e^\nu\epsilon & 0 & 0 & 0 \\ 0 & -e^\lambda p & 0 & 0 \\ 0 & 0 & -r^2 p & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta p \end{bmatrix}.$$

When we raise the index of both sides of the Einstein equations we obtain

$$R^\alpha_\beta - \frac{1}{2}R\delta^\alpha_\beta = -\frac{8\pi G}{c^4}T^\alpha_\beta.$$

Now, we have a more simpler formula for T^α_β

$$(T^\alpha_\beta) = \begin{bmatrix} -\epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}.$$

Now, we will solve for the Christoffel symbols using the standard trick. Let us define a curve as a function of some parameter $0 \leq \tau \leq \tau_0$. the arclength of this curve ℓ is given by:

$$\ell = \int_0^{\tau_0} d\tau \sqrt{e^{\nu}\dot{t}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2 - e^{\lambda}\dot{r}^2},$$

where the dot refers to derivatives with respect to τ. We find the geodesics of this curve by applying the Euler-Lagrange equations to the integrand (we can square the integrand and still obtain the same geodesics in this case). Once we apply the Euler Lagrange equations to the integrand we compare them with the geodesic equation

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\lambda\nu}\frac{dx^{\lambda}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0$$

to find out the Christoffel symbols. We obtain (assuming λ and ν are functions of r and t alone) the following:

$$\begin{array}{lll}
\Gamma^t_{tt} = \frac{1}{2}\partial_t\nu & \Gamma^t_{rr} = \frac{1}{2}e^{\lambda-\nu}\partial_t\lambda & \Gamma^t_{rt} = \frac{1}{2}\partial_r\nu \\
\Gamma^r_{tt} = -\frac{1}{2}e^{\nu-\lambda}\partial_r\nu & \Gamma^r_{tr} = \frac{1}{2}\partial_t\lambda & \Gamma^r_{rr} = \frac{1}{2}\partial_r\lambda \\
\Gamma^r_{\theta\theta} = -re^{-\lambda} & \Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-\lambda} & \Gamma^{\theta}_{r\theta} = \frac{1}{r} \\
\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^{\phi}_{r\phi} = \frac{1}{r} & \Gamma^{\phi}_{\theta\phi} = \cot\theta,
\end{array}$$

all other components are either zero or can be determined by symmetry of the lower two indices.

Now for the Ricci Tensor $R_{\alpha\beta}$, we calculate:

$$\begin{array}{l}
R_{tt} = \frac{1}{2}[\partial_t^2\lambda + \frac{1}{2}(\partial_t\lambda)^2 - \frac{1}{2}\partial_t\nu\partial_t\lambda] + \frac{1}{2}e^{\nu-\lambda}[\partial_r^2\nu + \frac{1}{2}(\partial_r\nu)^2 - \frac{1}{2}\partial_r\nu\partial_r\lambda + \frac{2}{r}\partial_r\nu] \\
R_{rr} = -\frac{1}{2}[\partial_r^2\nu + \frac{1}{2}(\partial_r\nu)^2 - \frac{1}{2}\partial_r\lambda\partial_r\nu - \frac{2}{r}\partial_r\lambda] + \frac{1}{2}e^{\lambda-\nu}[\partial_t^2\lambda + \frac{1}{2}(\partial_t\lambda)^2 - \frac{1}{2}\partial_t\lambda\partial_t\nu] \\
R_{tr} = \frac{1}{r}\partial_t\lambda \\
R_{\theta\theta} = e^{-\lambda}[\frac{1}{2}r(\partial_r\lambda - \partial_r\nu) - 1] + 1 \\
R_{\phi\phi} = R_{\theta\theta}\sin^2\theta,
\end{array}$$

where, again all other components are zero or can be determined by the symmetry of the indices. Now the Ricci Scalar R is

$$R = e^{-\nu}R_{tt} - e^{-\lambda}R_{rr} - \frac{R_{\theta\theta}}{r^2} - \frac{R_{\phi\phi}}{r^2\sin^2\theta},$$

which is:

$$R = e^{-\lambda}\left[\partial_r^2\nu + \frac{1}{2}(\partial_r\nu)^2 - \frac{1}{2}\partial_r\lambda\partial_r\nu + \frac{2}{r}(\partial_r\nu - \partial_r\lambda) + \frac{2}{r^2}(1 - e^{\lambda})\right].$$

There is one more set of equations that arise from the condition that the covariant divergence of $T_{\mu\nu}$ must be zero (the Bianchi Identity) which means:

$$T^i_{j;i} = \frac{\partial T^i_j}{\partial x^i} + \Gamma^i_{\alpha i}T^{\alpha}_j - \Gamma^{\alpha}_{ij}T^i_{\alpha} = 0$$

the semicolon denotes a covariant derivative. The result is the two equations (we keep certain off-diagonal terms in T^α_β, namely, the T^t_r and T^r_t components for reasons that will become clear later):

$$\frac{\partial T^t_t}{\partial t} + \frac{\partial T^r_t}{\partial r} + \frac{1}{2}(T^t_t - T^r_r)\frac{\partial \lambda}{\partial t} + T^r_t\left[\frac{1}{2}\frac{\partial}{\partial r}(\lambda + \nu) + \frac{2}{r}\right] = 0,$$

$$\frac{\partial T^t_r}{\partial t} + \frac{\partial T^r_r}{\partial r} + \frac{1}{2}T^t_r\frac{\partial}{\partial t}(\lambda + \nu) + \frac{1}{2}(T^r_r - T^t_t)\frac{\partial \nu}{\partial r} + \frac{2}{r}(T^r_r - p) = 0,$$

where we have used $T^r_t = -e^{\nu-\lambda}T^t_r$ and $T^\theta_\theta = T^\phi_\phi = p$.

We will utilize these formulas later, but for the equation of stellar equilibrium, there is no time dependence for ν and λ (so the $R_{\alpha\beta}$ becomes diagonal). Also, we will use the subscript zero to denote the functions at stellar equilibrium. Now we have two independent field equations:

$$R^t_t - \frac{1}{2}R = -\frac{8\pi G}{c^4}T^t_t,$$

which implies

(1) $$-\frac{1}{r^2}\frac{d}{dr}(re^{-\lambda_0}) + \frac{1}{r^2} = \frac{8\pi G}{c^4}\epsilon_0$$

and

$$R^r_r - \frac{1}{2}R = -\frac{8\pi G}{c^4}T^r_r,$$

which implies

(2) $$\frac{e^{-\lambda_0}}{r}\frac{d\nu_0}{dr} = \frac{1}{r^2}(1 - e^{-\lambda_0}) + \frac{8\pi G}{c^4}p_0.$$

We now use the fact that the covariant divergence of T^α_β is zero (we modify the previous equation so that the non-diagonal components are zero):

(3) $$\frac{dp_0}{dr} + \frac{1}{2}(p_0 + \epsilon_0)\frac{d\nu_0}{dr} = 0.$$

We define the mass $M = M(r)$ as:

(4) $$M = \frac{4\pi}{c^2}\int_0^r \epsilon_0 \eta^2 d\eta$$

and integrating 1 we obtain

$$e^{-\lambda_0} = 1 - \frac{2GM}{rc^2},$$

which we insert into 2 and 3 to obtain:

(5) $$\frac{dp_0}{dr} = -\frac{1}{c^2}(p_0 + \epsilon_0)\left(\frac{GM}{r^2} + \frac{4\pi G p_0 r}{c^2}\right)\left(1 - \frac{2GM}{rc^2}\right)^{-1}.$$

Equations 4 and 5 are the Tolmann-Oppenheimer-Volkoff equations of stellar stability.

The Tolmann-Oppenheimer-Volkoff equations are not enough to close the system, we need an equation of state that will help determine the pressure in terms of the density. This is often taken to be the polytrope equation when modelling stars:

$$p = K\epsilon^{\gamma},$$

where K and γ are constants.

3 The Buchdahl Stability Limit

Obtaining the Buchdahl Stability limit is straightforward given the Tolmann-Oppenheimer-Volkoff equations. The criterion says that the radius R of the star is:

$$R \geq \frac{9}{8} R_S$$

where R_S is the Schwarzschild radius. One thing to note is that this stability criterion does not depend on any equation of state relating p and ϵ. This stability theorem relies on the following:

- The density, ϵ_0, and the pressure p_0 at the center of the star must be finite
- The density must decrease as a function of r, i.e. $\epsilon_0'(r)$ must be negative while inside the star
- The density must be zero outside the star's radius R
- e^{ν_0} and e^{λ_0} must be positive

which are all reasonable assumptions for a stable star.

In order to proceed we must make use of 2 and 3 rewritten as

$$\frac{8\pi G}{c^4} p_0 = \frac{1}{r}\left(1 - \frac{2GM}{c^2 r}\right)\frac{d\nu_0}{dr} - \frac{2GM}{c^2 r^3}, \tag{6}$$

and

$$\frac{1}{2}\frac{d\nu_0}{dr} = \frac{-1}{\epsilon_0 + p_0}\frac{dp_0}{dr}. \tag{7}$$

Defining the function

$$f = e^{\nu_0/2},$$

we have

$$\frac{f'}{f} = \frac{1}{2}\frac{d\nu_0}{dr},$$

from which it follows

$$\frac{d}{dr}\left[\frac{1}{r}\sqrt{1 - \frac{2GM}{c^2 r}}\, f'\right] = \frac{f}{\sqrt{1 - \frac{2GM}{c^2 r}}}\frac{d}{dr}\frac{GM}{c^2 r^3}. \tag{8}$$

Before deriving 8, let us see how to use it to derive the Buchdahl limit. Rewriting 6 in terms of f

$$\frac{8\pi G}{c^4}p_0 = \frac{2}{r}\left(1-\frac{2GM}{c^2r}\right)\frac{f'}{f} - \frac{2GM}{c^2r^3},$$

we note that since $\epsilon_0' < 0$ and $\epsilon_0(0) < \infty$ we have

$$\frac{M}{r^3} = \frac{1}{r^3}\int_0^r 4\pi\epsilon_0\eta^2 d\eta \leq \frac{4\pi\epsilon_0(0)}{3} < \infty$$

and since $e^{-\lambda_0} > 0$ we have

$$1-\frac{2GM}{c^2r} > 0,$$

so that

$$\frac{f'}{rf} < \infty,$$

(p_0 must be finite). Now writing

$$M = \frac{4\pi}{3}\bar{\epsilon_0}r^3,$$

where $\bar{\epsilon_0}$ is the average density which satisfies $\bar{\epsilon_0} > \epsilon_0$ (since $\epsilon_0' < 0$) we have

$$\frac{d}{dr}\frac{M}{r^3} = \frac{4\pi}{r}(\epsilon_0 - \bar{\epsilon_0}) < 0.$$

Hence, if we analyze the right hand side of 8 we see

$$\frac{d}{dr}\left(\frac{f'}{r}\sqrt{1-\frac{2GM}{c^2r}}\right) \leq 0$$

and is finite, upon integrating from a radius r to the radius of the star R we have

$$\frac{f'(R)}{R}\sqrt{1-\frac{2GM(R)}{c^2R}} - \frac{f'(r)}{r}\sqrt{1-\frac{2GM(r)}{c^2r}} < 0.$$

Now at $r=R$, f becomes the Schwarzschild exterior solution which means that

$$f(R) = \sqrt{1-\frac{2GM(R)}{c^2R}},$$

and

$$f'(R) = \frac{GM(R)}{c^2R^2\sqrt{1-\frac{2GM(R)}{c^2R}}}.$$

Therefore

$$\frac{GM(R)}{c^2R^3} \leq \frac{f'(r)}{r}\sqrt{1-\frac{2GM(r)}{c^2r}},$$

so that

$$f'(r) \geq \frac{r}{c^2R^3}\frac{GM(R)}{\sqrt{1-\frac{2GM(r)}{c^2r}}},$$

which we integrate from $r = 0$ to R

$$f(R) - f(0) \leq \frac{GM(R)}{c^2R^3}\int_0^R \frac{\eta d\eta}{\sqrt{1-\frac{2GM(\eta)}{c^2\eta}}},$$

rearranging and using the fact that

$$M(r) = \frac{4\pi}{3}\bar{\epsilon}_0(r)r^3 \geq \frac{M(R)r^3}{R^3}$$

we have

$$0 \leq f(0) \leq \sqrt{1-\frac{2GM(R)}{c^2R}} - \frac{GM(R)}{c^2R^3}\int_0^R \frac{\eta d\eta}{\sqrt{1-\frac{2GM(R)r^2}{c^2R^3}}}$$

giving us

$$\frac{3}{2}\sqrt{1-\frac{2GM(R)}{c^2R}} \geq \frac{1}{2}$$

which yields the Buchdahl Result

$$R \geq \frac{9}{8}R_S.$$

Now to prove 8, we begin by differentiating 6 to obtain

$$\frac{4\pi G}{c^2}p_0' = \frac{d}{dr}\left(\frac{1}{r}\left(1-\frac{2GM}{c^2r}\right)\frac{f'}{f}\right) - \frac{d}{dr}\frac{GM}{c^2r^3},$$

which using 7 we see that

$$-\frac{4\pi G f'(p_0+\epsilon_0)}{c^2 f} = \frac{d}{dr}\left(\frac{1}{r}\left(1-\frac{2GM}{c^2r}\right)\frac{f'}{f}\right) - \frac{d}{dr}\frac{GM}{c^2r^3}.$$

We note that

$$\begin{aligned}\frac{d}{dr}\left(\frac{1}{r}\left(\sqrt{1-\frac{2GM}{c^2r}}\right)^2\frac{f'}{f}\right) &= \frac{d}{dr}\left(\frac{\sqrt{1-\frac{2GM}{c^2r}}f'}{r}\right)\frac{\sqrt{1-\frac{2GM}{c^2r}}}{f} \\ &+\frac{f'\sqrt{1-\frac{2GM}{c^2r}}}{r}\frac{d}{dr}\left(\frac{\sqrt{1-\frac{2GM}{c^2r}}}{f}\right),\end{aligned}$$

and that

$$\begin{aligned}\frac{d}{dr}\left(\frac{\sqrt{1-\frac{2GM}{c^2r}}}{f}\right) &= \frac{r}{fc^2\sqrt{1-\frac{2GM}{c^2r}}}\left(-4\pi G\epsilon_0 + \frac{GM}{c^2r^3}\right) \\ &-\frac{f'}{f^2}\sqrt{1-\frac{2GM}{c^2r}},\end{aligned}$$

which when inserted into the second equation from the previous sentence, we obtain (after simplification):

$$-\frac{4\pi G f' p_0}{c^2\sqrt{1-\frac{2GM}{c^2 r}}} = \frac{d}{dr}\left(\frac{f'\sqrt{1-\frac{2GM}{c^2 r}}}{r}\right) - \frac{f}{\sqrt{1-\frac{2GM}{c^2 r}}}\frac{d}{dr}\left(\frac{GM}{c^2 r^3}\right)$$
$$-\frac{f'^2}{rf}\sqrt{1-\frac{2GM}{c^2 r}} + \frac{f' GM}{c^2 r^3\sqrt{1-\frac{2GM}{c^2 r}}},$$

and by finally using 6 this yields us 8.

4 Stability of the Star

In order to obtain the stability limit of the star, we must perturb the solution of the star in stellar equilibrium. The method that was used by Chandrasekhar was the following: he began with the Einstein field equations (from the previous section) and allowed a non-zero radial velocity for the fluid. The radial velocity of the fluid creates off-diagonal terms in T^α_β, and causes ν and λ to depend on time. Chandrasekhar accounted for each dependence on time by perturbing each variable λ, ν, p, and ϵ. He then proceeded to solve each perturbation to first order in terms of non-perturbed quantities. To obtain the equation of state, he used the conservation of particle number as a condition. Letting ξ denote the integral of the radial velocity w.r.t time, he obtains a differential equation of the form

$$-\frac{\partial^2 \xi}{\partial t^2} = A\xi$$

where A is a linear differential operator that is self adjoint, i.e. satisfies:

$$< \xi_i, A\xi_j > = < A\xi_i, \xi_j >$$

where $< \cdot, \cdot >$ is an inner product on function space; physicists will be familiar with the L^2 inner product used in quantum mechanics. Chandrasekhar then assumed ξ has time dependence of the form $e^{i\omega t}$, which we insert into the above equation to obtain a condition on ω^2. The Rayleigh-Ritz principle insures us that for any test function ξ satisfying the boundary conditions of the problem,

$$\omega^2 \leq \frac{< \xi, A\xi >}{< \xi, \xi >};$$

if this tells us that ω^2 is negative, then ξ grows without bound, so there is a perturbation that is unstable. This method is the same, in spirit, to the methods used in phase plane analysis to determine the stability of fixed points – except in this context the phase plane is instead function space.

4.1 The Perturbed Einstein Equations

We introduce a small radial oscillation to the star, which means that u^r is now non-zero. Now, λ, ν, p, and ϵ are all perturbed from stability, so we denote:

$$\lambda = \lambda_0 + \delta\lambda, \tag{9}$$

$$\nu = \nu_0 + \delta\nu, \tag{10}$$

$$p = p_0 + \delta p, \tag{11}$$

and

$$\epsilon = \epsilon_0 + \delta\epsilon \tag{12}$$

as the first order perturbations of each quantity.

To first order, there is no radial component dr in the metric when we calculate u^t or u^r there is only the component

$$ds^2 = e^{\nu} dt^2$$

which means

$$u^t = \frac{dt}{ds} = e^{-\nu/2}, \tag{13}$$

and

$$u^r = \frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = e^{-\nu/2}v, \tag{14}$$

where $v = \frac{dr}{dt}$.

Now we calculate the stress-energy tensor T^α_β to first order and obtain:

$$(T^\alpha_\beta) = \begin{bmatrix} p & e^{\lambda_0-\nu_0}(p_0+\epsilon_0)v & 0 & 0 \\ -(p_0+\epsilon_0)v & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \tag{15}$$

We will proceed to derive the Einstein equations one-by-one. First,

$$R^t_t - \frac{1}{2}R = -\frac{8\pi G}{c^4}T^t_t$$

yields (only to first order)

$$-\frac{1}{r^2}\frac{\partial}{\partial r}(re^{-\lambda}) + \frac{1}{r^2} = \frac{8\pi G}{c^4}\epsilon$$

we expand out the perturbations

$$-\frac{1}{r^2}\frac{\partial}{\partial r}(re^{-\lambda_0}e^{-\delta\lambda}) + \frac{1}{r^2} = \frac{8\pi G}{c^4}(\epsilon_0 + \delta\epsilon),$$

and we keep only the first order terms

$$1 - \frac{d}{dr}(re^{-\lambda_0}) + \frac{\partial}{\partial r}(re^{-\lambda_0}\delta\lambda) = \frac{8\pi r^2 G}{c^4}(\epsilon_0 + \delta\epsilon),$$

and then by using the equilibrium solution we simplify to the equation

$$\frac{\partial}{\partial r}(re^{-\lambda_0}\delta\lambda) = \frac{8\pi r^2 G}{c^4}\delta\epsilon. \tag{16}$$

Second,

$$R^r_r - \frac{1}{2}R = -\frac{8\pi G}{c^4}T^r_r$$

yields (only to first order)

$$-e^{-\lambda}\left(\frac{1}{r}\frac{\partial\nu}{\partial r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = -\frac{8\pi G}{c^4}p$$

we expand out the perturbations

$$-e^{-\lambda_0}e^{-\delta\lambda}\left(\frac{1}{r}\frac{d\nu_0}{dr} + \frac{1}{r}\frac{\partial}{\partial r}\delta\nu + \frac{1}{r}\right) + \frac{1}{r^2} = -\frac{8\pi G}{c^4}(p_0 + \delta p),$$

and we keep only the first order terms

$$-e^{-\lambda_0}\left(\frac{1}{r}\frac{d\nu_0}{dr} + \frac{1}{r}\frac{\partial}{\partial r}\delta\nu + \frac{1}{r^2}\right) + e^{-\lambda_0}\delta\lambda\left(\frac{1}{r}\frac{d\nu_0}{dr} + \frac{1}{r^2}\right) + \frac{1}{r^2} = -\frac{8\pi G}{c^4}(p_0 + \delta p),$$

and then using the equilibrium solution we simplify the equation

$$\frac{e^{-\lambda_0}}{r}\left(\frac{\partial}{\partial r}\delta\nu - \frac{d\nu_0}{dr}\delta\lambda - \frac{1}{r}\delta\lambda\right) = \frac{8\pi G}{c^4}\delta p. \tag{17}$$

Third, we use

$$R^r_t = -\frac{8\pi G}{c^4}T^r_t$$

yields (only to first order)

$$-\frac{e^{-\lambda}}{r}\frac{\partial\lambda}{\partial t} = \frac{8\pi G}{c^4}(p_0 + \epsilon_0)v$$

we expand out the perturbations and simultaneously use the equilibrium solution we simplify to obtain

$$\frac{e^{-\lambda_0}}{r}\frac{\partial}{\partial t}\delta\lambda = -\frac{8\pi G}{c^4}(p_0 + \epsilon_0)v. \tag{18}$$

Fourth, we use

$$\frac{\partial T^t_r}{\partial t} + \frac{\partial T^r_r}{\partial r} + \frac{1}{2}T^t_r\frac{\partial}{\partial t}(\lambda + \nu) + \frac{1}{2}(T^r_r - T^t_t)\frac{\partial\nu}{\partial r} + \frac{2}{r}(T^r_r - p) = 0,$$

yields (only to first order)

$$\frac{\partial}{\partial t}(e^{\lambda_0 - \nu_0}(p_0 + \epsilon_0)v) + \frac{\partial p}{\partial r} + \frac{1}{2}(p + \epsilon)\frac{\partial\nu}{\partial r} = 0$$

we expand out the perturbations (and notice that the equilibrium values have no time dependence)

$$e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\frac{\partial v}{\partial t}+\frac{dp_0}{dr}+\frac{\partial}{\partial r}\delta p+\frac{1}{2}(p_0+\epsilon_0+\delta\epsilon+\delta p)\left(\frac{d\nu_0}{dr}+\frac{\partial}{\partial r}\delta\nu\right)=0,$$

and then using the equilibrium solution we obtain

$$e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\frac{\partial v}{\partial t}+\frac{\partial}{\partial r}\delta p+\frac{1}{2}(\delta\epsilon+\delta p)\frac{d\nu_0}{dr}+\frac{1}{2}(p_0+\epsilon_0)\frac{\partial}{\partial r}\delta\nu=0. \tag{19}$$

We have thus obtained the four field equations we will use in the upcoming sections.

4.2 Solving for the Perturbations

First we must introduce a variable ξ such that

$$\frac{\partial\xi}{\partial t}=v. \tag{20}$$

Now 18 we integrate both sides with respect to time:

$$\frac{e^{-\lambda_0}}{r}\delta\lambda=-\frac{8\pi G}{c^4}(p_0+\epsilon_0)\xi. \tag{21}$$

We use 21 in 16 to obtain

$$-\frac{\partial}{\partial r}\left(\frac{8\pi rG}{c^4}(p_0+\epsilon_0)\xi\right)=\frac{8\pi r^2G}{c^4}\delta\epsilon,$$

which we simplify to obtain

$$\delta\epsilon=-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2(p_0+\epsilon_0)\xi). \tag{22}$$

And then we use 21 in 17 to obtain

$$\frac{e^{-\lambda_0}}{r}\frac{\partial}{\partial r}\delta\nu=\frac{8\pi G}{c^4}\left(\delta p-(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\xi\right). \tag{23}$$

Before we proceed we need an identity. If we subtract the first Einstein equation with the second ($R^t_t-\frac{1}{2}R$ by $R^r_r-\frac{1}{2}R$), we obtain at equilibrium

$$\frac{e^{-\lambda_0}}{r}\frac{d}{dr}(\lambda_0+\nu_0)=\frac{8\pi G}{c^4}(p_0+\epsilon_0). \tag{24}$$

This means that when we apply the identity 24, 21, and 23 into 19 we find

$$\begin{aligned}-e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\frac{\partial^2\xi}{\partial t^2} &= \frac{\partial}{\partial r}\delta p+\delta p\frac{d}{dr}\left(\frac{1}{2}\lambda_0+\nu_0\right)+\frac{1}{2}\delta\epsilon\frac{d\nu_0}{dr}\\ &\quad -\frac{1}{2}(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\left(\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi.\end{aligned} \tag{25}$$

As can be seen in 25, all that remains is determining δp in terms of the unperturbed quantities. We can solve our problem by obtaining the equation of state of the system.

4.3 Equation of State

The last necessary equation of state comes from the condition of conservation of particle number per unit volume, N. The way this is acheived is through a relativistic version of the continuity equation from fluid dynamics (we use a covariant divergence rather than ordinary divergence):

$$(Nu^k)_{;k} = 0$$

expanding this out we obtain

$$\frac{\partial}{\partial x^k}(Nu^k) + Nu^k \Gamma^\alpha_{\alpha k} = 0.$$

Since we are dealing with the perturbed case, only u^t and u^r are non-zero. Therefore the covariant divergence becomes

$$\frac{\partial}{\partial t}(Ne^{-\nu/2}) + \frac{\partial}{\partial r}(Nve^{-\nu_0/2}) + \frac{1}{2}Ne^{-\nu_0/2}\frac{\partial}{\partial t}(\lambda+\nu) + Ne^{-\nu_0/2}v\frac{\partial}{\partial r}\left(\frac{1}{2}(\lambda+\nu) + 2\log r\right) = 0.$$

Now following the same procedure of the previous sections, we introduce a perturbation

$$N = N_0(r) + \delta N(r,t), \tag{26}$$

and now expand the conservation equation (and use the fact that the equilibrium values are time-independent)

$$e^{-\nu_0/2}\frac{\partial}{\partial t}\delta N - \frac{1}{2}N_0 e^{-\nu_0/2}\frac{\partial}{\partial t}\delta\nu + \frac{\partial}{\partial r}(N_0 v e^{-\nu_0/2}) + \frac{2N_0 e^{-\nu_0/2}v}{r}$$
$$+\frac{1}{2}N_0 e^{-\nu_0/2}v\frac{d}{dr}(\lambda_0+\nu_0) + \frac{1}{2}N_0 e^{-\nu_0/2}\frac{\partial}{\partial t}(\delta\lambda+\delta\nu) = 0,$$

which we simplify to obtain

$$e^{-\nu_0/2}\frac{\partial}{\partial t}\delta N + \frac{1}{r^2}\frac{\partial}{\partial r}(N_0 r^2 v e^{-\nu_0/2}) + \frac{1}{2}N_0 e^{-\nu_0/2}\frac{\partial}{\partial t}\delta\lambda + \frac{1}{2}N_0 e^{-\nu_0/2}v\frac{d}{dr}(\lambda_0+\nu_0) = 0. \tag{27}$$

We proceed to integrate 27 to obtain δN

$$\delta N = -\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}(N_0 r^2 \xi e^{-\nu_0/2}) - \frac{1}{2}N_0\left(\delta\lambda + \xi\frac{d}{dr}(\lambda_0+\nu_0)\right) = 0,$$

the second term on the right hand side disappears on the account that 21 together with 24 implies

$$\delta\lambda = -\xi\frac{d}{dr}(\lambda_0+\nu_0).$$

so that

$$\delta N = -\xi\frac{dN_0}{dr} - N_0\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}(r^2 e^{-\nu_0/2}\xi). \tag{28}$$

Before proceeding, we must cast 22 in a different form by expanding it out

$$\delta\epsilon = -\xi\left(\frac{dp_0}{dr} + \frac{d\epsilon_0}{dr}\right) - (p_0+\epsilon_0)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\xi),$$

we then use 3 and rewrite $\delta\epsilon$

$$\delta\epsilon = -\xi\frac{d\epsilon_0}{dr} + \frac{1}{2}\xi(p_0+\epsilon_0)\frac{d\nu_0}{dr} - (p_0+\epsilon_0)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\xi),$$

which simplifies into

$$\delta\epsilon = -\xi\frac{d\epsilon_0}{dr} - (p_0+\epsilon_0)\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}(r^2 e^{-\nu_0/2}\xi). \tag{29}$$

We finally obtain the equation of state by assuming $N = N(\epsilon, p)$ which also says $p = (N, \epsilon)$ and so we solve for δp

$$\delta p = \frac{\partial p}{\partial N}\delta N + \frac{\partial p}{\partial \epsilon}\delta\epsilon.$$

Now using 28 and 29

$$\delta p = -\xi\left(\frac{\partial p}{\partial N}\frac{dN_0}{dr} + \frac{\partial p}{\partial \epsilon}\frac{d\epsilon_0}{dr}\right) - \left(\frac{\partial p}{\partial N}N_0 + \frac{\partial p}{\partial \epsilon}(p_0+\epsilon_0)\right)\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}(N_0 r^2\xi e^{-\nu_0/2})$$

we can simplify to obtain the final expression

$$\delta p = -\xi\frac{dp_0}{dr} - \gamma p_0\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}(r^2 e^{-\nu_0/2}\xi), \tag{30}$$

where

$$\gamma = p_0^{-1}\left(\frac{\partial p}{\partial N}N_0 + \frac{\partial p}{\partial \epsilon}(p_0+\epsilon_0)\right).$$

γ is expressed in this way because in the classical theory of gasses it refers to the ratio of the specific heats (specific heat at constant pressure to the specific heat at constant volume), called the adiabatic index.

4.4 The Variational Principle

Now that we have the expression for δp we can now proceed to obtain the necessary variational principle. Using 30 and 22 in 25

$$\begin{aligned}
-e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\frac{\partial^2\xi}{\partial t^2} \;=\; & \frac{d}{dr}\left(-\xi\frac{dp_0}{dr}\right) - \xi\frac{dp_0}{dr}\left(\frac{1}{2}\frac{d\lambda_0}{dr} + \frac{d\nu_0}{dr}\right) \\
& -\frac{1}{2}(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr} + \frac{1}{r}\right)\left(\frac{d\lambda_0}{dr} + \frac{d\nu_0}{dr}\right)\xi \\
& -e^{-(\lambda_0+2\nu_0)/2}\frac{d}{dr}\left(e^{(\lambda_0+2\nu_0)/2}\frac{\gamma p_0}{r^2}e^{\nu_0/2}\frac{d}{dr}(r^2 e^{-\nu_0/2}\xi)\right) \\
& -\frac{1}{2r^2}\frac{d}{dr}(r^2(p_0+\epsilon_0)\xi)\frac{d\nu_0}{dr}.
\end{aligned} \tag{31}$$

Now we substitute 3 into 31 term-by-term:

$$-\frac{d}{dr}\left(\xi\frac{dp_0}{dr}\right) = \frac{d}{dr}\left(\frac{\xi(p_0+\epsilon_0)}{2}\frac{d\nu_0}{dr}\right) = \frac{1}{2}\frac{d\nu_0}{dr}\frac{d}{dr}(\xi(p_0+\epsilon_0)) + \frac{1}{2}\frac{d^2\nu_0}{dr^2}\xi(p_0+\epsilon_0),$$

$$-\left(\frac{1}{2}\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi\frac{dp_0}{dr} = \frac{1}{2}\frac{d\nu_0}{dr}(p_0+\epsilon_0)\left(\frac{1}{2}\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi,$$

and we expand out the term:

$$-\frac{1}{2r^2}\frac{d}{dr}(r^2(p_0+\epsilon_0)\xi)\frac{d\nu_0}{dr} = -\frac{1}{2}\left(\frac{2}{r}(p_0+\epsilon_0)\xi + \frac{d}{dr}((p_0+\epsilon_0)\xi)\right)\frac{d\nu_0}{dr},$$

and we sum all of these three terms along with

$$-\frac{1}{2}(p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\left(\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi,$$

to obtain:

$$\frac{1}{2}(p_0+\epsilon_0)\left(\frac{d^2\nu_0}{dr^2} - \frac{1}{2}\frac{d\lambda_0}{dr}\frac{d\nu_0}{dr} - \frac{1}{r}\frac{d\lambda_0}{dr} - \frac{3}{r}\frac{d\nu_0}{dr}\right)\xi. \tag{32}$$

If we recall the second Einstein equation ($R^r_r - \frac{1}{2}R$) in equilibrium we have:

$$\frac{16\pi G}{c^4}p_0 e^{\lambda_0} = \frac{d^2\nu_0}{dr^2} - \frac{1}{2}\frac{d\lambda_0}{dr}\frac{d\nu_0}{dr} + \frac{1}{2}\left(\frac{d\nu_0}{dr}\right)^2 + \frac{1}{r}\frac{d}{dr}(\nu_0 - \lambda_0),$$

which we insert into 32:

$$\frac{8\pi G}{c^4}p_0 e^{\lambda_0}(p_0+\epsilon_0)\xi - \frac{1}{4}(p_0+\epsilon_0)\frac{d\nu_0}{dr}\left(\frac{8}{r}+\frac{d\nu_0}{dr}\right)\xi,$$

and using 3 we have

$$\frac{4}{r}\frac{dp_0}{dr}\xi + \frac{8\pi G}{c^4}e^{\lambda_0}p_0(p_0+\epsilon_0)\xi - \frac{1}{p_0+\epsilon_0}\left(\frac{dp_0}{dr}\right)^2\xi. \tag{33}$$

We have thus been able to simplify 31 into:

$$\begin{aligned} -e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\frac{\partial^2\xi}{\partial t^2} \quad = \quad & \frac{4}{r}\frac{dp_0}{dr}\xi - e^{-(\lambda_0+2\nu_0)/2}\frac{d}{dr}\left(e^{(\lambda_0+3\nu_0)/2}\frac{\gamma p_0}{r^2}\frac{d}{dr}(r^2e^{-\nu_0/2}\xi)\right) \\ & \frac{8\pi G}{c^4}e^{\lambda_0}p_0(p_0+\epsilon_0)\xi - \frac{1}{p_0+\epsilon_0}\left(\frac{dp_0}{dr}\right)^2\xi. \end{aligned} \tag{34}$$

To finish deriving the variational principle, we now let ξ have time dependence of the form $e^{i\omega t}$ (now ξ represents the amplitude of the oscillations), we multiply both sides of

34 with $r^2\xi\exp\left[\frac{1}{2}(\lambda_0+\nu_0)\right]$, and conclude by integrating over the range of r:

$$\begin{aligned}\omega^2\int_0^R e^{(3\lambda_0-\nu_0)/2}(p_0+\epsilon_0)r^2\xi^2dr &= 4\int_0^R e^{(\lambda_0+\nu_0)/2}r\frac{dp_0}{dr}\xi^2dr \\ +\int_0^R e^{(\lambda_0+3\nu_0)/2}\frac{\gamma p_0}{r^2}\left[\frac{d}{dr}(r^2e^{-\nu_0/2}\xi)\right]^2dr &- \int_0^R e^{(\lambda_0+\nu_0)/2}\left(\frac{dp_0}{dr}\right)^2\frac{r^2\xi^2}{p_0+\epsilon_0}dr \\ +\frac{8\pi G}{c^4}\int_0^R e^{(3\lambda_0+\nu_0)/2}p_0(p_0 &+ \epsilon_0)r^2\xi^2dr,\end{aligned} \tag{35}$$

this gives us the variational principle given the boundary conditions that $\xi=0$ at $r=0$ and $\delta p=0$ at $r=R$. The reader can varify that the differential operator acting on ξ is self-adjoint.

4.5 The Stability Limit

We now arrive to the point that we can derive the General Relativistic Stability result. In order to do this, we start with a solution to the Tolmann-Oppenheimer-Volkoff equations. This solution will be a simple one – we assume ϵ_0 is a constant. Thus equation 4 becomes:

$$M(r)=\frac{4\pi\epsilon_0 r^3}{3c^2},$$

and equation 5 yields:

$$\left(1-\frac{r^2}{\alpha^2}\right)p_0'=-\frac{r}{2\alpha^2\epsilon_0}(p_0+\epsilon_0)(3p_0+\epsilon_0),$$

where

$$\frac{1}{\alpha^2}=\frac{8\pi G\epsilon_0}{3c^4}.$$

We now solve the previous differential equation by integrating from the surface of the star at R to some radius r inside the star:

$$\int_{p_0(R)}^{p_0(r)}\frac{d\xi}{(\xi+\epsilon_0)(3\xi+\epsilon_0)}=-\int_R^r\frac{\eta d\eta}{2\alpha^2\epsilon_0(1-\frac{\eta^2}{\alpha^2})},$$

we integrate the left expression using partial fractions and use the fact that the pressure is zero at the surface of the star

$$\frac{1}{2\epsilon_0}\int_{p_0(R)}^{p_0(r)}\left[-\frac{1}{\xi+\epsilon_0}+\frac{3}{\epsilon_0+3\xi}\right]d\xi=\frac{1}{2\epsilon_0}\log\left(\frac{\epsilon_0+3p_0}{p_0+\epsilon_0}\right),$$

and integrating the right hand side yields

$$\frac{1}{2\epsilon_0}\log\left(\sqrt{\frac{1-\frac{r^2}{\alpha^2}}{1-\frac{R^2}{\alpha^2}}}\right)=\frac{1}{2\epsilon_0}\log\left(\frac{\zeta}{\zeta_0}\right),$$

where

$$\zeta = \sqrt{1 - \frac{r^2}{\alpha^2}},$$

and

$$\zeta_0 = \sqrt{1 - \frac{R^2}{\alpha^2}}.$$

Upon rearranging the expression we obtain:

$$p_0 = \epsilon_0 \frac{\zeta - \zeta_0}{3\zeta_0 - \zeta},$$

$$e^{\lambda_0} = \frac{1}{\zeta^2},$$

and

$$e^{\nu_0} = \frac{1}{4}(3\zeta_0 - \zeta)^2.$$

Notice that the condition that the pressure p_0 is positive implies $3\zeta_0 > 1$ which implies the Buchdahl limit.

Now we proceed to insert this solution (carefully) into 35:

$$4\alpha^2\omega^2\zeta_0 \int_0^{\psi_0} \frac{\psi^2\xi^2 d\psi}{\zeta^2(3\zeta_0 - \zeta)^2} = \zeta_0 \int_0^{\psi_0} \frac{2\zeta^2 - 1 - 9\zeta_0^2}{\zeta^3(3\zeta_0 - \zeta)^2}\psi^2\xi^2 d\psi + \frac{1}{8}\gamma \int_0^{\psi_0} (\zeta - \zeta_0)(3\zeta_0 - \zeta)^2 \left(\frac{d}{d\psi}(\psi^2 e^{-\nu_0/2}\xi)\right)^2 \frac{d\psi}{\psi^2\zeta}, \tag{36}$$

where

$$\psi = \frac{r}{\alpha}, \psi_0 = \frac{R}{\alpha}.$$

We used the fact that

$$\frac{d}{dr}p_0 = \frac{-2r\epsilon_0\zeta_0}{\alpha^2\zeta(3\zeta_0 - \zeta)^2}.$$

Now in 36 we select as a trial function

$$\xi = \psi e^{\nu_0/2} = \frac{1}{2}\psi(3\zeta_0 - \zeta),$$

and obtain

$$(\alpha\omega)^2\zeta_0 \int_0^{\psi_0} \frac{\psi^4}{\zeta^3} d\psi = \frac{1}{4}\zeta_0 \int_0^{\psi_0} (2\zeta^2 - 1 - 9\zeta_0^2)\frac{\psi^4}{\zeta^3} d\psi + \frac{9}{8}\gamma \int_0^{\zeta_0} (\zeta - \zeta_0)(3\zeta_0 - \zeta)^2 \frac{\psi^2}{\zeta} d\psi, \tag{37}$$

we now use a trigonometric substitution $y = \cos\theta$, $\psi = \sin\theta$, and $\theta_0 = \sin^{-1}(R/\alpha)$ to give

$$(38)\qquad (\alpha\omega)^2\cos\theta_0\int_0^{\theta_0}\frac{\sin^4\theta}{\cos^2\theta}d\theta = \frac{1}{4}\cos\theta_0\int_0^{\theta_0}(2\cos^2\theta - 1 - 9\cos^2\theta_0)\frac{\sin^4\theta}{\cos^2\theta}d\theta + \frac{9}{8}\gamma\int_0^{\theta_0}(\cos\theta-\cos\theta_0)(3\cos\theta_0-\cos\theta)^2\sin^2\theta d\theta.$$

As θ_0 approaches zero, the trial function tends to the true solution. Therefore we will divide both sides of 38 by $\cos\theta_0$ and expand each of the expressions in terms of θ_0: first,

$$(\alpha\omega)^2\int_0^{\theta_0}\frac{\sin^4\theta}{\cos^2\theta}d\theta = (\alpha\omega)^2\frac{\theta_0^5}{5} + \mathcal{O}(\theta_0^9),$$

second,

$$\begin{aligned}\frac{1}{4}\int_0^{\theta_0}(2\cos^2\theta-1-9\cos^2\theta_0)\frac{\sin^4\theta}{\cos^2\theta}d\theta &= \frac{1}{4}\int_0^{\theta_0}\left(2\left(1-\frac{\theta^2}{2}\right)^2 - 1 - 9\left(1-\frac{\theta_0^2}{2}\right)^2\right)\theta^4 d\theta + \mathcal{O}(\theta_0^9)\\ &= \frac{1}{4}\int_0^{\theta_0}d\theta\left(-8-2\theta^2+\frac{\theta^4}{2}+9\theta_0^2-\frac{9\theta_0^4}{4}\right)\theta^4 + \mathcal{O}(\theta_0^9) = -\frac{2}{5}\theta_0^5 + \frac{53}{140}\theta_0^7 + \mathcal{O}(\theta_0^9),\end{aligned}$$

finally,

$$\begin{aligned}\frac{9\gamma}{8}\int_0^{\theta_0}(\cos\theta-\cos\theta_0)(3\cos\theta_0-\cos\theta)^2\sin^2\theta d\theta &= \frac{9\gamma}{16}\int_0^{\theta_0}\theta^2(\theta_0^2-\theta^2)\left(2-\frac{3\theta_0^2}{2}+\frac{\theta^2}{2}\right)^2 d\theta + \mathcal{O}(\theta_0^9)\\ &= \frac{9\gamma}{16}\left(\frac{8}{15}\theta_0^5 - \frac{24}{35}\theta_0^7\right) + \mathcal{O}(\theta_0^9),\end{aligned}$$

which we put all together

$$(39)\qquad (\alpha\omega)^2 = \frac{1}{2}\left[(3\gamma-4) - \frac{1}{14}\theta_0^2(54\gamma-53)\right] + \mathcal{O}(\theta_0^9).$$

To figure out the stability condition, we note that instability occurs when ω^2 becomes negative so in 39

$$\gamma - \frac{4}{3} < \frac{1}{42}\theta_0^2(54\gamma-53) < \frac{19}{42}\theta_0^2,$$

where the inequality comes from the fact that the bound for Newtonian stability is $\gamma = 4/3$ so that to find a non-Newtonian instability we need $\gamma > 4/3$. We now expand the previous formula and obtain

$$(40)\qquad R < \frac{19}{42(\gamma-\frac{4}{3})}\frac{2GM}{c^2},$$

hence if γ is slightly larger than 4/3 then instability occurs at the radius shown in this expression. This instability is purely relatavistic in nature.

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