The enumeration of regions in the Shi arrangement with a given separating wall

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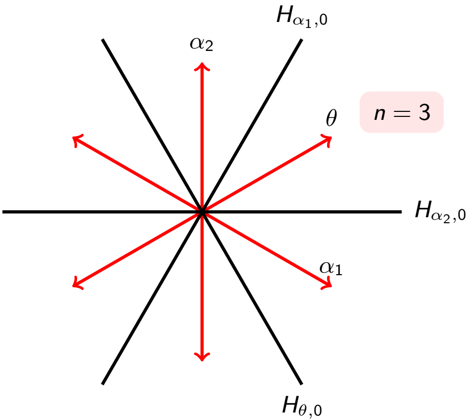
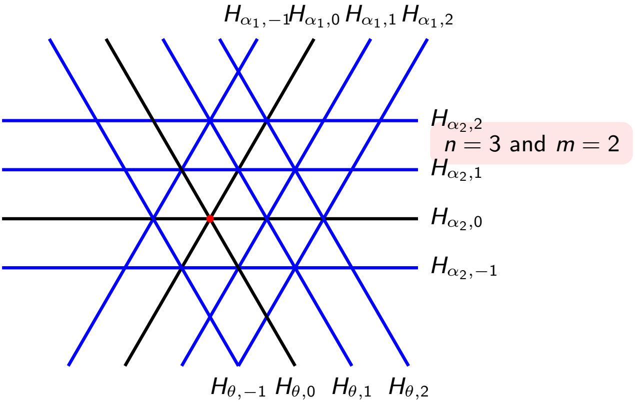
Roots and hyperplanes



The roots α_1, α_2, and θ and their reflecting hyperplanes.

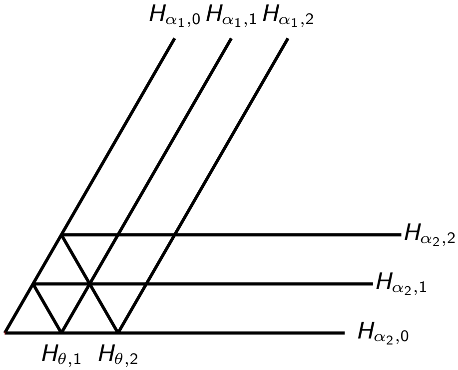
m-Shi arrangement



For any positive integers n and m, the extended Shi arrangement is

$$\{H_{\alpha_{ij},k} | k \in \mathbb{Z}, \ -m < k \leq m \text{ and } 1 \leq i \leq j \leq n-1\}$$

Regions in the dominant chamber

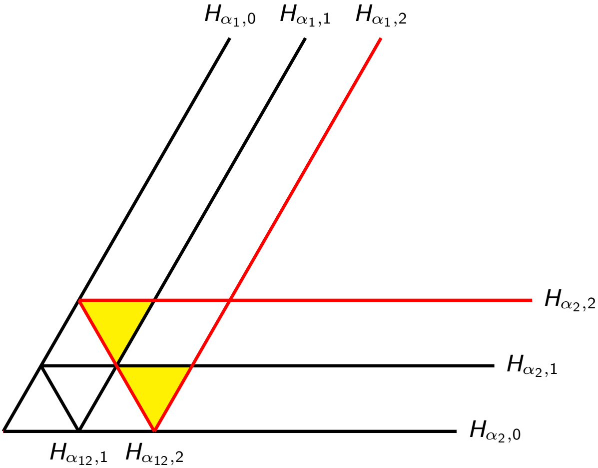


12 regions

$$C_{nm} = \frac{1}{nm+1}\binom{n(m+1)}{n}$$

$$C_{32} = 12$$

Regions on $H_{\theta,m}$

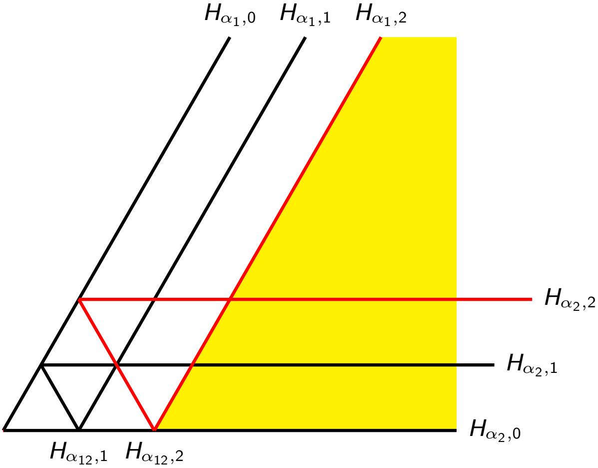


A hyperplane H separates two regions if they lie on opposite sides of it. A hyperplane H is a separating wall for a region R if H is a supporting hyperplane of R and H separates R from R_0.

There are two regions which have $H_{\theta,m}$ as a separating wall.

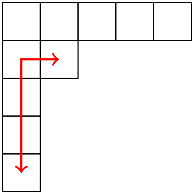
Problem

Given a hyperplane $H_{\alpha_{ij},m}$, how many dominant regions have it as a separating wall? In other words, the rest of the hyperplanes in the arrangement cut $H_{\alpha_{ij},m}$ into regions. How many?



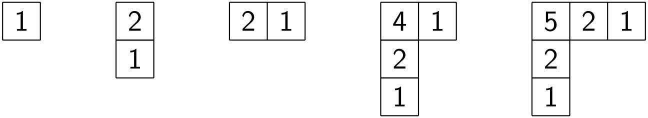
There are three regions which have $H_{\alpha_1,m}$ as a separating wall.

Use a bijection from cores to regions.

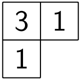


Hook $h_{21} = 5$

An *n*-core is an integer partition λ such that $n \nmid h_{ij}$ for all boxes (i, j) in λ. Some 3-cores. Boxes contain their hook numbers.

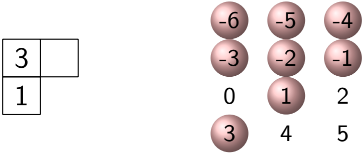


Not a 3-core.

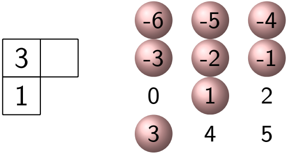


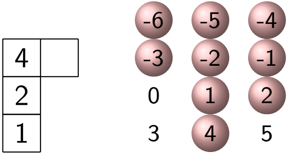
Abacus description of n-cores

The hooklengths from the first column of a partition λ, plus all negative integers, are a set of β-numbers for λ. Construct an n-abacus for a partition by putting its β-numbers on an n-runner abacus.

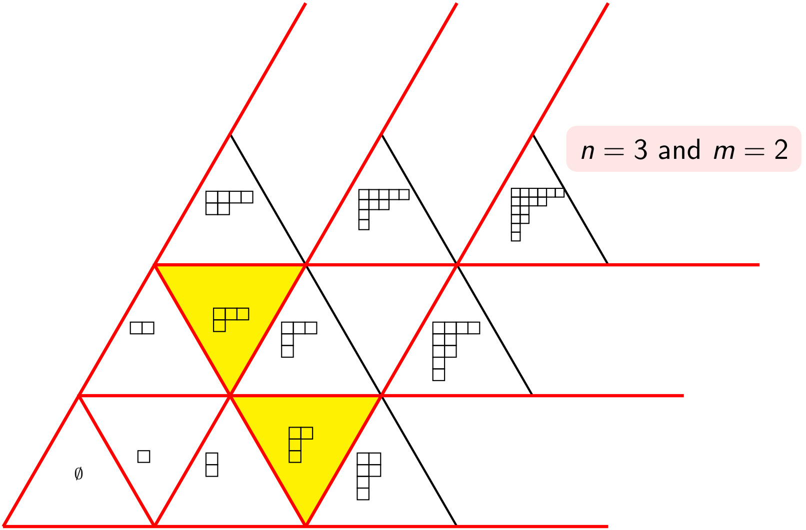


A partition λ is an n-core if and only if its abacus is flush.





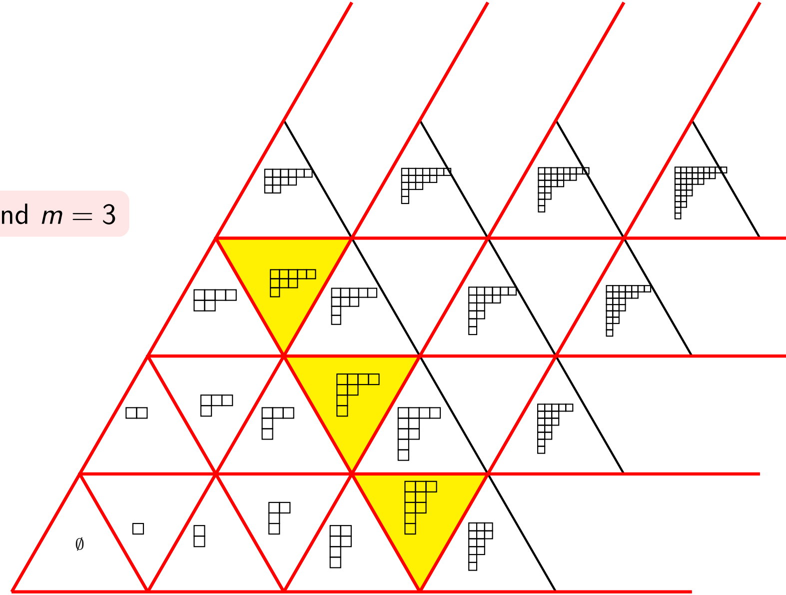
Regions on $H_{\theta,m}$



Shaded regions have hook $4 = n(m-1)+1$ in box (1,1).

Regions on $H_{\theta,m}$

$n = 3$ and $m = 3$



Shaded regions have hook $7 = n(m-1)+1$ in box (1,1).

Number of regions on $H_{\theta,m}$

λ is an n-core and box (1,1) has hook length $n(m-1)+1$. What does its abacus look like? Runner 0 has 0 beads, runner 1 has m beads, and all other runners have from 0 to $m-1$ beads. There are m^{n-2} such abacuses.

Base case done

What about the rest of the $H_{\alpha,m}$?

α_{14}	α_{13}	α_{12}	α_{11}
α_{24}	α_{23}	α_{22}	
α_{34}	α_{33}		
α_{44}			

Generating functions

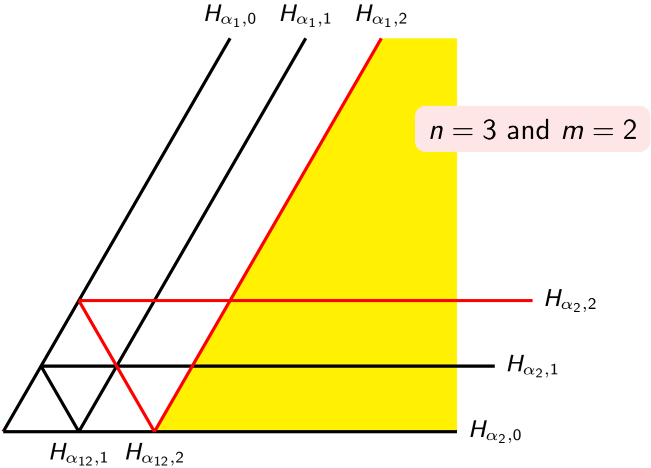
$$f^n_{\alpha m}(p,q) = \sum_{R \in \mathfrak{h}^n_{\alpha m}} p^{r(R)} q^{c(R)}.$$

$$r(R) = |\{(j,k) : R \text{ and } R_0 \text{ are separated by } H_{\alpha_{1j},k}$$
$$\text{where } 1 \le k \le m \text{ and } 1 \le j \le n-1\}|$$

$$c(R) = |\{(i,k) : R \text{ and } R_0 \text{ are separated by } H_{\alpha_{in-1},k}$$
$$\text{where } 1 \le k \le m \text{ and } 1 \le i \le n-1\}|.$$

$\mathfrak{h}^n_{\alpha m} = \{m$-Shi regions which which have $H_{\alpha,m}$ as a separating wall.$\}$

Regions on $H_{\alpha_1,m}$



$$f^3_{\alpha_1 2}(p, q) = p^4 q^2 (1 + q + q^2)$$

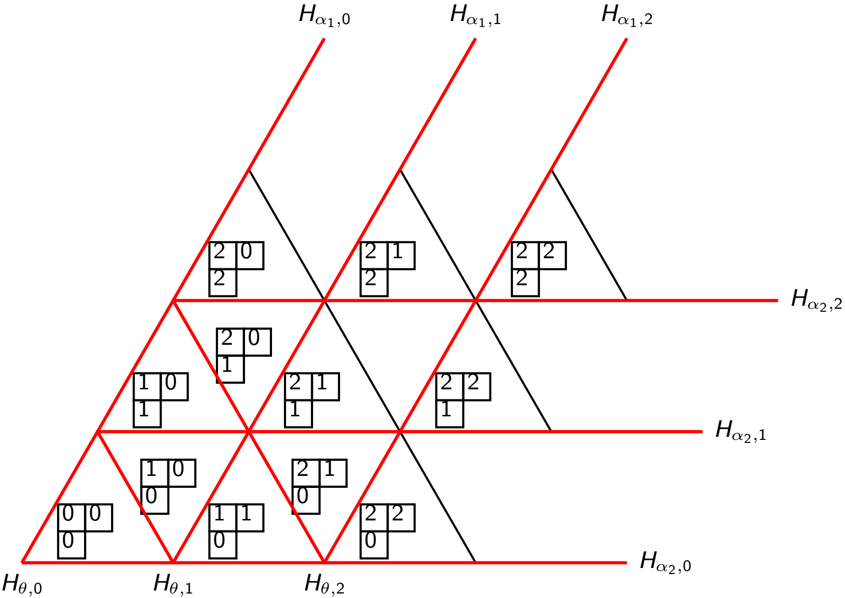
Region coordinates

For each region R keep track of all hyperplanes crossed in a triangular array. Let r_{ij} be the number of translates of $H_{\alpha_{ij}}$ which separate the region R from R_0.

r_{14}	r_{13}	r_{12}	r_{11}
r_{24}	r_{23}	r_{22}	
r_{34}	r_{33}		
r_{44}			

Coordinates-array for a Shi region, $n = 5$.

Shi tableaux and regions



Recursion

A region $R \in \mathcal{S}_{n,m}$ has $H_{\alpha_{ij},m}$ as a separating wall if and only if $r_{ij} = m$ and for all t such that $i \leq t < j$, $r_{it} + r_{t+1,j} = m - 1$.The conditions on the coordinates translate into a recursion on the generating function.

r_{14}	r_{13}	r_{12}	r_{11}
r_{24}	r_{23}	r_{22}	
r_{34}	r_{33}		
r_{44}			

Base case

$$\begin{aligned} f^n_{\theta m}(p,q) &= p^m q^m \left(p^{m-1} + p^{m-2}q + \ldots + pq^{m-2} + q^{m-1}\right)^{n-2} \\ &= [m]^{n-2}_{p,q} p^m q^m \end{aligned}$$

$$f^n_{\theta m}(p,q) = \sum_{\lambda:\lambda \text{ ``has'' } H_{\theta,m} \text{ as separating wall}} p^{\ell(\lambda)} q^{\lambda_1}$$

Recursion applied to generating function

The generating function recursion keeps track of the number of ways to attach a new first part on a core partition (new first column on Shi tableau) which corresponds to a boundary region from one dimension less.

$$\begin{aligned} f^{n}_{\alpha m}(p,q) &= \left(p^m(1+q+q^2+\cdots+q^{(n-1)m})f^{n-1}_{\alpha m}(p,q)\right)_{\leq q^{(n-1)m}} \\ &= \left(p^m[(n-1)m+1]_q f^{n-1}_{\alpha m}(p,q)\right)_{\leq q^{(n-1)m}} \end{aligned}$$

Symmetry

T is the Shi tableau for R and T' (conjugate) is the Shi tableau for R'.

$R' \Leftrightarrow$

r_{14}	r_{13}	r_{12}	r_{11}
r_{24}	r_{23}	r_{22}	
r_{34}	r_{33}		
r_{44}			

$R' \Leftrightarrow$

r_{14}	r_{24}	r_{34}	r_{44}
r_{13}	r_{23}	r_{33}	
r_{12}	r_{22}		
r_{11}			

$$R \in \mathfrak{h}^{n}_{\alpha_{ij} m} \text{ if and only if } R \in \mathfrak{h}^{n}_{\alpha_{n-j,n-i} m}.$$

In terms of generating functions, this becomes the following:

$$f^{n}_{\alpha_{ij} m}(p, q) = f^{n}_{\alpha_{n-j,n-i} m}(q, p).$$

Example

$n = 7$ and $m = 2$

α_{16}	α_{15}	α_{14}	α_{13}	α_{12}	α_{11}
α_{26}	α_{25}	α_{24}	α_{23}	α_{22}	
α_{36}	α_{35}	α_{34}	α_{33}		
α_{46}	α_{45}	α_{44}			
α_{56}	α_{55}				
α_{66}					

$f^{7}_{\alpha_{24}2}(p, q)$

Example

$n = 4$

α_{13}	α_{12}	α_{11}
α_{23}	α_{22}	
α_{33}		

$$f^{7}_{\alpha_{24}2}(p,q) = \left(p^2[13]_q f^{6}_{\alpha_{24}2}(p,q)\right)_{\leq q^{12}}$$

$$= \left(p^2[13]_q \left(p^2[11]_q f^{5}_{\alpha_{24}2}(p,q)\right)_{\leq q^{10}}\right)_{\leq q^{12}}$$

$$= \left(p^2[13]_q \left(p^2[11]_q f^{5}_{\alpha_{13}2}(q,p)\right)_{\leq q^{10}}\right)_{\leq q^{12}}$$

$$= \left(p^2[13]_q \left(p^2[11]_q \left(q^2[9]_p f^{4}_{\alpha_{13}2}(q,p)\right)_{\leq p^8}\right)_{\leq q^{10}}\right)_{\leq q^{12}}$$

$$= \left(p^2[13]_q \left(p^2[11]_q \left(q^2[9]_p \left(p^2 q^2 [2]^2_{p,q}\right)\right)_{\leq p^8}\right)_{\leq q^{10}}\right)_{\leq q^{12}}$$

$$f^{7}_{\alpha_{24}2}(1,1) = 781$$