COUNTING SHI REGIONS WITH A FIXED SEPARATING WALL

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Abstract. Athanasiadis introduced separating walls for a region in the extended Shi arrangement and used them to generalize the Narayana numbers. In this paper, we fix a hyperplane in the extended Shi arrangement for type A and calculate the number of dominant regions which have the fixed hyperplane as a separating wall; that is, regions where the hyperplane supports a facet of the region and separates the region from the origin.

1. Introduction

A hyperplane arrangement dissects its ambient vector space into regions. The regions have walls–hyperplanes which support facets of the region– and the walls may or may not separate the region from the origin. The regions in the extended Shi arrangement are enumerated by well-known sequences: all regions by the extended parking functions numbers, the dominant regions by the extended Catalan numbers, dominant regions with a given number of certain separating walls by the Narayana numbers. In this paper we study the extended Shi arrangement by fixing a hyperplane in it and calculating the number of regions for which that hyperplane is a separating wall. For example, suppose we are considering the mth extended Shi arrangement in dimension $n-1$, with highest root θ. Let $H_{\theta,m}$ be the mth translate of the hyperplane through the origin with θ as normal. Then we show there are m^{n-2} regions which abut $H_{\theta,m}$ and are separated from the origin by it.

At the heart of this paper is a well-known bijection from certain integer partitions to dominant alcoves (and regions). One particularly nice aspect of our work is that we are able to use the bijection to enumerate regions. We characterize the partitions associated to the regions in question by certain interesting features and easily count those partitions, whereas it is not clear how to count the regions directly.

We give two very different descriptions of this bijection, one combinatorial and one geometric. We can then prove several results in two ways, using the different descriptions.

We rely on work from several sources. Shi (1986) introduced what is now called the Shi arrangement while studying the affine Weyl group of type A, and Stanley (1998) extended it. We also use his study of alcoves in Shi (1987a). Richards (1996), on decomposition numbers for Hecke algebras, has been very useful. The Catalan numbers have been extended and generalized; see Athanasiadis (2005) for the history. Fuss-Catalan numbers is another name for the extended Catalan numbers. The Catalan numbers can be written as a sum of Narayana numbers. Athanasiadis (2005) generalized the Narayana numbers. He showed they enumerated several types of objects; one of them was the number of dominant Shi regions with a fixed

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number of separating walls. This led us to investigate separating walls. All of our work is for type A, although Shi arrangements, Catalan numbers, and Narayana numbers exist for other types.

In Section 2, we introduce notation, define the Shi arrangement, certain partitions, and the bijection between them which we use to count regions. In Section 3, we characterize the partitions assigned to the regions which have $H_{\theta,m}$ as separating wall. In order to enumerate the regions which have other separating walls, we must use a generating function, which we introduce in Section 4. Finally, in Section 5, we give a recursion for the generating functions from Section 4, which enables us to count the regions which have other separating walls $H_{\alpha,m}$.

2. Preliminaries

Here we introduce notation and review some constructions.

2.1. **Root system notation.** Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the standard basis of $\mathbb{R}^n$ and $\langle\,|\,\rangle$ be the bilinear form for which this is an orthonormal basis. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$ is a basis of

$$V = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i = 0\}.$$

We let $\alpha_{ij} = \alpha_i + \ldots + \alpha_j$, the highest root $\alpha_{1,n-1} = \theta$, and note that $\alpha_{ii} = \alpha_i$ and $\alpha_{ij} = \varepsilon_i - \varepsilon_{j+1}$.

The elements of $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ are called roots and we say a root α is positive, written $\alpha > 0$, if $\alpha \in \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. We let $\Delta^- = -\Delta^+$ and say $\alpha < 0$ if $\alpha \in \Delta^-$. Then Π is the set of simple roots. As usual, we let $Q = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$ be identified with the root lattice of type A_{n-1} and $Q^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geq 0}\alpha_i$.

2.2. **Extended Shi arrangements.** A *hyperplane arrangement* is a set of hyperplanes, possibly affine hyperplanes, in V. We are interested in certain sets of hyperplanes of the following form. For each $\alpha \in \Delta^+$, we define its reflecting hyperplane

$$H_{\alpha,0} = \{v \in V \mid \langle v \mid \alpha \rangle = 0\}$$

and for $k \in \mathbb{Z}$, $H_{\alpha,0}$'s kth translate,

$$H_{\alpha,k} = \{v \in V \mid \langle v \mid \alpha \rangle = k\}.$$

Note $H_{-\alpha,-k} = H_{\alpha,k}$ so we usually take $k \in \mathbb{Z}_{\geq 0}$. Then the extended Shi arrangement, here called the *m-Shi arrangement*, is the collection of hyperplanes

$$\mathcal{H}_m = \{H_{\alpha,k} \mid \alpha \in \Delta^+, -m < k \leq m\}.$$

This arrangement is defined for crystallographic root systems of all finite types.

Regions of the m-Shi arrangement are the connected components of the hyperplane arrangement complement $V \setminus \bigcup_{H \in \mathcal{H}_m} H$.

We denote the closed half-spaces $\{v \in V \mid \langle v \mid \alpha \rangle \geq k\}$ and $\{v \in V \mid \langle v \mid \alpha \rangle \leq k\}$ by ${H_{\alpha,k}}^+$ and ${H_{\alpha,k}}^-$ respectively. The *dominant* chamber of V is $V \cap \bigcap_{i=1}^{n-1} {H_{\alpha_i,0}}^+$ and is also referred to as the fundamental chamber in the literature. This paper primarily concerns regions and alcoves in the dominant chamber.

A *dominant region* of the m-Shi arrangement is a region that is contained in the dominant chamber. We call the collection of dominant regions in the m-Shi arrangement $\mathcal{S}_{n,m}$.

Each connected component of

$$V \setminus \bigcup_{\substack{\alpha\in\Delta^+\\ k\in\mathbb{Z}}} H_{\alpha,k}$$

is called an *alcove* and the *fundamental alcove* is $\mathcal{A}_0$, the interior of $H_{\theta,1}{}^- \cap \bigcap_{i=1}^{n-1} H_{\alpha_i,0}{}^+$, where $\theta = \alpha_1 + \cdots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$. A *dominant alcove* is one contained in the dominant chamber. Denote the set of dominant alcoves by $\mathfrak{A}_n$.

A *wall* of a region is a hyperplane in $\mathcal{H}_m$ which supports a facet of that region or alcove. Two open regions are *separated* by a hyperplane H if they lie in different closed half-spaces relative to H. Please see Athanasiadis (2005) or Humphreys (1990) for details. We study dominant regions with a fixed separating wall. A *separating wall* for a region R is a wall of R which separates R from $\mathcal{A}_0$.

2.3. **The affine symmetric group.**

Definition 2.1. *The affine symmetric group, denoted $\widehat{\mathfrak{S}}_n$, is defined as*

$$\begin{aligned}\widehat{\mathfrak{S}}_n = \langle s_1, \dots, s_{n-1}, s_0 \mid s_i^2 = 1, \quad & s_i s_j = s_j s_i \text{ if } i \not\equiv j \pm 1 \mod n,\\ & s_i s_j s_i = s_j s_i s_j \text{ if } i \equiv j \pm 1 \mod n\rangle\end{aligned}$$

for $n > 2$, but $\widehat{\mathfrak{S}}_2 = \langle s_1, s_0 \mid s_i^2 = 1\rangle$.

The affine symmetric group contains the symmetric group $\mathfrak{S}_n$ as a subgroup. $\mathfrak{S}_n$ is the subgroup generated by the s_i, $0 < i < n$. We identify $\mathfrak{S}_n$ as permutations of $\{1, \dots, n\}$ by identifying s_i with the simple transposition $(i, i+1)$.

The affine symmetric group $\widehat{\mathfrak{S}}_n$ acts freely and transitively on the set of alcoves. We thus identify each alcove $\mathcal{A}$ with the unique $w \in \widehat{\mathfrak{S}}_n$ such that $\mathcal{A} = w^{-1}\mathcal{A}_0$. Each simple generator s_i, $i > 0$, acts by reflection with respect to the simple root α_i. In other words, it acts by reflection over the hyperplane $H_{\alpha_i,0}$. The element s_0 acts as reflection with respect to the affine hyperplane $H_{\theta,1}$.

More specifically, the action on V is given by

$$\begin{aligned} s_i(a_1, \dots, a_i, a_{i+1}, \dots, a_n) &= (a_1, \dots, a_{i+1}, a_i, \dots, a_n) \quad \text{for } i \neq 0, \text{ and}\\ s_0(a_1, \dots, a_n) &= (a_n + 1, a_2, \dots, a_{n-1}, a_1 - 1).\end{aligned}$$

Note $\mathfrak{S}_n$ preserves $\langle\ |\ \rangle$, but $\widehat{\mathfrak{S}}_n$ does not.

2.4. **Shi coordinates and Shi tableaux.** Every alcove $\mathcal{A}$ can be written as $w^{-1}\mathcal{A}_0$ for a unique $w \in \widehat{\mathfrak{S}}_n$ and additionally, for each $\alpha \in \Delta^+$, there is a unique integer k_α such that $k_\alpha < \langle \alpha \mid x\rangle < k_\alpha + 1$ for all $x \in \mathcal{A}$. Shi characterized the integers k_α which can arise in this way and the next lemma gives the conditions for type A.

Lemma 2.2 (Shi (1987a))**.** *Let $\{k_{\alpha_{ij}}\}_{1\le i\le j\le n-1}$ be a set of $\binom{n}{2}$ integers. There exists a $w \in \widehat{\mathfrak{S}}_n$ such that*

$$k_{\alpha_{ij}} < \langle \alpha_{ij} \mid x\rangle < k_{\alpha_{ij}} + 1$$

for all $x \in w^{-1}\mathcal{A}_0$ if and only if

$$k_{\alpha_{it}} + k_{\alpha_{t+1,j}} \le k_{\alpha_{ij}} \le k_{\alpha_{it}} + k_{\alpha_{t+1,j}} + 1,$$

for all t such that $i \le t < j$.

From now on, except in the discussion of Proposition 4.3, we write k_{ij} for $k_{\alpha_{ij}}$. These $\{k_{ij}\}_{1\leq i\leq n-1}$ are the *Shi coordinates* of the alcove. We arrange the coordinates for an alcove $\mathcal{A}$ in the Young's diagram (see Section 2.5) of a staircase partition $(n-1, n-2, \ldots, 1)$ by putting k_{ij} in the box in row i, column $n-j$. See Krattenthaler et al. (2002) for a similar arrangement of sets indexed by positive roots. For a dominant alcove, the entries are nonincreasing along rows and columns and are nonnegative.

We can also assign coordinates to regions in the Shi arrangement. In each region of the m-Shi hyperplane arrangement, there is exactly one "representative," or m-minimal, alcove closest to the fundamental alcove $\mathcal{A}_0$. See Shi (1987b) for $m=1$ and Athanasiadis (2005) for $m \geq 1$. Let $\mathcal{A}$ be an alcove with Shi coordinates $\{k_{ij}\}_{1\leq i\leq n-1}$ and suppose it is the m-minimal alcove for the region R. We define coordinates $\{e_{ij}\}_{1\leq i\leq j\leq n-1}$ for R by $e_{ij} = \min(k_{ij}, m)$.

Again, we arrange the coordinates for a region R in the Young's diagram (see Section 2.5) of a staircase partition $(n-1, n-2, \ldots, 1)$ by putting e_{ij} in the box in row i, column $n-j$. For dominant regions, the entries are nonincreasing along rows and columns and are nonnegative.

Example 2.3. *For $n=5$, the coordinates are arranged*

k_{14}	k_{13}	k_{12}	k_{11}
k_{24}	k_{23}	k_{22}	
k_{34}	k_{33}		
k_{44}			

e_{14}	e_{13}	e_{12}	e_{11}
e_{24}	e_{23}	e_{22}	
e_{34}	e_{33}		
e_{44}			

Example 2.4. *The dominant chamber for the 2-Shi arrangement for $n=3$ is illustrated in Figure 1 The yellow region has coordinates $e_{12}=2$, $e_{11}=1$, and $e_{22}=2$. Its 2-minimal alcove has coordinates $k_{12}=3$, $k_{11}=1$, and $k_{22}=2$.*

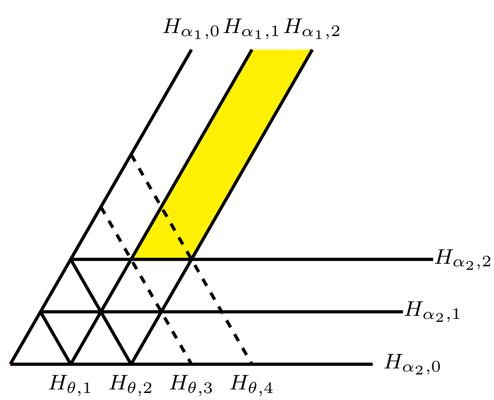

FIGURE 1. $\mathcal{S}_{3,2}$ consists of 12 regions

Denote the *Shi tableau* for the alcove $\mathcal{A}$ by $T_{\mathcal{A}}$ and for the region R by T_R.

Both Richards (1996) and Athanasiadis (2005) characterized the Shi tableaux for dominant m-Shi regions.

Lemma 2.5. *Let $T = \{e_{ij}\}_{1 \le i \le j \le n-1}$ be a collection of integers such that $0 \le e_{ij} \le m$. Then T is the Shi tableau for a region $R \in \mathcal{S}_{n,m}$ if and only if*

(2.1)
$$e_{ij} = \begin{cases} e_{it} + e_{t+1,j} \text{ or } e_{it} + e_{t+1,j} + 1 & \text{if } m-1 \ge e_{it} + e_{t+1,j} \text{ for } t = i, \ldots, j-1 \\ m & \text{otherwise} \end{cases}$$

Proof. Athanasiadis (2005) defined co-filtered chains of ideals as decreasing chains of ideals in the root poset

$$\Delta^+ = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_m$$

in Δ^+ such that

$$(I_i + I_j) \cap \Delta^+ \subseteq I_{i+j}, \tag{2.2}$$

and

$$(J_i + J_j) \cap \Delta^+ \subseteq J_{i+j}, \tag{2.3}$$

where $I_k = I_m$ for $k > m$ and $J_i = \Delta^+ \setminus I_i$. He gave a bijection between co-filtered chains of ideals and m-minimal alcoves for $R \in \mathcal{S}_{n,m}$. Given such a chain, let $e_{uv} = k$ if $\alpha_{uv} \in I_k$, $\alpha_{uv} \notin I_{k+1}$, and $k < m$ and let $e_{uv} = m$ if $\alpha_{uv} \in I_m$. Then conditions (2.2) and (2.3) translate into (2.1). □

Lemma 3.9 from Athanasiadis (2005) is crucial to our work here. He characterizes the co-filtered chains of ideals for which $H_{\alpha,m}$ is a separating wall. We translate that into our set-up in Lemma 2.6, using entries from the Shi Tableau.

Lemma 2.6 (Athanasiadis (2005))**.** *A region $R \in \mathcal{S}_{n,m}$ has $H_{\alpha_{uv},m}$ as a separating wall if and only if $e_{uv} = m$ and for all t such that $u \le t < v$, $e_{ut} + e_{t+1,v} = m - 1$.*

2.5. **Partitions.** A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonnegative integers. $\lambda_1, \lambda_2, \ldots$ are called the *parts* of λ. We identify a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with its Young diagram, that is the array of boxes with coordinates $\{(i,j) : 1 \le j \le \lambda_i \text{ for all } \lambda_i\}$. The *conjugate* of λ is the partition λ' whose diagram is obtained by reflecting λ's diagram about the diagonal. The *length* of a partition λ, $\ell(\lambda)$, is the number of positive parts of λ.

2.5.1. *Core partitions.* The (k,l)-*hook* of any partition λ consists of the (k,l)-box of λ, all the boxes to the right of it in row k together with all the nodes below it and in column l. The *hook length* h^λ_{kl} of the box (k,l) is the number of boxes in the (k,l)-hook. Let n be a positive integer. An n-*core* is a partition λ such that $n \nmid h^\lambda_{(k,l)}$ for all $(k,l) \in \lambda$. We let $\mathcal{C}_n$ denote the set of partitions which are n-cores.

2.5.2. $\widehat{\mathfrak{S}}_n$ *action on cores.* There is a well-known action of $\widehat{\mathfrak{S}}_n$ on n-cores which we will briefly describe here; please see Misra and Miwa (1990), Lascoux (2001), Lapointe and Morse (2005), Berg et al. (2009), or Fishel and Vazirani (2010), for more details and history.

The Young's diagram of a partition λ is made up of boxes. We say the box in row i and column j has *residue* r if $j - i \equiv r \mod n$. A box not in the Young's diagram of λ is called *addable* if we obtain a partition when we add it to λ. In other words, the box $(i, j+1)$ is addable if $\lambda_i = j$ and either $i = 1$ or $\lambda_{i-1} > \lambda_i$. A box in the Young diagram of λ is called *removable* if we obtain a partition when we remove it from λ. It is well-known (see for example Fishel and Vazirani (2010) or Lapointe and Morse (2005)) that the following action of $s_i \in \widehat{\mathfrak{S}}_n$ on n-cores is well-defined.

Definition 2.7. $\widehat{\mathfrak{S}}_n$ *action n-core partitions:*

(1) *If λ has an addable box with residue r, then $s_r(\lambda)$ is the n-core partition created by adding all addable boxes of residue r to λ.*
(2) *If λ has an removable box with residue r, then $s_r(\lambda)$ is the n-core partition created by removing all removable boxes of residue r from λ.*
(3) *If λ has neither removable nor addable boxes of residue r, then $s_r(\lambda)$ is λ.*

2.6. **Abacus diagrams.** In Section 3, we use a bijection, called Ψ, to describe certain regions. We will need abacus diagrams to define Ψ. We associate to each partition λ its abacus diagram. When λ is an n-core, its abacus has a particularly nice form.

The *β-numbers* for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ are the hook lengths from the boxes in its first column:

$$\beta_k = h^{\lambda}_{(k,1)}.$$

Each partition is determined by its β-numbers and $\beta_1 > \beta_2 > \cdots > \beta_{\ell(\lambda)} > 0$.

An *n-abacus diagram*, or abacus diagram when n is clear, is a diagram with integer entries arranged in n columns labeled $0, 1, \ldots, n-1$. The columns are called *runners*. The horizontal cross-sections or rows will be called *levels* and runner k contains the integer entry $qn + r$ on level q where $-\infty < q < \infty$. We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and ∞ at the bottom, and we always put runner 0 in the leftmost position, increasing to runner $n-1$ in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called *beads*. The *level* of a bead labeled by $qn + r$ is q and its runner is r. Entries which are not circled will be called *gaps*. Two abacus diagrams are equivalent if one can be obtained by adding a constant to each entry of the other.

See Example 2.9 below.

Given a partition λ its abacus is any abacus diagram equivalent to the one with beads at entries $\beta_k = h^{\lambda}_{(k,1)}$ and all entries $j \in \mathbb{Z}_{<0}$.

Given the original n-abacus for the partition λ with beads at $\{\beta_k\}_{1 \le k \le \ell(\lambda)}$, let b_i be one more than the largest level number of a bead on runner i; that is, the level of the first gap. Then $(b_0, \ldots, b_{n-1})$ is the *vector of level numbers* for λ.

The *balance number* of an abacus is the sum over all runners of the largest level of a bead in that runner. An abacus is *balanced* if its balance number is zero. There is a unique n-abacus which represents a given n-core λ for each balance number. In particular, there is a unique n-abacus for λ with balance number 0.

Remark 2.8. *It is well-known that λ is an n-core if and only if all its n-abacus diagrams are flush, that is to say whenever there is a bead at entry j there is also a bead at $j - n$. Additionally, if $(b_0, \ldots, b_{n-1})$ is the vector of level numbers for λ, then $b_0 = 0$, $\sum_{i=0}^{n-1} b_i = \ell(\lambda)$, and since there are no gaps, $(b_0 \ldots, b_{n-1})$ describes λ completely.*

Example 2.9. *Both abacus diagrams in Figure 2 represent the 4-core $\lambda = (5, 2, 1, 1, 1)$. The levels are indicated to the left of the abacus and below each runner is the largest level number of a bead in that runner. The boxes of the Young diagram of λ have been filled with their hooklengths. The diagram on the left is balanced. The diagram* *on the right is the original diagram, where the beads are placed at the β-numbers and negative integers. The vector of level numbers for λ is $(0, 3, 1, 1)$.*

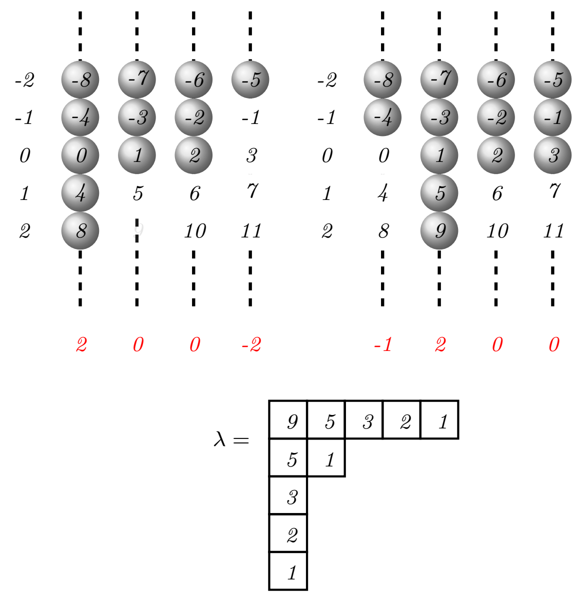

FIGURE 2. The abacus represents the 4-core λ.

2.7. **Bijections.** We describe here two bijections, Ψ and Φ, from the set of n-cores to dominant alcoves. We neither use nor prove the fact that $\Psi = \Phi$.

2.7.1. *Combinatorial description.* Ψ is a slightly modified version of the bijection given in Richards (1996). Given an n-core λ, let $(b_0 = 0, b_1, \ldots, b_{n-1})$ be the level numbers for its abacus. Now let $\tilde{p}_i = b_{i-1}n + i - 1$, which is the entry of the first gap on runner i, for i from 1 to n, and then let $p_1 = 0 < p_2 < \cdots < p_n$ be the $\{\tilde{p}_i\}$ written in ascending order. Finally we define $\Psi(\lambda)$ to be the alcove whose Shi coordinates are given by

$$k_{ij} = \lfloor \frac{p_{j+1} - p_i}{n} \rfloor$$

for $1 \leq i \leq j \leq n - 1$.

Example 2.10. *We continue Example 2.9. We have $n = 4$, $\lambda = (5, 2, 1, 1, 1)$, and $(b_0, b_1, b_2, b_3) = (0, 3, 1, 1)$. Then $\tilde{p}_1 = 0$, $\tilde{p}_2 = 13$, $\tilde{p}_3 = 6$, and $\tilde{p}_4 = 7$ and $p_1 = 0$, $p_2 = 6$, $p_3 = 7$, and $p_4 = 13$. Thus $\Psi(\lambda)$ is the alcove with the following Shi tableau.*

$k_{13} = 3$	$k_{12} = 1$	$k_{11} = 1$
$k_{23} = 1$	$k_{22} = 0$	
$k_{33} = 1$		

Proposition 2.11. *The map Ψ from n-cores to dominant alcoves is a bijection.*

Proof. We first show that we indeed produce an alcove by the process above. By Lemma 2.2, it is enough to show that $k_{it} + k_{t+1,j} \leq k_{ij} \leq k_{it} + k_{t+1,j} + 1$ for all t such that $1 \leq t < j$.

$k_{ij} = \lfloor \frac{p_{j+1} - p_i}{n} \rfloor$ implies that

$$k_{ij} = \frac{p_{j+1} - p_i}{n} - B_{ij} \text{ where } 0 \leq B_{ij} < 1. \tag{2.4}$$

Let t be such that $0 \leq t < j$. Using (2.4), we have

$$k_{it} + k_{t+1,j} = \frac{p_{t+1} - p_i}{n} - \frac{p_{j+1} - p_{t+1}}{n} - B_{ij} - B_{j+1,t+1}.$$

Now let $A = B_{ij} + B_{j+1,t+1}$, so that $0 \leq A < 2$. We have

$$k_{it} + k_{t+1,j} + A = \frac{p_{j+1} - p_i}{n}.$$

Thus

$$\lfloor k_{it} + k_{t+1,j} + A \rfloor = \lfloor \frac{p_{j+1} - p_i}{n} \rfloor = k_{ij}$$

or, since k_{it} and $k_{t+1,j}$ are integers,

$$k_{it} + k_{t+1,j} + \lfloor A \rfloor = k_{ij}. \tag{2.5}$$

Combining (2.5) with $\lfloor A \rfloor$ is equal to 0 or 1 shows that the conditions in Lemma 2.2 are satisfied and we have the Shi coordinates of an alcove. Since each $k_{ij} \geq 0$, it is an alcove in the dominant chamber.

Now we reverse the process described above to show that Ψ is a bijection. Let $\{k_{ij}\}_{1 \leq i \leq j \leq n-1}$ be the Shi coordinates of a dominant alcove. Write $p_i = nq_i + r_i$ for the intermediate values $\{p_i\}$, which we first calculate. Then $p_1 = q_1 = r_1 = 0$ and $q_i = k_{1,i-1}$. We must now determine $r_2, \ldots, r_n$, a permutation of $1, \ldots, n-1$. However, since

$$k_{ij} = \begin{cases} q_{j+1} - q_i & \text{if } r_{j+1} > r_i \\ q_{j+1} - q_i - 1 & \text{if } r_{j+1} < r_i. \end{cases}, \tag{2.6}$$

we can determine the inversion table for this permutation, using k_{ij} for $2 \leq i \leq j \leq n-1$ and $q_1, \ldots, q_n$. Indeed,

$$\begin{aligned} \mathrm{Inv}(r_{j+1}) &= |\{r_i \mid 1 \leq i < j+1 \text{ and } r_i > r_{j+1}\}| \\ &= |\{(k_{1j}, k_{1,i-1}, k_{ij}) \mid k_{1j} = k_{1,i-1} + k_{ij} + 1\}|. \end{aligned} \tag{2.7}$$

Therefore, we can compute $r_2, \ldots, r_n$ and therefore $p_1, p_2, \ldots, p_n$. We can now sort the $\{p_i\}$ according to their residue mod n, giving us $\tilde{p}_1, \ldots, \tilde{p}_n$; from this, $(b_0, \ldots, b_{n-1})$. Note that $(b_0, \ldots, b_{n-1})$ is a permutation of $q_1, \ldots, q_n$. □

Example 2.12. *We continue Examples 2.9 and 2.10 here. Suppose we are given that $n = 4$ and the alcove coordinates $k_{13} = 3$, $k_{12} = 1$, $k_{11} = 1$, $k_{23} = 1$, $k_{22} = 0$, and $k_{33} = 1$. That is,*

$$T_R = \begin{array}{|c|c|c|} \hline k_{13} & k_{12} & k_{11} \\ \hline k_{23} & k_{22} & \multicolumn{1}{c}{} \\ \cline{1-2} k_{33} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|} \hline 3 & 1 & 1 \\ \hline 1 & 0 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

We demonstrate Ψ^{-1} and calculate (b_0, b_1, b_2, b_3) and thereby the 4-core λ. We have $q_1 = 0$, $q_2 = 1$, $q_3 = 1$, and $q_4 = 3$, and $r_1 = 0$, from k_{13}, k_{12}, and k_{11}. We must determine r_2, r_3, r_4, a permutation of $1, 2, 3$.

Using (2.7), we know $\mathrm{Inv}(r_4) = 2$, *since* $k_{13} = k_{11}+k_{23}+1$ *and* $k_{13} = k_{12}+k_{33}+1$.

$\mathrm{Inv}(r_3) = 0$, *since* $k_{12} \neq k_{11} + k_{22} + 1$.

$\mathrm{Inv}(r_2) = 0$, *always.*

Therefore we have $r_3 = 3$, $r_2 = 2$, and $r_4 = 1$, which means $b_1 = q_4 = 3$, $b_2 = q_2 = 1$, and $b_3 = q_3 = 1$.

Remark 2.13. *The column (or row) sums of the Shi tableau of an alcove give us a partition whose conjugate is $(n-1)$-bounded, as in the bijections of Lapointe and Morse (2005) or Björner and Brenti (1996)*

2.7.2. *Geometric description.* The bijection Φ associates an n-core to an alcove through the $\widehat{\mathfrak{S}}_n$ action described in Sections 2.5.2 and 2.3. The map $\Phi : w\emptyset \mapsto w^{-1}\mathcal{A}_0$ for $w \in \widehat{\mathfrak{S}}_n$ a minimal length coset for $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$, is a bijection. In Fishel and Vazirani (2010), it is shown that the m-minimal alcoves of Shi regions in $\mathcal{S}_{n,m}$ correspond, under Φ, to n-cores which are also $(nm+1)$-cores.

3. Separating wall $H_{\theta,m}$

Separating walls were defined in Section 2.2 as a wall of a region which separates the region from $\mathcal{A}_0$. Equivalently for alcoves, $H_{\alpha,k}$ is a separating wall for the alcove $w^{-1}\mathcal{A}_0$ if there is a simple reflection s_i, where $0 \leq i < n$, such that $w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$ and $(s_i w)^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^-$. We want to count the regions which have $H_{\alpha,m}$ as a separating wall, for any $\alpha \in \Delta^+$. We do this by induction and the base case will be $\alpha = \theta$. Our main result in this section characterizes the regions which have $H_{\theta,m}$ as a separating wall by describing the n-core partitions associated to them under the bijections Ψ and Φ described in Section 2.7.

Theorem 3.1. *Let $\Psi : \mathcal{C}_n \to \mathfrak{A}_n$ be the bijection described in Section 2.7.1, let $R \in \mathcal{S}_{n,m}$ have m-minimal alcove $\mathcal{A}$, and let λ be the n-core such that $\Psi(\lambda) = \mathcal{A}$. Then $H_{\theta,m}$ is a separating wall for the region R if and only if $h^{\lambda}_{11} = n(m-1)+1$.*

Proof. Let $\vec{b}(\lambda) = (b_0, b_1, \ldots, b_{n-1})$ be the vector of level numbers for the n-core λ, so $b_o = 0$. We first note that $h_{11} = \beta_1 = n(m-1)+1$ if and only if $b_1 = m$ and $b_i < m$ for $1 < i \leq n-1$.

Now suppose that $H_{\theta,m}$ is a separating wall for the region R. Let $\{e_{ij}\}$ be the coordinates of R and let $\{k_{ij}\}$ be the coordinates of $\mathcal{A}$. By Lemma 2.6, we know that $e_{1,n-1} = m$ and $e_{1t} + e_{t+1,n-1} = m-1$, for all t such that $1 \leq t < n-1$. Therefore for all e_{ij} except $e_{1,n-1}$, we have $e_{ij} \leq m-1$, so that $e_{ij} = k_{ij}$. Since $k_{1t} + k_{t+1,n-1} \leq k_{1,n-1} \leq k_{1t} + k_{t+1,n-1} + 1$, we have that $k_{1,n-1} \leq m$, so indeed the Shi coordinates of R are the same as the coordinates of $\mathcal{A}$.

Consider the proof of Proposition 2.11 where we describe Ψ^{-1}, but in this situation. We see that $\{q_i\}_{1 \leq i \leq n-1}$, a nonincreasing rearrangement of $(b_1 \ldots, b_{n-1})$, is made up of m and $n-2$ nonnegative integers strictly less than m. So we need only show that $b_1 = m$, in view of our first remark of the proof. Combining (2.7) with the facts that if $H_{\theta,m}$ is a separating wall for a region then $e_{ij} = k_{ij}$ and, then by Lemma 2.6, $k_{1,n-1} = k_{1,i-1} + k_{i,n+1} + 1$ for all i such that $2 \le i \le n$, we have $\mathrm{Inv}(r_n) = n-1$. This implies that $r_n = 1$, so that $b_1 = q_n = k_{1,n-1} = m$.

Conversely, suppose that $h^\lambda_{11} = n(m-1)+1$, so that $b_1 = m$ and $b_i \le m-1$ for $1 < i \le n-1$. Then $\tilde{p}_2 = nm+1$ and $\tilde{p}_i = nb_{i-1} + i - 1 \le n(m-1) + i - 1 \le n(m-1)+n-1 = nm-1$. Therefore, $p_1 = 0$ and $p_n = nm+1$ and $p_i \le nm-1$, so that $q_1 = 0$, $q_n = m$, $r_n = 1$, and $q_i \le m-1$ and thus $k_{1,n-1} = m$ and $k_{1i} \le m-1$. By specializing (2.6) to $j = n-1$, we have

$$k_{i,n-1} = \begin{cases} q_n - q_i & \text{if } r_n > r_i \\ q_n - q_i - 1 & \text{if } r_n < r_i. \end{cases} \tag{3.1}$$

Then, by (3.1), $k_{i,n-1} = q_n - q_i - 1$, so that

$$k_{1,i-1} + k_{i,n-1} = q_i + q_n - q_i - 1 = m - 1.$$

Since $k_{ij} \le m$ for $1 \le i \le j \le n-1$, $k_{ij} = e_{ij}$ and the conditions in Lemma 2.6 that $H_{\theta,m}$ be a separating wall are fulfilled. □

We can also look at the regions which have $H_{\theta,m}$ as a separating wall in terms of the geometry directly. Theorem 3.5 is an alternate version of Theorem 3.1.

Proposition 3.2. *Let λ be an n-core and $w \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ be of minimal length such that $\lambda = w\emptyset$. Let $k = \lambda_1 + \frac{n-1}{2}$. Let $\gamma = \alpha_{1,n-1} + \alpha_{2,n-1} + \cdots + \alpha_{n-1,n-1}$.*

(1) *Then the affine hyperplane $H_{\gamma,k}$ passes through the corresponding alcove $w^{-1}\mathcal{A}_0$. More precisely, $\langle w^{-1}(\frac{1}{n}\rho) \mid \gamma\rangle = k$.*
(2) *Then the affine hyperplane H_{γ,λ_1} passes through the corresponding alcove $w^{-1}\mathcal{A}_0$. More precisely, $\langle w^{-1}(\Lambda_r) \mid \gamma\rangle = \lambda_1$, where $r \equiv \lambda_1 \mod n$.*

Proof. First, recall that $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha = (\frac{n-1}{2}, \frac{n-1}{2} - 1, \ldots, \frac{1-n}{2})$.

Hence $\frac{1}{n}\rho \in \mathcal{A}_0$ and so $w^{-1}(\frac{1}{n}\rho) \in w^{-1}\mathcal{A}_0$. Let $\eta = \sum_i \varepsilon_i$. Recall $V = \eta^\perp$ as for all $(a_1, \ldots, a_n) \in V$ we have $\sum_i a_i = 0$. Observe that for all $v \in V$, $\langle v \mid \gamma\rangle = \langle v \mid \eta - n\varepsilon_n\rangle = \langle v \mid -n\varepsilon_n\rangle$. So it suffices to show $\langle w^{-1}(\frac{1}{n}\rho) \mid \varepsilon_n\rangle = -\frac{k}{n}$.

Recall we may write $w = t_\beta u$ where $\beta \in Q$ and $u \in \mathfrak{S}_n$, where t_β is translation by β. Please see Humphreys (1990) for details. Then $w^{-1} = u^{-1}t_{-\beta} = t_{u^{-1}(-\beta)}u^{-1}$ satisfies $u^{-1}(-\beta) \in Q^+$.

Write $\lambda_1 = nq - (n-r)$ with $0 \le n-r < n$. Then $1 \le r \le n$, $q = \lceil \frac{\lambda_1}{n} \rceil$, and $r \equiv \lambda_1 \mod n$. Let a_i be the level of the first gap in runner i of the balanced abacus diagram for λ and write $\vec{n}(\lambda) = (a_1, a_2, \ldots, a_n)$. It is worth noting that $\vec{n}(\lambda) = w(0, \ldots, 0)$. By (Berg et al., 2009, Prop 3.2.13), the largest entry of $\vec{n}(\lambda)$ is $a_r = q$ and the rightmost occurrence of q occurs in the r^{th} position. Hence the smallest entry of $-\vec{n}(\lambda)$ is $-q$ and its rightmost occurrence is also in position r. Since $u^{-1} \in \mathfrak{S}_n$ is of minimal length such that $u^{-1}(-\vec{n}(\lambda)) \in Q^+$, we have that $u^{-1}(\varepsilon_r) = \varepsilon_n$.

Now we compute

$$\begin{aligned}
\langle w^{-1}(\frac{1}{n}\rho) \mid \varepsilon_n\rangle &= \langle t_{u^{-1}(-\vec{n}(\lambda))}u^{-1}(\frac{1}{n}\rho) \mid \varepsilon_n\rangle \\
&= \langle u^{-1}(-\vec{n}(\lambda)) \mid \varepsilon_n\rangle + \langle u^{-1}(\frac{1}{n}\rho) \mid \varepsilon_n\rangle \\
&= \langle u^{-1}(-\vec{n}(\lambda)) \mid \varepsilon_n\rangle + \langle \frac{1}{n}\rho \mid u(\varepsilon_n)\rangle \\
&= -q + \langle \frac{1}{n}\rho \mid \varepsilon_r\rangle = -q + \frac{1}{n}(\frac{n-1}{2} - (r-1)) \\
&= -\frac{1}{n}(nq - \frac{n-1}{2} + r - 1) = -\frac{1}{n}(nq - (n-r) + n - 1 - \frac{n-1}{2}) \\
&= -\frac{1}{n}(nq - (n-r) + \frac{n-1}{2}) = -\frac{1}{n}(\lambda_1 + \frac{n-1}{2}) \\
&= -\frac{k}{n}.
\end{aligned}$$

For the second statement, note the fundamental weight $\Lambda_j \in V$ has coordinates given by

$$\Lambda_j = \frac{1}{n}((n-j)(\varepsilon_1 + \cdots + \varepsilon_j) - j(\varepsilon_{j+1} + \cdots + \varepsilon_n)).$$

So $\Lambda_j \in H_{\alpha_i,0}$ for $i \neq j$, $\Lambda_j \in H_{\alpha_j,1}$, and the $\{\Lambda_j \mid 1 \leq j \leq n\} \cup \{0\}$ are precisely the vertices of $\mathcal{A}_0$. For the notational consistency of this statement and others below, we will adopt the convention that $\Lambda_0 = 0$ (which is consistent with considering $0 \in H_{\theta,0} = H_{\alpha_0,1}$). Hence we have that $w^{-1}(\Lambda_j) \in w^{-1}\mathcal{A}_0$.

As above we compute

$$\begin{aligned}
-\frac{1}{n}\langle w^{-1}(\Lambda_r) \mid \gamma\rangle &= \langle w^{-1}(\Lambda_r) \mid \varepsilon_n\rangle \\
&= -q + \langle \Lambda_r \mid \varepsilon_r\rangle = -q + \frac{1}{n}(n-r) \\
&= -\frac{1}{n}(nq - (n-r)) = -\frac{1}{n}(\lambda_1).
\end{aligned}$$

□

Proposition 3.3. *Let λ be an n-core and $w \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ be of minimal length such that $\lambda = w\emptyset$. Let $K = \ell(\lambda) + \frac{n-1}{2}$. Let $\Gamma = \alpha_{1,n-1} + \alpha_{1,n-2} + \cdots + \alpha_{1,1}$.*

1. *Then the affine hyperplane $H_{\Gamma,K}$ passes through the corresponding alcove $w^{-1}\mathcal{A}_0$. More precisely, $\langle w^{-1}(\frac{1}{n}\rho) \mid \Gamma\rangle = K$.*
2. *Then the affine hyperplane $H_{\Gamma,\ell(\lambda)}$ passes through the corresponding alcove $w^{-1}\mathcal{A}_0$. More precisely, $\langle w^{-1}(\Lambda_{s-1}) \mid \Gamma\rangle = \ell(\lambda)$, where $1 - s \equiv \ell(\lambda)$ mod n.*

Proof. First note $\langle v \mid \Gamma\rangle = \langle v \mid n\varepsilon_1\rangle$ for all $v \in V$, so it suffices to compute $\langle w^{-1}(\frac{1}{n}\rho) \mid n\varepsilon_1\rangle$.

Next, note $\Gamma = n\varepsilon_1$.

Write $\ell(\lambda) = nM + (1-s)$ with $1 \leq s \leq n$, so $-M = -\left\lceil \frac{\ell(\lambda)}{n} \right\rceil$. By Berg et al. (2009), the smallest entry of $\vec{n}(\lambda) = (a_1, a_2, \ldots, a_n)$ is $a_s = -M$ and the leftmost occurrence of $-M$ occurs in the s^{th} position. Hence the largest entry of $-\vec{n}(\lambda)$ is

M and its leftmost occurrence is also in position s. Then for u as above, it is clear $u(\varepsilon_1) = \varepsilon_s$. So, by a similar computation as above,

$$\begin{aligned}
\langle w^{-1}(\frac{1}{n}\rho) \mid \Gamma\rangle &= n\langle w^{-1}(\frac{1}{n}\rho) \mid \varepsilon_1\rangle \\
&= n\langle u^{-1}(-\vec{n}(\lambda)) \mid \varepsilon_1\rangle + n\langle u^{-1}(\frac{1}{n}\rho) \mid \varepsilon_1\rangle \\
&= nM + \langle \rho \mid \varepsilon_s\rangle = nM + \frac{n-1}{2} - (s-1) \\
&= \ell(\lambda) + \frac{n-1}{2} = K.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\langle w^{-1}(\Lambda_{s-1}) \mid \varepsilon_1\rangle &= M + \langle \Lambda_{s-1} \mid \varepsilon_s\rangle = M + \frac{1}{n}(-(s-1)) \\
&= \frac{1}{n}(nM + (1-s)) = \frac{1}{n}\ell(\lambda).
\end{aligned}$$

□

Taking subscripts $\mod n$ we have $\langle w^{-1}(\Lambda_{\lambda_1}) \mid \gamma\rangle = \lambda_1$ and $\langle w^{-1}(\Lambda_{-\ell(\lambda)}) \mid \Gamma\rangle = \ell(\lambda)$.

Corollary 3.4. $n\langle w^{-1}(\frac{1}{n}\rho) \mid \theta\rangle = \lambda_1 + \ell(\lambda) + n - 1$.

Note that when $\lambda \neq \emptyset$, the above quantity is $h_{11}^{\lambda} + n$ where h_{11}^{λ} is the hooklength of the first box. (One could also set $h_{11}^{\emptyset} = -1$.)

Theorem 3.5. *Let $\Phi : \mathcal{C}_n \to \mathfrak{A}_n$ be the bijection described in Section 2.7.2, let $R \in \mathcal{S}_{n,m}$ have m-minimal alcove $\mathcal{A}$, and let λ be the n-core such that $\Phi(\lambda) = \mathcal{A}$. Then $H_{\theta,m}$ is a separating wall for the region R if and only if $h_{11}^{\lambda} = n(m-1)+1$.*

Proof. Let r, s, q, M, and w be as in Propositions 3.2 and 3.3. Suppose that $H_{\theta,m}$ is a separating wall for R and let i be such that $w^{-1}\mathcal{A}_0 \subseteq {H_{\theta,m}}^+$ and $w^{-1}s_i\mathcal{A}_0 \subseteq {H_{\theta,m}}^-$. Recall $\Lambda_j \in H_{\alpha_i,0}$ for all $j \neq i$, and $\Lambda_i \in H_{\alpha_i,1}$. Hence $w^{-1}(\Lambda_j) \in H_{\theta,m}$ but $w^{-1}(\Lambda_i) \in H_{\theta,m+1}$. In fact, this configuration of vertices characterizes separating walls.

Note

$$\langle \Lambda_j \mid \varepsilon_s - \varepsilon_r\rangle = \begin{cases} 1 & \text{if } s \leq j < r \\ -1 & \text{if } s > j \geq r \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

By Propositions 3.2 and 3.3, $\langle w^{-1}(\Lambda_j) \mid \theta\rangle = M + q + \langle \Lambda_j \mid \varepsilon_s - \varepsilon_r\rangle$. Because $H_{\theta,m}$ is a separating wall, this yields $M + q + \langle \Lambda_j \mid \varepsilon_s - \varepsilon_r\rangle = m + \delta_{i,j}$. We must consider two cases. First, $M + q = m$ and $\langle \Lambda_j \mid \varepsilon_s - \varepsilon_r\rangle = \delta_{i,j}$. In other words, by (3.2) $s \leq j < r$ implies $j = i$. More precisely, $r - s = 1$, $s = i$, and $\varepsilon_s - \varepsilon_r = \alpha_i$. In the second case, $M + q - 1 = m$ and $\langle \Lambda_j \mid \varepsilon_s - \varepsilon_r\rangle = \delta_{i,j} - 1$. In other words, $s > j \geq r$ for all $1 \leq j < n$ (and recall $\langle \Lambda_0 \mid \varepsilon_s - \varepsilon_r\rangle = \langle 0 \mid \varepsilon_s - \varepsilon_r\rangle = 0$). More precisely, $r - s = 1 - n$ and $\varepsilon_s - \varepsilon_r = -\theta$.

Putting this all together for $\lambda \neq \emptyset$,

$$\begin{aligned} h_{11}^{\lambda} &= \ell(\lambda) + \lambda_1 - 1 \\ &= (nM + 1 - s) + (nq - (n - r)) - 1 \\ &= n(M+q) - n + (r - s) = \begin{cases} nm - n + 1 \\ n(m+1) - n + (1-n) \end{cases} \\ &= n(m-1) + 1. \end{aligned}$$

Conversely, if $h_{11}^{\lambda} = n(m-1)+1$, then by the computation above $n(M+q) - n + (r-s) = nm - n + 1$, which forces $n(M+q-1-m+1) = 1+s-r$. Note $2-n \leq 1+s-r \leq n$. If $1+s-r < n$, divisibility forces $0 = 1+s-r = M+q-m$. In other words, $\varepsilon_s - \varepsilon_r = \alpha_i$ for $i = s$, and we compute as above that $\langle w^{-1}(\Lambda_j) \mid \theta \rangle = M + q + \delta_{i,j}$ showing $H_{\theta,m}$ is a separating wall. If instead $1+s-r = n$, this forces $M+q-m = 1$ and $\varepsilon_s - \varepsilon_r = \theta$. Hence $\langle w^{-1}(\Lambda_j) \mid \theta \rangle = M+q-1 = m$ for all $j < n$, but $\langle w^{-1}(0) \mid \theta \rangle = M+q = m+1$, so that $H_{\theta,m}$ is again a separating wall for $w^{-1}\mathcal{A}_0$.

As a side note, similar calculations show that $h_{11}^{\lambda} = n(m-1)-1$ if and only if either $M+q = m$ and $r-s = -1$, or $M+q = m-1$ and $r-s = n-1$. In both cases $H_{\theta,m}$ will not be a separating wall for $w^{-1}\mathcal{A}_0$, but will be a separating wall for $w^{-1}s_i\mathcal{A}_0$ where $i = s-1$. One vertex of $w^{-1}\mathcal{A}_0$ lies in $H_{\theta,m-1}$ and the rest in $H_{\theta,m}$.

□

We have the following corollary to Theorem 3.1 and Theorem 3.5.

Corollary 3.6. *There are m^{n-2} regions in $\mathcal{S}_{n,m}$ which have $H_{\theta,m} = H_{\alpha_{1n-1},m}$ as a separating wall.*

Proof. There are m^{n-2} vectors of level numbers $(b_0, b_1, \ldots, b_{n-1})$ such that $b_0 = 0$, $b_1 = m$, and $0 \leq b_i \leq m-1$ for $2 \leq i \leq n-1$. □

There are direct explanations for Corollary 3.6, but we need Theorem 3.1 and Theorem 3.5 to develop our recursions, where we need to know more than the number of regions which have $H_{\theta,m}$ as a separating wall. We use the number of hyperplanes which separate each region from the origin.

4. Generating functions

We use $\mathfrak{h}_{\alpha k}^{n}$ to denote the set of regions in $\mathcal{S}_{n,m}$ which have $H_{\alpha,k}$ as a separating wall. See Figure 3. In the language of Athanasiadis (2005), these are the regions whose corresponding co-filtered chain of ideals have α as an indecomposable element of rank k.

In this section, we present a generating function for regions in $\mathfrak{h}_{\alpha k}^{n}$. In Section 5, we discuss a recursion for regions. The recursion is found by adding all possible first columns to Shi tableaux for regions in $\mathcal{S}_{n-1,m}$ to create all Shi tableaux for regions in $\mathcal{S}_{n,m}$. The generating function keeps track of the possible first columns and rows. We use two statistics r() and c() on regions in the extended Shi arrangement. Let $R \in \mathcal{S}_{n,m}$ and define

$$\mathrm{r}(R) = |\{(j,k) : R \text{ and } \mathcal{A}_0 \text{ are separated by } H_{\alpha_{1j},k} \text{ and } 1 \leq k \leq m\}|$$

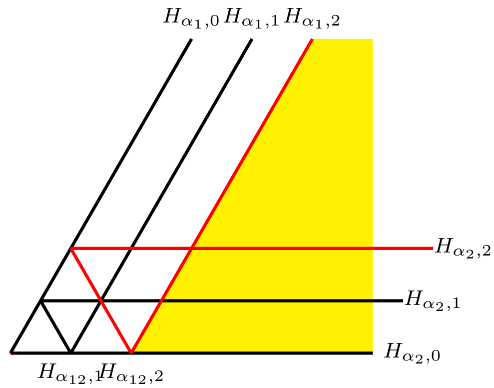

FIGURE 3. There are three regions in $\mathfrak{h}^3_{\alpha_1 2}$

and

$$\mathrm{c}(R) = |\{(i,k) : R \text{ and } \mathcal{A}_0 \text{ are separated by } H_{\alpha_{in-1},k} \text{ and } 1 \leq k \leq m\}|.$$

$\mathrm{r}(R)$ counts the number of translates of $H_{\alpha_{1j},0}$ which separate R from $\mathcal{A}_0$, for $1 \leq j \leq n-1$. Similarly for $\mathrm{c}(R)$ and translates of $H_{\alpha_{i,n-1},0}$.

The generating function is

$$f^n_{\alpha_{ij}m}(p,q) = \sum_{R \in \mathfrak{h}^n_{\alpha_{ij}m}} p^{\mathrm{c}(R)} q^{\mathrm{r}(R)}.$$

Example 4.1. $f^3_{\alpha_1 2}(p,q) = p^4q^2 + p^4q^3 + p^4q^4$.

We let $[k]_{p,q} = \sum_{j=0}^{k-1} p^j q^{k-1-j}$ and $[k]_q = [k]_{1,q}$. We will also need to truncate polynomials and the notation we use for that is

$$\left(\sum_{j=0}^{j=n} a_j q^j \right)_{\leq q^N} = \sum_{j=0}^{j=N} a_j q^j.$$

The statistics are related to the n-core partition assigned by Φ to the m-minimal alcove for the region.

Proposition 4.2. *Let λ be an n-core with vector of level numbers $(b_0, \ldots, b_{n-1})$ and suppose $\Psi(\lambda) = R$ and $R \in \mathfrak{h}^n_{\theta m}$. Then $r(R) = m + \sum_{i=2}^{n-1} b_i = \ell(\lambda)$ and $c(R) = m + \sum_{i=2}^{n-1}(m-1-b_i) = \lambda_1$.*

Proof. Let λ, $(b_0, \ldots, b_{n-1})$, and R be as in the statement of the claim. Let $\{e_{ij}\}$ be the region coordinates for R and $\{k_{ij}\}$ be the coordinates of R's m-minimal alcove, and let $\{p_i\}$ and $\{\tilde{p}_i\}$ be as in the definition of Ψ. Then

$$\begin{aligned} \mathrm{r}(R) &= e_{1,n-1} + e_{1,n-2} + \ldots + e_{11} \\ &= k_{1,n-1} + k_{1,n-2} + \ldots + k_{11} \\ &= \lfloor \frac{p_n}{n} \rfloor + \ldots + \lfloor \frac{p_1}{n} \rfloor \\ &= \lfloor \frac{\tilde{p}_n}{n} \rfloor + \ldots + \lfloor \frac{\tilde{p}_1}{n} \rfloor \\ &= \sum_{i=0}^{n-1} b_i \end{aligned}$$

The second part of the claim follows since $\mathrm{c}(R) = k_{1,n-1} + k_{2,n-1} + \ldots + k_{n-1,n-1}$ and $k_{i,n-1} = (m-1) - k_{1,i-1}$ for $R \in \mathfrak{h}^n_{\theta m}$ and $2 \leq i \leq n-1$. □

We can also relate the statistics r() and c() to the n-core partition corresponding under Φ to the m-minimal alcove of the region R.

For now, let $k_{w,\alpha}$ be the Shi coordinate of $w^{-1}\mathcal{A}_0$. Note $k_{w,\alpha} < \langle w^{-1}(\frac{1}{n}\rho) \mid \alpha\rangle < k_{w,\alpha} + 1$, so $k_{w,\alpha} = \left\lfloor \langle w^{-1}(\frac{1}{n}\rho) \mid \alpha\rangle \right\rfloor$.

Proposition 4.3. *Let λ be an n-core and $w \in \widehat{\mathfrak{S}}_n/\mathfrak{S}_n$ be of minimal length such that $\lambda = w\emptyset$.*

(1) *Then $\sum_i k_{w,\alpha_{i,n-1}} = \lambda_1$.*
(2) *Then $\sum_j k_{w,\alpha_{1,j}} = \ell(\lambda)$.*

Proof. Consider

$$\begin{aligned} \langle w^{-1}(\frac{1}{n}\rho) \mid \alpha_{i,n}\rangle &= \langle u^{-1}(-\beta) \mid \alpha_{i,n}\rangle + \langle u^{-1}(\frac{1}{n}\rho) \mid \alpha_{i,n}\rangle \\ &= \langle u^{-1}(-\beta) \mid \alpha_{i,n}\rangle + \langle \frac{1}{n}\rho \mid \varepsilon_{u(i)} - \varepsilon_{u(n)}\rangle. \\ &= \langle u^{-1}(-\beta) \mid \alpha_{i,n}\rangle + \langle \frac{1}{n}\rho \mid \varepsilon_{u(i)} - \varepsilon_r\rangle. \end{aligned}$$

Note $\langle u^{-1}(-\beta) \mid \alpha_{i,n}\rangle \in \mathbb{Z}$ and $\left\lfloor \langle \frac{1}{n}\rho \mid \varepsilon_{u(i)} - \varepsilon_r\rangle \right\rfloor = 0$ if $u(i) < r$, but $\left\lfloor \langle \frac{1}{n}\rho \mid \varepsilon_{u(i)} - \varepsilon_r\rangle \right\rfloor = -1$ if $u(i) > r$. Hence

$$\sum_i \left\lfloor \langle u^{-1}(\frac{1}{n}\rho) \mid \alpha_{i,n}\rangle \right\rfloor = -(n-r).$$

We then compute

$$\begin{aligned} \sum_i k_{w,\alpha_{i,n}} &= r - n + \sum_i \langle u^{-1}(-\beta) \mid \alpha_{i,n}\rangle \\ &= r - n + \langle u^{-1}(-\beta) \mid \gamma\rangle = r - n + qn \\ &= \lambda_1 \end{aligned}$$

by the computations in the proof of Proposition 3.2.

Likewise, $\left\lfloor \langle u^{-1}(\frac{1}{n}\rho) \mid \alpha_{1,j}\rangle \right\rfloor = \left\lfloor \langle \frac{1}{n}\rho \mid \varepsilon_s - \varepsilon_{u(j)}\rangle \right\rfloor = -1$ if $u(j) < s$ and zero otherwise. As above,

$$\begin{aligned} \sum_j k_{w,\alpha_{1,j}} &= -(s-1) + \sum_j \langle u^{-1}(-\beta) \mid \alpha_{1,j}\rangle \\ &= 1 - s + \langle u^{-1}(-\beta) \mid \Gamma\rangle = 1 - s + Mn \\ &= \ell(\lambda) \end{aligned}$$

by the computations in the proof of Proposition 3.3. □

We thus obtain another corollary to Theorem 3.1.

Corollary 4.4.

$$f^n_{\theta,m}(p,q) = p^m q^m [m]^{n-2}_{p,q}.$$

Proof. Corollary 4.4 follows from Theorem 3.1 or 3.5, Proposition 4.2, and the abacus representation of n-cores which have the prescribed hook length.

$$\begin{aligned}
f^n_{\theta,m}(p,q) &= \sum_{R\in\mathfrak{h}^n_{\theta m}} p^{\mathrm{c}(R)} q^{\mathrm{r}(R)} \\
&= \sum_{\substack{\lambda \text{ is an } n-\text{core} \\ h^\lambda_{11}=n(m-1)+1}} p^{m+\sum_{i=2}^{n-1} b_i} q^{m+\sum_{i=2}^{n-1}(m-1-b_i)} \\
&= \sum_{\substack{(b_2,\ldots,b_{n-1}) \\ 0\le b_i\le m-1}} p^m q^m \left(\prod_{i=2}^{n-1} p^{b_i} q^{m-1-b_i}\right) \\
&= p^m q^m (p^{m-1} + p^{m-2}q + \cdots + pq^{m-2} + q^{m-1})^{n-2} \\
&= p^m q^m [m]^{n-2}_{p,q}.
\end{aligned}$$

□

Corollary 3.6 can be derived from Corollary 4.4 by evaluating at $p = q = 1$.

5. Arbitrary separating wall

The next few lemmas provide an inductive method for determining whether or not $R \in \mathcal{S}_{n,m}$ is an element of $\mathfrak{h}^n_{\alpha_{2,n-1}m}$.

Given a Shi tableau $T_R = \{e_{ij}\}_{1\le i\le j\le n-1}$, where $R \in \mathcal{S}_{n,m}$, let $\tilde{T}_R$ be the tableau with entries $\{e_{ij}\}_{1\le i\le j\le n-2}$. That is, $\tilde{T}_R$ is T_R with the first column removed.

Example 5.1. *Suppose* $R \in \mathcal{S}_{5,m}$ *and*

$$T_R = \begin{array}{|c|c|c|c|} \hline e_{14} & e_{13} & e_{12} & e_{11} \\ \hline e_{24} & e_{23} & e_{22} & \\ \cline{1-3} e_{34} & e_{33} & & \\ \cline{1-2} e_{44} & & & \\ \cline{1-1} \end{array}. \quad \textit{Then } \tilde{T}_R = \begin{array}{|c|c|c|} \hline e_{13} & e_{12} & e_{11} \\ \hline e_{23} & e_{22} & \\ \cline{1-2} e_{33} & & \\ \cline{1-1} \end{array}$$

The next lemma tells us that $\tilde{T}_R$ is always the Shi tableau for a region in one less dimension.

Lemma 5.2. *If* T_R *is the tableau of a region* $R \in \mathcal{S}_{n,m}$ *and* $1 \le u \le v \le n-1$, *then* $\tilde{T}_R = T_{\tilde{R}}$ *for some* $\tilde{R} \in \mathcal{S}_{n-1,m}$.

Proof. This follows from Lemma 2.5. □

Lemma 5.3. *Let* T_R *be the Shi tableau for the region* $R \in \mathcal{S}_{n,m}$ *and let* $\tilde{R}$ *be defined by* $T_{\tilde{R}} = \tilde{T}_R$, *where* $\tilde{R} \in \mathcal{S}_{n-1,m}$ *by Lemma 5.2. Then* $R \in \mathfrak{h}^n_{\alpha_{i,n-2}m}$ *if and only if* $\tilde{R} \in \mathfrak{h}^{n-1}_{\alpha_{i,n-2}m}$.

Proof. This follows from Lemma 2.6. □

In terms of generating functions, Lemma 5.3 states:

$$f^n_{\alpha_{i,n-2}m}(p,q) = \sum_{R \in \mathfrak{h}^n_{\alpha_{i,n-2}m}} p^{\mathrm{c}(R)} q^{\mathrm{r}(R)} = \sum_{R_1 \in \mathfrak{h}^{n-1}_{\alpha_{i,n-2}m}} \sum_{\substack{R \in \mathcal{S}_{n,m} \\ \tilde{R}=R_1}} p^{\mathrm{c}(R)} q^{\mathrm{r}(R)} \tag{5.1}$$

If $R_1 \in \mathfrak{h}^{n-1}_{\alpha_{i,n-n}}$ and $R \in \mathcal{S}_{n,m}$ are such that $\tilde{R} = R_1$, then, since $e_{i,n-2} = m$ in the Shi tableau for R_1, $\mathrm{r}(R) = \mathrm{r}(R_1) + m$ and $\mathrm{c}(R) = \mathrm{c}(R_1) + k$, for some k. We need to establish the possible values for k.

We will use Proposition 3.5 from Richards (1996) to do this. His "pyramids" correspond to our Shi tableaux for regions, with his e and w being our n and $m+1$. He does not mention hyperplanes, but with the conversion ${}_u a_v = m - e_{u+1,v}$ his conditions in Proposition 3.4 become our conditions in Lemma 2.5.

In our language, his Proposition 3.5 becomes

Lemma 5.4 (Richards (1996))**.** *Let $\mu_1, \mu_2, \ldots, \mu_n$ be non-negative integers with*

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0 \textit{ and } \mu_i \leq (n-i)m.$$

Then there is a unique region $R \in \mathcal{S}_{n,m}$ with Shi tableau $T_R = \{e_{ij}\}_{1 \leq i \leq j \leq n-1}$ such that

$$\mu_j = \mu_j(R) = \sum_{i=1}^{n-j} e_{i,n-j} \textit{ for } 1 \leq j \leq n-1$$

We include his proof for completeness.

Proof. By Lemma 2.5, we have $e_{ij} \geq e_{i+1,j}$ and $e_{ij} \geq e_{i,j-1}$ for $1 \leq i < j \leq n-1$, which, combined with $0 \leq e_{ij} \leq m$, means that the column sums $\mu_j = \sum_{i=1}^{j} e_{ij}$ form a partition such that $0 \leq \mu_j \leq m(n-j)$.

We use induction on n to show that given such a partition μ, there is at most one region whose Shi tableau has column sums μ. It is clearly true for $n = 2$. Let $n > 2$ and suppose we had two regions R_1 with coordinates $\{e_{ij}\}_{1 \leq i \leq j \leq n-1}$ and R_2 with coordinates $\{f_{ij}\}_{1 \leq i \leq j \leq n-1}$ such that

$$\mu_j = \sum_{i=1}^{j} e_{ij} = \sum_{i=1}^{j} f_{ij}.$$

By induction $e_{ij} = f_{ij}$ for $1 \leq i \leq j \leq n-2$. Let u be the least index such that $e_{u,n-1} \neq f_{u,n-1}$ and assume $e_{u,n-1} < f_{u,n-1}$. Then since $\sum_{i=1}^{n-1} e_{i,n-1} = \sum_{i=1}^{n-1} f_{i,n-1}$, we have that $e_{v,n-1} > f_{v,n-1}$ for some v such that $u < v \leq n-1$. Then since $f_{u,n-1} \leq f_{u,v-1} + f_{v,n-1} + 1$ by Lemma 2.5 and $f_{u,v-1} = e_{u,v-1}$ by induction, we have

$$e_{u,n-1} < f_{u,n-1} \leq e_{u,v-1} + f_{v,n-1} + 1 \leq e_{u,v-1} + e_{v,n-1}.$$

This contradicts Lemma 2.5 applied to R_1.

However, there are $\frac{1}{mn+1}\binom{(m+1)n}{n}$ dominant Shi regions by Shi (1997) for $m = 1$ and Athanasiadis (2004) for $m > 1$ and it is well-known that there are also $\frac{1}{mn+1}\binom{(m+1)n}{n}$ partitions μ such that $0 \leq \mu_i \leq m(n-i)$, so we are done. $\square$

Example 5.5. *Consider R_1, R_2, and R_3 in $\mathcal{S}_{3,2}$ with tableaux*

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 2 & 2 & \\ \cline{1-2} 2 & & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 2 & 2 & \\ \cline{1-2} 1 & & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 2 & 2 & \\ \cline{1-2} 0 & & \\ \cline{1-1}\end{array}$$

respectively. Then $\tilde{R}_1 = \tilde{R}_2 = \tilde{R}_2 = R$, where R is the region in $\mathcal{S}_{2,2}$ with tableau

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

Let $\alpha = \alpha_{ij}$, where $1 \leq i \leq j \leq n-2$ in the following. Lemma 5.4 means for all pairs (R_1, k), where $R_1 \in \mathcal{S}_{n-1,m}$, $T_{R_1} = \{e_{ij}\}_{1 \leq i \leq j \leq n-2}$, and k is an integer such that $\sum_{i=1}^{n-2} e_{i,n-2} \leq k \leq (n-1)m$, there is a region $R \in \mathcal{S}_{n,m}$ such that $\tilde{R} = R_1$ and the first column sum of R's Shi tableau is k. Additionally, by Lemma 5.3, we have $R_1 \in \mathfrak{h}^{n-1}_{\alpha m}$ if and only if $R \in \mathfrak{h}^{n}_{\alpha m}$. On the other hand, given $R \in \mathcal{S}_{n,m}$ with Shi tableau $T_R = \{e_{ij}\}_{1 \leq i \leq j \leq n-1}$, let k be the first column sum of T_R. Then by Lemma 5.2 and the fact that $e_{i,n-1} \geq e_{i,n-2}$ for $1 \leq i \leq n-2$, the pair $(\tilde{R}, k)$ is such that $\tilde{R} \in \mathcal{S}_{n-1,m}$ and the first column sum of $T_{\tilde{R}}$ is not more than k. Again, by Lemma 5.3, we have $\tilde{R} \in \mathfrak{h}^{n-1}_{\alpha m}$ if and only if $R \in \mathfrak{h}^{n}_{\alpha m}$.

We continue (5.1), keeping in mind that $\mathrm{c}(R_1)$ is the first column sum for T_{R_1}. For ease of reading, write α for $\alpha_{i,n-2}$ in the following calculation.

$$\begin{aligned}
f^n_{\alpha m}(p,q) &= \sum_{R \in \mathfrak{h}^n_{\alpha m}} p^{\mathrm{c}(R)} q^{\mathrm{r}(R)} \\
&= \sum_{R_1 \in \mathfrak{h}^{n-1}_{\alpha m}} \sum_{\substack{R \in \mathcal{S}_{n,m} \\ \tilde{R} = R_1}} p^{\mathrm{c}(R)} q^{\mathrm{r}(R)} \\
&= \sum_{R_1 \in \mathfrak{h}^{n-1}_{\alpha m}} \sum_{\substack{k \\ \mathrm{c}(R_1) \leq k \leq n(m-1)}} p^{k} q^{\mathrm{r}(R_1)+m} \\
&= \sum_{R_1 \in \mathfrak{h}^{n-1}_{\alpha m}} \sum_{\substack{k' \\ 0 \leq k' \leq n(m-1)-\mathrm{c}(R_1)}} p^{\mathrm{c}(R_1)+k'} q^{\mathrm{r}(R_1)+m} \\
&= \left(\sum_{R_1 \in \mathfrak{h}^{n-1}_{\alpha m}} \sum_{\substack{k' \\ 0 \leq k' \leq n(m-1)}} p^{\mathrm{c}(R_1)+k'} q^{\mathrm{r}(R_1)+m} \right)_{\leq p^{(n-1)m}} \\
&= \left(\sum_{R_1 \in \mathfrak{h}^{n-1}_{\alpha m}} \sum_{\substack{k' \\ 0 \leq k' \leq n(m-2)}} p^{\mathrm{c}(R_1)+k'} q^{\mathrm{r}(R_1)+m} \right)_{\leq p^{(n-1)m}} \\
&\qquad \text{since } \mathrm{c}(R_1) \geq m \\
&= \left(q^m \left(\sum_{R_1 \in \mathfrak{h}^{n-1}_{\alpha m}} p^{\mathrm{c}(R_1)} q^{\mathrm{r}(R_1)} \right) \left(\sum_{\substack{k' \\ 0 \leq k' \leq n(m-2)}} p^{k'} \right) \right)_{\leq p^{(n-1)m}} \\
&= \left(q^m [(n-2)m+1]_p f^{n-1}_{\alpha m}(p,q) \right)_{\leq p^{(n-1)m}} .
\end{aligned}$$

The result of the above calculation is that

$$f^n_{\alpha m}(p,q) = \left(q^m[(n-2)m+1]_p f^{n-1}_{\alpha m}(p,q)\right)_{\leq p^{(n-1)m}} \tag{5.2}$$

when $\alpha = \alpha_{i,n-2}$.

The next proposition will provide a method for determining whether or not $H_{\alpha_{1n-j},m}$ is a separating wall for R. Given a Shi tableau $T = \{e_{ij}\}_{1\leq i\leq j\leq n-1}$ for a region in $\mathcal{S}_{n,m}$, let T' be its *conjugate* given by $T' = \{e'_{ij}\}_{1\leq i\leq j\leq n-1}$, where $e'_{ij} = e_{n-j,n-i}$.

Example 5.6.

By Lemma 2.5, T' will also be Shi tableau of a region in $\mathcal{S}_{n,m}$. Additionally, by Lemma 2.6, we have the following proposition.

Proposition 5.7. *Suppose the regions R and R' are related by*

$$(T_R)' = T_{R'}.$$

Then $R \in \mathfrak{h}^n_{\alpha_{ij}m}$ if and only if $R' \in \mathfrak{h}^n_{\alpha_{n-j,n-i}m}$.

In terms of generating functions, this becomes the following:

$$f^n_{\alpha_{ij}m}(p,q) = f^n_{\alpha_{n-j,n-i}m}(q,p). \tag{5.3}$$

We will now combine Theorem 3.1, Proposition 5.3, and Proposition 5.7 to produce an expression for the generating function for regions with a given separating wall.

Given a polynomial $f(p,q)$ in two variables, let $\phi_{k,m}(f(p,q))$ be the polynomial

$$(q^m[m(k-2)+1]_p f(p,q))_{\leq p^{(k-1)m}}$$

and let $\rho(f)$ be the original polynomial with p and q reversed: $f(q,p)$. Then (5.2) is

$$f^n_{\alpha_{ij}m}(p,q) = \phi_{n,m}(f^{n-1}_{\alpha_{ij}m}(p,q))$$

for $j = n-2$ and (5.3) is

$$f^n_{\alpha_{ij}m}(p,q) = \rho(f^n_{\alpha_{n-j,n-i}m}(p,q)).$$

Finally, the full recursion is

Theorem 5.8.

$$f^n_{\alpha_{uv}m}(p,q) = \phi_{n,m}(\phi_{n-1,m}(\ldots\phi_{v+2,m}(\rho(\phi_{v+1,m}(\ldots(\phi_{v-u+3,m}(p^mq^m[m]^{v-u}_{p,q})\ldots).$$

The idea behind the theorem is that, given a root α_{uv} in dimension $n-1$, we remove columns using Lemma 5.4 until we are in dimension $(v+1)-1$, then we conjugate, then remove columns again until our root is $\alpha_{1,v-u+1}$ and we are in dimension $(v-u+2)-1$.

Example 5.9. *We would like to know how many elements there are in $\mathfrak{h}^7_{\alpha_{24}2}$; that is, how many dominant regions in the 2-Shi arrangement for $n=7$ have $H_{\alpha_{24},2}$ as a separating wall. In order to make this readable, we omit the m subscript, since it is always 2 in this calculation.*

$$
\begin{aligned}
f^{7}_{\alpha_{24}2}(p,q) &= \left(q^2[11]_p f^6_{\alpha_{24}}(p,q)\right)_{\le p^{12}} \\
&= \left(q^2[11]_p \left(q^2[9]_p f^5_{\alpha_{24}}(p,q)\right)_{\le p^{10}}\right)_{\le p^{12}} \\
&= \left(q^2[11]_p \left(q^2[9]_p f^5_{\alpha_{13}}(q,p)\right)_{\le p^{10}}\right)_{\le p^{12}} \\
&= \left(q^2[11]_p \left(p^2[9]_p \left(q^2[7]_q f^4_{\alpha_{13}}(q,p)\right)_{\le q^8}\right)_{\le q^{10}}\right)_{\le q^{12}} \\
&= \left(q^2[11]_p \left(p^2[9]_p \left(q^2[7]_q \left(p^2q^2[2]^2_{p,q}\right)\right)_{\le q^8}\right)_{\le p^{10}}\right)_{\le p^{12}}
\end{aligned}
$$

After expanding this polynomial and evaluating at $p = q = 1$, we see there are 781 regions in the dimension 7 2-Shi arrangement which have $H_{\alpha_{24},2}$ as a separating wall.

Future work

It would be interesting to expand this problem by considering a given set of more than one separating walls. That is, given a set $\mathcal{H}_{\Delta'} = \{H_{\alpha,m}, \alpha \in \Delta' \subseteq \Delta\}$ of hyperplanes in the Shi arrangement, find the number of regions having all the hyperplanes in $\mathcal{H}_{\Delta'}$ as separating walls.

We would again be able to define a similar generation function, use the functions $\phi_{k,m}$ and ρ corresponding to truncation and conjugation of the Shi tableaux, but we should be able to compute the generating function for a suitably chosen base case.

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