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VANISHING INTEGRALS OF MACDONALD AND KOORNWINDER POLYNOMIALS

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Abstract.
When one expands a Schur function in terms of the irreducible characters of the symplectic (or orthogonal) group, the coefficient of the trivial character is 0 unless the indexing partition has an appropriate form. A number of q,t-analogues of this fact were conjectured in [10]; the present paper proves most of those conjectures, as well as some new identities suggested by the proof technique. The proof involves showing that a nonsymmetric version of the relevant integral is annihilated by a suitable ideal of the affine Hecke algebra, and that any such annihilated functional satisfies the desired vanishing property. This does not, however, give rise to vanishing identities for the standard nonsymmetric Macdonald and Koornwinder polynomials; we discuss the required modification to these polynomials to support such results.

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1. Introduction

Whenever one considers an identity of Schur functions, it is natural to consider whether that identity admits a q, t-analogue; that is, whether there is a corresponding identity for Macdonald polynomials. One such (classical) identity arises in the representation theory of real Lie groups, or equivalently in the theory of compact symmetric spaces.

Theorem. *[7] For any integer $n \geq 0$ and partition λ with at most n parts, the integral*

$$\int_{O \in O(n)} s_\lambda(O) dO$$

(with respect to Haar measure on the orthogonal group) vanishes unless $\lambda = 2\mu$ for some μ (that is, unless every part of λ is even). Similarly, for n even, the integral

$$\int_{S \in Sp(n)} s_\lambda(S) dS$$

vanishes unless $\lambda = \mu^2$ for some μ.

Recall the Schur function s_λ is a symmetric polynomial in n variables which gives the character of an irreducible (polynomial) representation of $U(n)$ ($GL(n)$). (More precisely, s_λ refers to a symmetric function in infinitely many variables which when specialized $s_\lambda(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ is as stated. For our purposes, only the finite version is relevant.) The character's value on a matrix is given by evaluating the Schur function at the matrix's eigenvalues. The above theorem describes which representations have $O(n)$ ($Sp(n)$) invariants–exactly those indexed by partitions all of whose parts are even (occur with even multiplicity).

The symmetric function interpretation of this theorem is that if one expands s_λ in terms of the irreducible characters of $O(n)$ ($Sp(n)$), the coefficient of the trivial character is 0 unless $\lambda = 2\mu$ ($\lambda = \mu^2$). This formulation has a nice q, t-analogue in several cases.

Remark. The nonzero values of the integral are in this case all equal to 1; this will fail upon passing to the Macdonald analogue, although in all cases for which we can compute the nonzero values, said values are at least "nice" (i.e., expressible as a ratio of products of binomials).

This can in turn be restated in terms of the eigenvalue densities of the orthogonal and symplectic groups. For the symplectic group, this is particularly simple (integrating over the torus T instead of the whole group):

$$\int s_\lambda(z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_n, z_n^{-1}) \prod_{1\le i\le n} |z_i - 1/z_i|^2 \prod_{1\le i<j\le n} |z_i + 1/z_i - z_j - 1/z_j|^2 \, dT$$

vanishes unless $\lambda = \mu^2$ for some μ. For the orthogonal group, the situation is more complicated, as the orthogonal group has two components, and the structure of the eigenvalues on a given component depends significantly on the parity of the dimension; we thus obtain four different integrals:

$$\int s_\lambda(\dots, z_i^{\pm 1}, \dots) \prod_{1\le i<j\le n} |z_i + 1/z_i - z_j - 1/z_j|^2 \, dT \tag{1.1}$$

$$\int s_\lambda(\dots, z_i^{\pm 1}, \dots, \pm 1) \prod_{1\le i\le n-1} |z_i - 1/z_i|^2 \prod_{1\le i<j\le n-1} |z_i + 1/z_i - z_j - 1/z_j|^2 \, dT \tag{1.2}$$

$$\int s_\lambda(\dots, z_i^{\pm 1}, \dots, 1) \prod_{1\le i\le n} |1 - z_i|^2 \prod_{1\le i<j\le n} |z_i + 1/z_i - z_j - 1/z_j|^2 \, dT \tag{1.3}$$

$$\int s_\lambda(\dots, z_i^{\pm 1}, \dots, -1) \prod_{1\le i\le n/2} |1 + z_i|^2 \prod_{1\le i<j\le n} |z_i + 1/z_i - z_j - 1/z_j|^2 \, dT \tag{1.4}$$

where the first two integrals correspond to the two components of $O(2n)$, and the last two integral correspond to the two components of $O(2n+1)$, and the claim is that each integral vanishes unless all ($2n$ or $2n+1$) parts of λ have the same parity.

In [10], q, t-analogues of each of these integrals were conjectured; that is, suitable choices of density were found such that specializing a Macdonald polynomial as above then integrating against the appropriate density gives 0 unless the partition satisfies the appropriate condition. In particular, the q, t-analogue of the symplectic vanishing integral (which we will prove in section 3) reads as follows.

Theorem. *For any integer $n \ge 0$, and partition λ with at most $2n$ parts, and any complex numbers q, t with $|q|, |t| < 1$, the integral*

$$\int P_\lambda(\dots, z_i^{\pm 1}, \dots; q, t) \prod_{1\le i\le n} \frac{(z_i^{\pm 2}; q)}{(t z_i^{\pm 2}; q)} \prod_{1\le i<j\le n} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)}{(t z_i^{\pm 1} z_j^{\pm 1}; q)} \, dT$$

vanishes unless $\lambda = \mu^2$ for some μ.

The proof below then suggests other statements along these lines, some of which are conjectured in [10], but some of which are new. Conversely, although we prove most of the conjectures of [10], Conjectures 3 and 5 of that paper remain open.

In many of these other identities, we relate a Macdonald or Koornwinder polynomial with one value of parameters q, t to polynomials in which q or t is replaced by its square or square root and thus these identities can be viewed as "quadratic" identities in the sense of basic hypergeometric series.

Similarly, one of the special cases of this Theorem proved in [10] was shown to be equivalent to a quadratic transformation for a univariate hypergeometric series. Thus in a sense these identities can be viewed as multivariate analogues of quadratic transformations.

There is a fundamental obstruction in using the affine Hecke algebra approach to directly proving the orthogonal cases which here only follow from the observation of [10] that the symplectic and orthogonal identities are equivalent by a sort of duality. Etingof (personal communication) has suggested an alternate approach using the construction of Macdonald polynomials in Etingof-Kirillov [4]. This approach works (for Jack polynomials; there are some technical difficulties in extending it to the quantum group case) for the orthogonal but *not* symplectic vanishing identities, and like our approach, it also gives no information about the nonzero values of the integrals. Presumably others of the identities we prove below could be proved in similar ways, where for Koornwinder polynomials we must use the construction of Oblomkov-Stokman [8].

In [12] the Koornwinder polynomials are generalized to a family of bi-orthogonal abelian functions. It is thus natural to conjecture that the vanishing identities should extend to the elliptic level. At present, this is somewhat problematic as neither the (double) affine Hecke algebra approach nor the Oblomkov-Stokman construction have been extended to this setting.

It is also worth noting that a *different* limit of the bi-orthogonal abelian functions gives ordinary symmetric Macdonald polynomials (as orthogonal polynomials) [11], suggesting that our Macdonald polynomial identities should also be limits of elliptic vanishing identities. It is likely that taking different limits of a single elliptic vanishing identity could give both a Macdonald and a Koornwinder identity. A particularly likely example are the identities (4.1) and (4.7).

Identities (4.5) and its dual (4.6) below can be generalized using a third approach that actually works on both cases. This will be discussed in a future paper.

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2. Conventions and Notation

A *partition* with $\leq n$ parts is a nonincreasing integer tuple $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$. We write $|\lambda| = \sum_{i=1}^n \lambda_i$ or as $\lambda \vdash \sum_{i=1}^n \lambda_i$. We also let $\ell(\lambda) = \max\{k \geq 0 \mid \lambda_k \neq 0\}$ so for instance, above we are taking $\ell(\lambda) \leq n$. We will denote the zero (or empty) partition by 0, when clear in context. We can picture a partition λ as a Ferrer's diagram: a collection of $|\lambda|$ cells whose coordinates we label (i,j) with $1 \leq j \leq \lambda_i$. So we can refer to a cell as $(i,j) \in \lambda$. We write λ' for the conjugate partition, which corresponds to a Ferrer's diagram with cells having coordinates (j,i).

A tuple $\nu = (\nu_1, \dots, \nu_n)$ of non-negative integers is called a *composition* of $|\nu| = \sum_i \nu_i$. We will denote by ν^+ the partition obtained by writing the parts of ν in nonincreasing order.

Given a partition μ, we write $\lambda = \mu^2$ if $\lambda_{2i-1} = \lambda_{2i} = \mu_i$. In particular, the parts of μ^2 occur with even multiplicity. We write $\lambda = 2\mu$ if $\lambda_i = 2\mu_i$, so each part of 2μ is even. Note that if $\lambda = \mu^2$ then the transposed partition $\lambda' = 2\mu'$.

We define

$$(a;q) = \prod_{k\geq 0}(1 - aq^k)$$

and $(a_1, a_2, \ldots, a_\ell; q) = (a_1; q)(a_2; q)\cdots(a_\ell; q)$. As an example, we set $(x_i^{\pm 1}x_j^{\pm 1}; q) = (x_ix_j, x_ix_j^{-1}, x_i^{-1}x_j, x_i^{-1}x_j^{-1}; q) = (x_ix_j; q)(x_ix_j^{-1}; q)(x_i^{-1}x_j; q)(x_i^{-1}x_j^{-1}; q)$. We write $(a; q)$ for what is often denoted $(a; q)_\infty$ in the literature, but as every q-symbol we use is infinite, there is no risk of confusion.

We also define

$$C^0_\mu(x; q, t) = \prod_{(i,j)\in\mu}(1 - q^{j-1}t^{1-i}x) = \prod_{1\le i\le \ell(\mu)} \frac{(t^{1-i}x; q)}{(q^{\mu_i}t^{1-i}x; q)},$$

$$\begin{aligned} C^-_\mu(x; q, t) &= \prod_{(i,j)\in\mu}(1 - q^{\mu_i - j}t^{\mu'_j - i}x) \\ &= \prod_{1\le i\le \ell(\mu)} \frac{(x; q)}{(q^{\mu_i}t^{\ell(\mu)-i}x; q)} \prod_{1\le i<j\le \ell(\mu)} \frac{(q^{\mu_i-\mu_j}t^{j-i}x; q)}{(q^{\mu_i-\mu_j}t^{j-i-1}x; q)}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} C^+_\mu(x; q, t) &= \prod_{(i,j)\in\mu}(1 - q^{\mu_i + j-1}t^{2-\mu'_j - i}x) \\ &= \prod_{1\le i\le \ell(\mu)} \frac{(q^{\mu_i}t^{2-\ell(\mu)-i}x; q)}{(q^{2\mu_i}t^{2-2i}x; q)} \prod_{1\le i<j\le \ell(\mu)} \frac{(q^{\mu_i+\mu_j}t^{3-j-i}x; q)}{(q^{\mu_i+\mu_j}t^{2-j-i}x; q)}. \end{aligned} \tag{2.2}$$

Similar to the q-symbols, we let $C^{0,\pm}_\mu(a_1, a_2, \ldots, a_\ell; q) = C^{0,\pm}_\mu(a_1; q)\cdots C^{0,\pm}_\mu(a_\ell; q)$. We refer to [10] for more details about these expressions and relations that hold among them (in particular those expressing $C^{0,\pm}_{\mu^2}(x; q, t)$ or $C^{0,\pm}_{2\mu}(x; q, t)$ in terms of $C^{0,\pm}_\mu(x; q^2, t)$ and $C^{0,\pm}_\mu(x; q, t^2)$).

It will be convenient in the sequel to use a plethystic substitution notation slightly different from that in the literature. When we write $g([r_k])$ for symmetric functions $g, r_k, k \ge 1$ we mean the image of g under the homomorphism $p_k \mapsto r_k$ where the p_k are the power sum symmetric functions. We take the convention $p_x = 0$ if $x \notin \{1, 2, 3, \ldots\}$. We abbreviate the case $r_{2k+1} = 0$, $r_{2k} = p_k$ by $g([2p_{k/2}])$. This is plethystic notation for the specialization $g(\cdots, \pm\sqrt{x_i}, \cdots)$.

2.1. The extended affine symmetric group $\widetilde{S}_n$

$\widetilde{S}_n = S_n \ltimes \mathbb{Z}^n$ can be identified with the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(x+n) = w(x) + n$ for all $x \in \mathbb{Z}$; if we also include bijections such that $w(x+n) = w(x) - n$, we obtain a group $\widetilde{S}^+_n = (S_n \times \mathbb{Z}_2) \ltimes \mathbb{Z}^n$, which is also an extended affine Weyl group (see section 6). The length 0 subgroup of $\widetilde{S}^+_n$ is generated by $\pi(x) = x + 1$ and $\iota(x) = n + 1 - x$.

$\widetilde{S}_n$ has generators $s_0, s_1, \ldots, s_{n-1}, \pi$ where

$$s_j(i) = \begin{cases} i & i \not\equiv j, j+1 \mod n \\ i+1 & i \equiv j \mod n \\ i-1 & i \equiv j+1 \mod n. \end{cases}$$

By convention we will view these bijections as acting on $\mathbb{Z}$ from the *right.*

It is easy to see these generators satisfy the type A braid relations

$$\begin{aligned} s_i s_j &= s_j s_i && i-j \not\equiv \pm 1 \pmod n \\ s_i s_j s_i &= s_j s_i s_j && i-j \equiv \pm 1 \pmod n; n>2, \end{aligned}$$

and

$$\pi s_i \pi^{-1} = s_{i-1}.$$

and quadratic relation $s_i^2 = 1$.

The *extended affine Hecke algebra* H_n *of type* A is defined to be the $\mathbb{C}(q,t)$-algebra with generators T_0,T_1,...T_{n-1},π, subject to the braid relations

$$\begin{aligned} T_i T_j &= T_j T_i && i-j \not\equiv \pm 1 \pmod n \\ T_i T_j T_i &= T_j T_i T_j && i-j \equiv \pm 1 \pmod n; n>2, \end{aligned}$$

and the quadratic relation

$$(T_i - t)(T_i + 1) = 0,$$

and

$$\pi T_i \pi^{-1} = T_{i-1}.$$

Given a reduced word $u = s_{i_1} s_{i_2} \cdots s_{i_k}$, we write $T_u = T_{i_1} T_{i_2} \cdots T_{i_k}$, which is independent of reduced word expression by the relations above. Note $T_u T_v = T_{uv}$ if $\ell(u) + \ell(v) = \ell(uv)$.

Observe that on specializing $t = 1$ we recover the group algebra $\mathbb{C}(q)\widetilde{S}_n$ whose generators we typically denote $\{s_0, s_1, \dots, s_{n-1}, \pi\}$.

Given an automorphism $\phi : H_n \to H_n$ and right module $\mathcal{L}$, we write $\mathcal{L}^\phi$ for the twisted module with action $v \cdot h := v(\phi(h))$. In the case $\phi(h) = T_u h T_u^{-1}$ we write $\mathcal{L}^u$ for $\mathcal{L}^\phi$.

We will write $\overline{T}_i = T_i + 1 - t$. Note $\overline{T}_i T_i = t$. (This is *not* Lusztig's bar involution.)

We have another presentation of H_n given by $T_1, T_2, \dots, T_{n-1}$, $Y_1^{\pm 1}, Y_2^{\pm 1}, \dots, Y_n^{\pm 1}$ with additional relations

$$\begin{aligned} Y_i Y_j &= Y_j Y_i && \forall i,j \\ T_i Y_j &= Y_j T_i && j \neq i, i+1 \\ T_i Y_i^{-1} T_i &= t Y_{i+1}^{-1} && 1 \le i < n, \end{aligned}$$

where we can also express the final one as $T_i Y_{i+1} = Y_i \overline{T}_i$.

This presentation relates to the first via:

$$Y_1 = T_1 \dots T_{n-1} \pi, \tag{2.3}$$

$$Y_2 = T_2 \dots T_{n-1} \pi \overline{T}_1, \tag{2.4}$$

$$\vdots$$

$$Y_n = \pi \overline{T}_1 \dots \overline{T}_{n-1}. \tag{2.5}$$

(This disagrees with the convention that $tY_{i+1} = T_i Y_i T_i$, but has the advantage of making dominant weights map to positive words!) That is, for a partition λ, $Y^\lambda = Y_1^{\lambda_1} Y_2^{\lambda_2} \cdots Y_n^{\lambda_n}$ simplifies in the other generators to a word involving only T_i and π and not involving $\overline{T}_i$.

H_n acts on the space of polynomials $V = \mathbb{C}(q^{1/n}, t)[x_1, \dots, x_n, (x_1 x_2 \cdots x_n)^{-1/n}]$ via:

$$T_i f = tf + \frac{x_{i+1} - tx_i}{x_{i+1} - x_i}(f^{s_i} - f) \tag{2.6}$$

$$T_0 f = tf + \frac{x_1 - tqx_n}{x_1 - qx_n}(f^{s_0} - f) \tag{2.7}$$

$$(\pi f)(x_1, \dots x_n) = f(qx_n, x_1, \dots x_{n-1}), \tag{2.8}$$

where $f^{s_i}(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i+1}, x_i, \dots, x_n)$ and $f^{s_0}(x_1, \dots, x_n) = f(qx_n, \dots, q^{-1}x_1)$. Observe

$$T_i 1 = t \tag{2.9}$$

$$\pi 1 = 1 \tag{2.10}$$

$$Y_i 1 = t^{n-i}. \tag{2.11}$$

Observe the T_i act trivially on $(x_1 x_2 \cdots x_n)^{-1/n}$, but π multiplies it by $q^{-1/n}$.

Given a partition λ or more generally a dominant weight of $SL_n \times GL_1$, i.e., a nonincreasing sequence of rational numbers with integer sum and integer differences, we can associate a monomial in V, namely $\prod_i x_i^{\lambda_i}$. This generates a $\widetilde{S}_n$-submodule of V. This however is not invariant under H_n, but if we sum the spaces corresponding to all weights weakly dominated by λ then the space is invariant under H_n, and similarly for weights strictly dominated by λ. The quotient of these two modules gives a deformation to H_n of the $\widetilde{S}_n$-submodule associated to λ. In this space, the commuting operators Y_i have joint eigenvalues which are simply permutations of the sequence

$$\dots q^{\lambda_i} t^{n-i} \dots$$

Generically, this deformation is a submodule of V, and thus the corresponding eigenfunctions are polynomials, namely the nonsymmetric Macdonald polynomials.

It will be convenient to introduce a special notation for dominant weights with negative parts. If μ, ν are partitions with $\ell(\mu) + \ell(\nu) \leq n$, then $\mu\bar{\nu}$ is the dominant weight vector of $SL_n \times GL_1$ $(\mu_1, \dots, \mu_{\ell(\mu)}, 0, \dots, 0, -\nu_{\ell(\nu)}, \dots, -\nu_1)$. We extend this to dominant weights with rational coefficients in the obvious way.

Recall if L is a functional on the polynomial space (or on any left module), then $h \in H_n$ acts on the right via $(L \cdot h)(f) := L(hf)$.

This representation has the following interpretation in terms of the double affine Hecke algebra (while we do not use or even define the double affine Hecke algebra here, it is worth noting we can view these problems in a larger context). Our affine Hecke algebra is a subalgebra of the double affine Hecke algebra, and it has a "trivial" module, which is the one dimensional module on which $\pi - 1$ and all $T_i - t$ vanish. If we induce this trivial module up to the double affine Hecke algebra and then restrict it back down, V sits inside the restriction. The Mackey formula thus gives us a decomposition of V into irreducibles (when q, t are generic) which we describe explicitly below.

If we specialize q, t to complex numbers such that $|q|, |t| < 1$, then the nonsymmetric density

$$\Delta_S = \Delta_S^{(n)}(q,t) = \prod_{i<j} \frac{(x_i/x_j, qx_j/x_i; q)}{(tx_i/x_j, qtx_j/x_i; q)}$$

is defined and can be integrated over the unit torus. Moreover it is a standard result of Macdonald polynomials theory that if $f \in \mathbb{C}(q^{1/n},t)[x_1,\dots,x_n,(x_1\cdots x_n)^{-1/n}]$ then

$$\frac{\int f\Delta_S\,dT}{\int \Delta_S\,dT} \in \mathbb{C}(q^{1/n},t), \tag{2.12}$$

where dT is Haar measure on the unit torus. That is, there exists a rational function in $q^{1/n}, t$ that agrees with the above for any specialization such that the integrals are defined. Similar comments apply to all the integrals we consider which can thus be considered either as analytic quantities with appropriately specialized parameters or as algebraic quantities with generic parameters. In particular, the normalized integral $\frac{\int f\Delta_S\,dT}{\int \Delta_S\,dT} = [E_0]f$ where E_0 is the nonsymmetric Macdonald polynomial corresponding to the empty partition. A similar statement holds in the other cases.

Above, we used the notation

$$[f_\mu]g$$

for the coefficient of f_μ in the expansion of g, where $\{f_\mu\}$ is a given a basis of some space of functions and g is another function in that space. It should be clear in all cases in which we use this notation which basis is intended.

Note that π is self-adjoint and the T_i are adjoint to T_{n-i} with respect to the inner product this density defines:

$$\langle f,g\rangle = \int f(x_1,\dots,x_n)g(\frac{1}{x_n},\dots,\frac{1}{x_1})\Delta_S\,dT.$$

An equivalent way of stating this uses the fact that $H_n \otimes H_n$ has a natural action on $\mathbb{C}(q^{1/n},t)[y_1,\dots,y_n,z_1,\dots,z_n,(y_1z_1\cdots y_nz_n)^{-1/n}]$ and says that the linear functional

$$L(h) = \int h(x_1,\dots,x_n,\frac{1}{x_n},\dots,\frac{1}{x_1})\Delta_S^{(n)}(q,t)\,dT$$

is annihilated by the ideal $\langle \pi\otimes 1 - 1\otimes\pi,\ T_i\otimes 1 - 1\otimes T_{n-i}, (1\le i\le n)\rangle$. As we will see below, such annihilation gives rise to vanishing identities. In this case, we obtain the (standard) fact that for weights λ and μ

$$\int P_\lambda(x_1,\dots,x_n)P_\mu(\frac{1}{x_n},\dots,\frac{1}{x_1})\widetilde{\Delta}_S\,dT$$

vanishes if $\lambda\neq\mu$. Here, $\widetilde{\Delta}_S$ is the symmetric density

$$\widetilde{\Delta}_S = \widetilde{\Delta}_S^{(n)}(q,t) = \prod_{i<j}\frac{(x_i/x_j, x_j/x_i; q)}{(tx_i/x_j, tx_j/x_i; q)} = \prod_{i\neq j}\frac{(x_i/x_j;q)}{(tx_i/x_j;q)}$$

which up to scalar is the symmetrization of Δ_S.

The operators Y_i are *not* self-adjoint and more generally the ideal does not contain elements of the form $Y_i \otimes 1 - c1 \otimes Y_j^{\pm 1}$. However, if we conjugate by $1 \otimes T_{w_0}$ it will contain $Y_i \otimes 1 - 1 \otimes Y_i$ and this implies orthogonality of Y-eigenvectors with respect to the conjugated inner product. With respect to the original inner product, we find that the eigenfunctions of the Y_i are orthogonal to the images of those functions under $T_{w_0}^{-1}$. This is precisely the orthogonality of nonsymmetric Macdonald polynomials given in [1]. (To be precise, Cherednik shows that the nonsymmetric Macdonald polynomials (a.k.a. the eigenfunctions of the Y_i) are orthogonal to the polynomials modified by the substitution $q \to \frac{1}{q}$, $t \to \frac{1}{t}$, but this turns out to be equivalent.) It follows that the symmetric Macdonald polynomials are orthogonal with respect to this density and hence the symmetrized density.

2.2. The extended affine hyperoctahedral group $\widetilde{C}_n$

We also consider $\widetilde{C}_n = C_n \ltimes \mathbb{Z}^n$, which can be identified with the centralizer in $\widetilde{S}_{2n}$ of the element ι of $\widetilde{S}_{2n}^{+}$, or equivalently as the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(i+2n) = w(i)+2n$ and $w(2n+1-i) = 2n+1-w(i)$. $\widetilde{C}_n$ has generators $s_0, s_1, \dots, s_n$, where

$$s_j(i) = \begin{cases} i & i \not\equiv j, j+1, 2n-j, 2n+1-j \mod 2n \\ i+1 & i \equiv j, 2n-j \mod 2n \\ i-1 & i \equiv j+1, 2n+1-j \mod 2n. \end{cases}$$

It is easy to see these generators satisfy the type C braid relations

$$\begin{aligned} &s_i s_j = s_j s_i && |i-j| > 1 \\ &s_i s_j s_i = s_j s_i s_j && |i-j| = 1, i, j \neq 0, n \\ &s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \\ &s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n \end{aligned}$$

and quadratic relation $s_i^2 = 1$.

For $n > 1$, the *affine Hecke algebra H_n^C of type BC* is defined to be the $\mathbb{C}(q,t,a,b,c,d)$-algebra with generators T_0,T_1,..., T_n, subject to the type C braid relations

$$\begin{aligned} T_i T_j &= T_j T_i && |i-j| > 1 \\ T_i T_j T_i &= T_j T_i T_j && |i-j| = 1, i, j \neq 0, n \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0 \\ T_n T_{n-1} T_n T_{n-1} &= T_{n-1} T_n T_{n-1} T_n \end{aligned}$$

and the quadratic relations

$$\begin{gathered} (T_0 + 1)(T_0 + cd/q) = 0, \\ (T_i + 1)(T_i - t) = 0, i \neq 0, n \\ (T_n + 1)(T_n + ab) = 0. \end{gathered}$$

In fact, the algebra H_n^C can be defined for $n = 1$ and all considerations below will work in that case. If $n = 1$ there are simply no braid relations, only quadratic ones. We omit the details.

The diagram automorphism of affine C_n gives rise to an action of the involution σ on H_n^C given by

$$\begin{aligned} \sigma T_i \sigma^{-1} &= T_{n-i} \\ \sigma a \sigma^{-1} &= c/\sqrt{q} \\ \sigma b \sigma^{-1} &= d/\sqrt{q} \end{aligned}$$

If the action of σ on scalars is trivial, i.e., $c = a\sqrt{q}$, $d = b\sqrt{q}$, then we can enlarge H_n^C to an *extended* affine Hecke algebra as in section 6. In general, we can view σ as giving an intertwiner between Hecke algebras with different parameters.

For $1 \leq i < n$, we will write $\overline{T}_i = T_i + 1 - t$. Note $\overline{T}_i T_i = t$. We set $\overline{T}_n = T_n + 1 + ab$ so that $\overline{T}_n T_n = -ab$ and $\overline{T}_0 = T_0 + 1 + \frac{cd}{q}$ so $\overline{T}_0 T_0 = -\frac{cd}{q}$.

As with type A, we have another presentation of H_n^C given by generators $T_1, \ldots, T_n$, $Y_1^{\pm 1}, Y_2^{\pm 1}, \ldots, Y_n^{\pm 1}$ with additional relations

$$\begin{aligned} Y_i Y_j &= Y_j Y_i \qquad \forall i, j \\ T_i Y_j &= Y_j T_i \qquad j \neq i, i+1 \\ T_i Y_i^{-1} T_i &= t Y_{i+1}^{-1} \qquad 1 \leq i < n \\ T_n Y_n^{-1} T_n &= -t^{2-2n} \frac{q}{cd} Y_n - t^{1-n} (\frac{q}{cd} + 1) T_n. \end{aligned}$$

This presentation relates to the first via:

$$Y_1 = T_1 \ldots T_n \ldots T_1 T_0 \tag{2.13}$$
$$Y_2 = T_2 \ldots T_n \ldots T_1 T_0 \overline{T}_1 \tag{2.14}$$
$$\vdots$$
$$Y_n = T_n \ldots T_1 T_0 \overline{T}_1 \ldots \overline{T}_{n-1}. \tag{2.15}$$

There is also an intertwiner

$$Y_\omega = \prod_{1 \leq i \leq n} T_n \ldots T_i \sigma$$

which satisfies $Y_i Y_\omega = Y_\omega Y_i$ (but note that the two Y_i live in different Hecke algebras) and $Y_\omega^2 = Y_1 Y_2 \ldots Y_n$. The significance of the intertwiner Y_ω is that the symmetric version: $Y_\omega (1 + Y_1^{-1})(1 + Y_2^{-1}) \cdots (1 + Y_n^{-1})$ takes a Koornwinder polynomial with parameters a, b, c, d to a multiple of the corresponding Koornwinder polynomial with parameters $c/\sqrt{q}, d/\sqrt{q}, a\sqrt{q}, b\sqrt{q}$. In fact, this is precisely the difference operator which was the fundamental tool of [10]. This is one reason why in section 6 we consider such a general version of extended affine Weyl groups.

When computing in H_n^C or more generally in the braid group, $B(\widetilde{C}_n)$, one helpful tool is the natural injection $H_n^C \to H_{2n}$ such that

$$T_i \mapsto T_i T_{2n-i}$$
$$T_0 \mapsto T_0$$
$$T_n \mapsto T_n$$

and such that σ acts as conjugation by π^n. Under this mapping, the natural lifting of the Y operators to the braid group behaves as follows:

$$Y_i \mapsto Y_i Y_{2n-i}^{-1} \tag{2.16}$$
$$Y_\omega \sigma^{-1} \mapsto \prod_{1\le i\le n} Y_i \pi^{-n}. \tag{2.17}$$

The Hecke algebra H_n^C and the intertwiner σ act on the vector space of Laurent polynomials $\mathbb{C}(q^{1/2}, t, a, b, c, d)[x_1^{\pm 1}, \dots, x_n^{\pm n}]$ via:

$$T_0 f = -(cd/q)f + \frac{(1-c/x_1)(1-d/x_1)}{1-q/x_1^2}(f^{s_0} - f) \tag{2.18}$$
$$T_i f = tf + \frac{x_{i+1} - tx_i}{x_{i+1} - x_i}(f^{s_i} - f) \tag{2.19}$$
$$T_n f = -abf + \frac{(1-ax_n)(1-bx_n)}{1-x_n^2}(f^{s_n} - f) \tag{2.20}$$
$$(\sigma f)(a,b,c,d;x_1,\dots x_n) = f(c/\sqrt{q}, d/\sqrt{q}, a\sqrt{q}, b\sqrt{q}; \sqrt{q}/x_n, \dots \sqrt{q}/x_1) \tag{2.21}$$

Recall that for $i \neq 0, n$, $f^{s_i}(x_1,\dots,x_i,x_{i+1},\dots,x_n) = f(x_1,\dots,x_{i+1},x_i,\dots,x_n)$, and $f^{s_0}(x_1,\dots,x_n) = f(q/x_1, x_2 \dots, x_n)$, $f^{s_n}(x_1,\dots,x_n) = f(x_1,\dots,x_{n-1},1/x_n)$.

In particular, in the space corresponding to monomials for the partition λ, the joint eigenvalues of the operators $Y_i, (abcdt^{2n-2}q^{-1})Y_i^{-1}$ are (signed) permutations of the sequence

$$\dots q^{\lambda_i} t^{2n-1-i}(abcd/q) \dots q^{-\lambda_i} t^{i-1}$$

The nonsymmetric density is

$$\Delta_K = \Delta_K^{(n)}(a,b,c,d;q,t) =$$
$$\prod_{1\le i\le n} \frac{(x_i^2, qx_i^{-2};q)}{(ax_i, bx_i, cx_i, dx_i, aqx_i^{-1}, bqx_i^{-1}, cx_i^{-1}, dx_i^{-1};q)} \prod_{1\le i<j\le n} \frac{(x_i x_j^{\pm 1}, qx_i^{-1}x_j^{\pm 1};q)}{(tx_i x_j^{\pm 1}, qtx_i^{-1}x_j^{\pm 1};q)}$$

Here we integrate over the unit torus with parameters specialized to have norm < 1. As with the S_n case, the normalized integral of any polynomial with rational function coefficients meromorphically continues to a rational function.

The operators T_i are self-adjoint with respect to the induced inner product.

The corresponding symmetric density is

$$\widetilde{\Delta}_K = \widetilde{\Delta}_K^{(n)}(a,b,c,d;q,t) =$$
$$\frac{(q;q)^n}{2^n n!} \prod_{1\le i\le n} \frac{(x_i^{\pm 2};q)}{(ax_i^{\pm 1}, bx_i^{\pm 1}, cx_i^{\pm 1}, dx_i^{\pm 1};q)} \prod_{1\le i<j\le n} \frac{(x_i^{\pm 1}x_j^{\pm 1};q)}{(tx_i^{\pm 1}x_j^{\pm 1};q)},$$

with normalization [5]

$$\int \widetilde{\Delta}_K^{(n)}(a,b,c,d;q,t)\,dT = \prod_{0\le j<n} \frac{(t, t^{2n-2-j}abcd;q)}{(t^{j+1}, t^j ab, t^j ac, t^j ad, t^j bc, t^j bd, t^j cd;q)}.$$

Again normalized integrals over this density can be taken by computing the constant coefficient in the expansion with respect to Koornwinder polynomials $K_\lambda^{(n)}$ [6], giving a rational function in q,t,a,b,c,d. In fact, one of the main results of [10] is that this integral essentially depends algebraically on n. More precisely, it is shown there that there exists a functional $I_K(f;q,t,T;a,b,c,d)$ on the space of ordinary symmetric functions such that

$$I_K(f;q,t,t^n;a,b,c,d) = [K_0^{(n)}(\cdots z_i\cdots;q,t;a,b,c,d)]f(\cdots z_i^{\pm 1}\cdots) = \frac{\int f\widetilde{\Delta}_K^{(n)}\,dT}{\int \widetilde{\Delta}_K^{(n)}\,dT}$$

for all integers $n\ge 0$. This can also be viewed as taking coefficients with respect to a basis $\widetilde{K}_\lambda(\,;q,t,T;a,b,c,d)$ of the space of symmetric functions over $\mathbb{C}(q,t,T,a,b,c,d)$ with the property that for all integers n such that $n\ge \ell(\lambda)$, $\widetilde{K}_\lambda(\cdots z_i^{\pm 1}\cdots;q,t,t^n;a,b,c,d) = K_\lambda^{(n)}(\cdots z_i\cdots;q,t;a,b,c,d)$. (These $\widetilde{K}_\lambda$ transform nicely under an analogue of Macdonald's involution and so we can use them to prove dual results in several cases.)

3. A $U(2n)/Sp(2n)$ vanishing integral

Theorem 3.1. *For any integer $n\ge 0$, and partition λ with at most $2n$ parts, and any complex numbers q, t with $|q|,|t|<1$, the integral*

$$\begin{aligned}\int P_\lambda^{(2n)}(z_1^{\pm 1},\dots,z_n^{\pm 1};q,t)\widetilde{\Delta}_K^{(n)}(\sqrt{t},-\sqrt{t},\sqrt{qt},-\sqrt{qt};q,t)\,dT\\ = \int P_\lambda^{(2n)}(z_1^{\pm 1},\dots,z_n^{\pm 1};q,t)\prod_{1\le i\le n}\frac{(z_i^{\pm 2};q)}{(tz_i^{\pm 2};q)}\prod_{1\le i<j\le n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)}{(tz_i^{\pm 1}z_j^{\pm 1};q)}\,dT \qquad (3.1)\end{aligned}$$

vanishes unless $\lambda=\mu^2$ for some μ.

Proof. Consider the following linear functional on the space of polynomials in $2n$ variables:

$$\begin{aligned}L(f) :=\\ \int f(z_1,\dots,z_n,1/z_n,\dots,1/z_1)\prod_{1\le i\le n}\frac{(z_i^2, qz_i^{-2};q)}{(tz_i^2, qtz_i^{-2};q)}\prod_{1\le i<j\le n}\frac{(z_iz_j^{\pm 1}, qz_i^{-1}z_j^{\pm 1};q)}{(tz_iz_j^{\pm 1}, qtz_i^{-1}z_j^{\pm 1};q)}\,dT\\ = \int f(z_1,z_2,\dots,z_n,1/z_n,\dots,1/z_2,1/z_1)\Delta_K^{(n)}(\sqrt{t},-\sqrt{t},\sqrt{qt},-\sqrt{qt};q,t)\,dT\end{aligned}$$

If f is symmetric, then we can freely symmetrize the density; since the density is recognizable as a special case of the nonsymmetric Koornwinder density, it symmetrizes to the symmetric density above. In other words, it will suffice to show that $L(P_\lambda(\,;q,t)) = 0$ unless $\lambda = \mu^2$.

Since nonsymmetric Macdonald polynomials of type C are orthogonal with respect to the density $\Delta_K^{(n)}\,(\sqrt{t}, -\sqrt{t},\ \sqrt{qt}, -\sqrt{qt}; q, t)$, we can interpret the result as saying when we expand type A Macdonald polynomials in terms of those of type C the coefficient of the trivial one is zero unless $\lambda = \mu^2$. (In the notation of section 2, $[E_0^C]E_\lambda^A = 0$ unless $\lambda = \mu^2$.)

The advantage of passing to this nonsymmetric functional is that we can use the affine Hecke algebra. Indeed, a straightforward calculation gives the following facts about the interaction between L and the Hecke algebra:

$$L(T_0 f) = tL(f) \tag{3.2}$$
$$L(T_n f) = tL(f) \tag{3.3}$$
$$L(T_i f) = L(T_{2n-i} f), \quad 1 \le i \le n-1. \tag{3.4}$$

But in fact, for generic q and t, any linear functional satisfying these three conditions will also satisfy the vanishing property "$L(P_\lambda(;q,t)) = 0$ unless $\lambda = \mu^2$".

The calculation to verify (3.2), (3.3), (3.4) is very similar to that of computing the adjoint of T_i with respect to $\langle\, , \rangle$. A sample computation is given here: First recall $(T_n - t)f = \frac{x_{n+1} - tx_n}{x_{n+1} - x_n}(f^{s_n} - f)$ by (2.6). After specializing as above, this will become $\frac{z_n^{-1} - tz_n}{z_n^{-1} - z_n} g_1$ where g_1 is a Laurent polynomial sent to $-g_1$ under the change of variables $z_n \leftrightarrow z_n^{-1}$. Observe the density

$$\Delta = \prod_{1 \le i \le n} \frac{(z_i^2, qz_i^{-2}; q)}{(tz_i^2, qtz_i^{-2}; q)} \prod_{1 \le i < j \le n} \frac{(z_i z_j^{\pm 1}, qz_i^{-1} z_j^{\pm 1}; q)}{(tz_i z_j^{\pm 1}, qtz_i^{-1} z_j^{\pm 1}; q)} = \frac{(1 - z_n^2)}{(1 - tz_n^2)} g_2$$

where g_2 is symmetric under the change of variables $z_n \leftrightarrow z_n^{-1}$. Hence $L((T_n - t)f) = \int \frac{z_n^{-1} - tz_n}{z_n^{-1} - z_n} g_1 \frac{(1 - z_n^2)}{(1 - tz_n^2)} g_2 = \int g_1 g_2 = \int (-g_1) g_2$ as we are integrating over the torus T and hence get the same integral under the change of variables $z_n \leftrightarrow z_n^{-1}$. This shows $L((T_n - t)f) = 0$.

Let $V_{\le\lambda}$ be the space of polynomials spanned by monomials $x^\nu := x_1^{\nu_1} x_2^{\nu_2} \dots$ where ν is a composition of $|\lambda|$ dominated by λ (i.e., such that the corresponding partition ν^+ is dominated by λ); similarly let $V_{<\lambda}$ be the space spanned by monomials *strictly* dominated by λ. Both subspaces are invariant under the action of the Hecke algebra, and we may thus consider the spaces $\mathcal{L}_\lambda$ of functionals on $V_{\le\lambda}/V_{<\lambda}$ satisfying (3.2),(3.3),(3.4). If we can show that $\mathcal{L}_\lambda = 0$ unless $\lambda = \mu^2$, we will be done, since $V_{\le\lambda}/V_{<\lambda}$ is isomorphic (for generic q, t) to the invariant subspace generated by $P_\lambda(;q,t)$ (with basis given by $E_\nu(;q,t)$ with $\nu^+ = \lambda$).

Fix a partition λ *not* of the form μ^2. Now, the monomials in the orbit of x^λ form a basis of $V_{\le\lambda}/V_{<\lambda}$ (for all nonzero q, t, not just generic q, t), and in that basis, the action of the Hecke algebra has coefficients in $\mathbb{Z}[q^{\pm 1}, t]$. Thus $\mathcal{L}_\lambda$ is the solution space of a system of linear equations with coefficients in $\mathbb{Z}[q^{\pm 1}, t]$. Now, the generic dimension of such a solution space is bounded above by the dimension under any specialization. Therefore, it will suffice to find *one* such specialization for which the claim holds, and thus the dimension of the solution space is 0 for generic q, t.

In particular, take $t = 1$, so that the affine Hecke algebra is just the group algebra of $\widetilde{S}_n$ with the corresponding action on polynomials. Then any functional $L \in \mathcal{L}_\lambda$ is invariant under the subgroup generated by s_0, s_n, and $s_i s_{2n-i}$ for $1 \le i \le n-1$. This is precisely the subgroup of elements invariant under the involution $s_i \mapsto s_{2n-i}$, and thus in particular contains a number of translations, which act diagonally on monomials. It follows immediately that $L(x^\nu) = 0$ unless $\nu_i = \nu_{2n+1-i}$ for $1 \le i \le n$. Since ν is simply a permutation of λ, the claim follows. □

Remark. It similarly follows that for generic q, t, $\dim(\mathcal{L}_{\mu^2}) \le 1$. In fact, since the integrals

$$\int m_{\mu^2}(\dots z_i^{\pm 1} \dots)\, dT$$

are nonzero, we can also conclude that $\dim(\mathcal{L}_{\mu^2}) \ge 1$ for *all* q, t (since we have exhibited a linear functional that specializes to a nonzero functional.) This implies for a wide class of irreducible representations we have a multiplicity one condition, i.e., that there exists at most a 1−dimensional space of linear functionals satisfying the above invariance conditions (3.2),(3.3),(3.4). This is in a sense a deformation of the fact that $(U(2n), Sp(2n))$ is a Gelfand pair, together with the identification of which representations are spherical. This appears to be true for general representations, but we do not consider that question here.

Now, as it stands, this argument is somewhat unsatisfactory; it would be nice to avoid the step of specialization to $t = 1$. Certainly, there is a natural analogue of the subgroup of translations inside the affine Hecke algebra; unfortunately, the conditions on L do *not* imply any sort of invariance with respect to the standard commutative subalgebra. Related to this is the fact that the obvious corresponding statement for nonsymmetric Macdonald polynomials does not hold; that is, for $t = 1$, the conditions on L suffice to make $L(E_\nu(;q,1)) = 0$ unless $\nu_i = \nu_{2n+1-i}$ for $1 \le i \le n$, but this is *not* true for $t \ne 1$. The key to resolving both of these issues is the fact that, although the standard commutative subalgebra is in some sense canonical (or, at least, is one of two canonical choices), it is not the only reasonable choice; we will consider this in more detail in the sections below. Equivalently, we can leave the commutative subalgebra alone and transform the functional, thus conjugating the ideal of equations on the functional. This gives nice identities for nonsymmetric Macdonald polynomials but makes the resulting functional extremely complicated. We consider this approach in sections 5 and 7.

Another flaw, which is intrinsic in the way we use the affine Hecke algebra, is that we obtain no information about the nonzero values of the integral. Indeed, the conditions on L determine it only up to a scalar multiple for each μ. In this case, the nonzero values were already determined (conditional on the vanishing result) in [10]; but for some of the other vanishing results we prove below, it is still an open question to determine the nonzero values.

For the present case, however, we have (in the notation of [10], and recalling the argument given there)

Corollary 3.2. *For any integer $n \geq 0$ and any partition μ with at most n parts,*

$$\frac{1}{Z}\int P_{\mu^2}(\dots,z_i^{\pm 1},\dots)\prod_{1\le i\le n}\frac{(z_i^{\pm 2};q)}{(tz_i^{\pm 2};q)}\prod_{1\le i<j\le n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)}{(tz_i^{\pm 1}z_j^{\pm 1};q)}\,dT$$
$$=\frac{1}{Z}\int P_{\mu^2}(\dots,z_i^{\pm 1},\dots)\widetilde{\Delta}_K^{(n)}(\sqrt{t},-\sqrt{t},\sqrt{qt},-\sqrt{qt};q,t)\,dT$$
$$=\frac{C^0_\mu(t^{2n};q,t^2)C^-_\mu(qt;q,t^2)}{C^0_\mu(qt^{2n-1};q,t^2)C^-_\mu(t^2;q,t^2)},$$

where the normalization Z is chosen to make the integral 1 when $\mu = 0$.

Proof. Let L be the above linear functional on symmetric functions (so that we are computing the value of $L(P_{\mu^2}(\,;q,t)))$; we have already established that $L(P_\lambda(\,;q,t)) = 0$ unless λ is of the form μ^2. Now, consider the value

$$L((e_1 - e_{2n-1})P_\lambda(\,;q,t)),$$

where e_i is the elementary symmetric function. On the one hand, this is 0, since the mere act of specializing to $z_i^{\pm 1}$ annihilates $e_1 - e_{2n-1}$. On the other hand, we can expand $(e_1 - e_{2n-1})P_\lambda(\,;q,t)$ as a linear combination of Macdonald polynomials using the Pieri identity; if we throw out those polynomials annihilated by L, at most two terms remain. Together with the identity

$$L(P_{1^n+\lambda}(\,;q,t)) = L(e_{2n}P_\lambda(\,;q,t)) = L(P_\lambda(\,;q,t)),$$

we obtain an identity of the form

$$L(P_{\mu^2}(\,;q,t)) = C_{\mu/\nu}(\,;q,t)L(P_{\nu^2}(\,;q,t)),$$

where ν is obtained by removing a single square from the diagram of μ. The claim then follows by induction in $|\mu|$. □

Remark. For many of the vanishing integrals considered below, either the linear functional fails to factor through a homomorphism, or the homomorphism it does factor through does not have any useful elements in its kernel, and we thus cannot apply the "Pieri trick" described above to obtain the nonzero values.

Remark. In this case, the normalization Z simplifies as

$$\prod_{1\le j\le n}\frac{(t,t^{2n-j};q)}{(t^{2j+2},-qt^{j+1};q)(-qt^{2j+2};q^2)}.$$

This is of course simply Macdonald's constant term identity of type C.

The final flaw in the above argument is that it *only* applies to the symplectic case of the vanishing integral. The point is that in the symplectic case, the condition on compositions ν translates to a very simple condition on the action of translations for $t = 1$, namely that certain translations should act in the same way on the monomial

x^ν. For the orthogonal case, the corresponding condition on eigenvalues of translations is actually Zariski dense; in particular, it cannot be detected by any finitely generated ideal of the Hecke algebra. It turns out, however, that one can deduce the orthogonal vanishing integrals from the symplectic vanishing integral, using the fact that both can be viewed as statements about the algebra of symmetric functions, related by a slightly modified Macdonald involution. (Note, in particular that the conjugate partition to one of the form μ^2 is of the form 2ν.)

We thus obtain the following corollary, dual to Corollary 3.2. For the details, see section 8 of [10]. Each of the four integrals is with respect to an appropriate special case of the normalized Koornwinder density; we denote such an n-dimensional integral as $I_K^{(n)}(f;q,t;a,b,c,d)$.

Corollary 3.3. *For all integers $n \geq 0$ and partitions λ with at most n parts,*

$$\frac{1}{2}I_K^{(n)}(P_\lambda(\dots,z_i^{\pm 1},\dots;q,t);q,t;\pm 1,\pm\sqrt{t}) + \frac{1}{2}I_K^{(n-1)}(P_\lambda(\dots,z_i^{\pm 1},\dots,\pm 1;q,t);q,t;\pm t,\pm\sqrt{t}) = 0$$

unless λ is of the form 2μ, in which case the value is

$$\frac{C^0_\mu(t^{2n};q^2,t)C^-_\mu(q;q^2,t)}{C^0_\mu(qt^{2n-1};q^2,t)C^-_\mu(t;q^2,t)}$$

Similarly,

$$\frac{1}{2}I_K^{(n)}(P_\lambda(\dots,z_i^{\pm 1},\dots,1;q,t);q,t;t,-1,\pm\sqrt{t}) + \frac{1}{2}I_K^{(n)}(P_\lambda(\dots,z_i^{\pm 1},\dots,-1;q,t);q,t;1,-t,\pm\sqrt{t}) = 0$$

unless λ is of the form 2μ, in which case the value is

$$\frac{C^0_\mu(t^{2n+1};q^2,t)C^-_\mu(q;q^2,t)}{C^0_\mu(qt^{2n};q^2,t)C^-_\mu(t;q^2,t)}.$$

Remark. Note that again the nonzero values follow via an application of the Pieri identity from the fact that the linear functionals vanish where required. Etingof (personal communication) has pointed out a direct proof of the orthogonal vanishing integrals in the Jack polynomial limit, using the construction of [4] for Jack polynomials; presumably the Macdonald polynomial analogue of the construction can be used to obtain the orthogonal vanishing integral for Macdonald polynomials. Etingof's argument also gives no information about the nonzero values, and just as the nature of the Hecke algebra made it impossible to use our argument in the orthogonal case, the nature of the Etingof-Kirillov construction of Jack polynomials makes it impossible to use Etingof's argument in the symplectic case.

4. Other Vanishing Integrals

In this section, we list the remaining vanishing results. For each result, we list the functional L that gives the vanishing integral, and the associated right ideal I in the Hecke algebra (of type A or C) that kills L. We also give the subgroup S of the braid group that leaves the functional invariant in the classical limit, as this motivated many of the relevant definitions. In fact, in each case the subgroup S of the braid group lies over the commutator of an involution in the extended affine Weyl group. For instance, each ideal I is generated by elements

$$t_\sigma^{-1/2} T_\sigma - \chi(\sigma)$$

where σ is a generator of S and χ is the character of $B(\widetilde{W})$ given by its action on the constant polynomials. (Here t_σ is the parameter associated to T_σ, i.e., $(T_\sigma+1)(T_\sigma-t_\sigma) = 0$.)

In each case we argue as in section 3, that is, we exhibit a specialization of the parameters such that (a) in that specialization a nonzero functional annihilated by I exists only if λ is of the stated form, and (b) if λ is of the stated form then there is unique such nonzero functional (which can be obtained by specializing the appropriate integral). Since the space of functionals annihilated by an ideal can only get bigger under specialization, this implies (a) that if λ is *not* of the appropriate form, then for generic parameters no such functional exists, and (b) if λ is of the appropriate form, such a functional exists and is generically unique.

Note that this uniqueness is on a partition by partition basis. It is quite possible (and indeed we give examples below) for there to be multiple nice functionals on the space of *all* polynomials that are all annihilated by the same ideal and thus satisfy the same vanishing conditions. (See for instance sections 4.1, 4.2 below.) For the $\widetilde{S}_n$ cases, this specialization is simply $t = 1$. For the $\widetilde{C}_n$ cases, we must moreover take $a = 1$, $b = -1$, $c = \sqrt{q}$, $d = -\sqrt{q}$. In all cases this has the effect of turning the Hecke algebra into a group algebra and the (nonsymmetric) Macdonald and Koornwinder polynomials into monomials and the density trivial, at which point the functional is easy to evaluate.

One consequence of this global non-uniqueness is that in order to determine the nonzero values of such a functional, it is not enough to know how the affine Hecke algebra acts. One must in fact consider more carefully the explicit structure of the functional, e.g., as in the Pieri trick used above (or more generally, how the double affine Hecke algebra interacts with the functional). Even if such a calculation could be pushed through, this would still leave the nontrivial task of deducing the values on the symmetric Macdonald and Koornwinder polynomials from the nonsymmetric ones.

In each case there is an associated family of chambers (see section 7) such that the elements of the form Y_ν^C contained in S imply the appropriate vanishing theorem. The fact that these chambers are not the standard chamber implies that the standard nonsymmetric Macdonald polynomials do not satisfy vanishing results, but the nonstandard E_λ^C do. This is so because a different choice of chamber C twists the Hecke algebra module by an inner automorphism that results in an isomorphic module, which is easily seen from the fact that the irreducible modules $\mathcal{L}_\lambda$ we consider have distinct central characters. In any event, while the nonstandard choice of chamber C gives a different family of nonsymmetric Macdonald polynomials E_λ^C (as they are the eigenfunctions of commuting operators Y^C) the symmetric Macdonald polynomials stay the same. ($P_\lambda^C = P_\lambda$;

symmetric functions in the Y^C are always central.) Further discussion on E^C_λ, Y^C are in section 7.

4.1. Macdonald polynomial results: $\tilde{S}_{2n}$

Case 1. This case was discussed in section 3 above.

Theorem 4.1.

$$\int P_\lambda(\dots z_i^{\pm 1}\dots;q,t)\prod_{1\le i\le n}\frac{(z_i^{\pm 2};q)}{(tz_i^{\pm 2};q)}\prod_{1\le i<j\le n}\frac{(z_i^{\pm 1}z_j^{\pm 1};q)}{(tz_i^{\pm 1}z_j^{\pm 1};q)}\,dT=0 \tag{4.1}$$

unless $\lambda=\mu^2$*, in which case (when suitably normalized) it is*

$$\frac{C^0_\mu(t^{2n};q,t^2)C^-_\mu(qt;q,t^2)}{C^0_\mu(qt^{2n-1};q,t^2)C^-_\mu(t^2;q,t^2)}.$$

This is the case $T=t^n$ of the symmetric function identity

$$I_K(P_\lambda(;q,t);q,t,T;\pm\sqrt{t},\pm\sqrt{qt})=\begin{cases}\frac{C^0_\mu(T^2;q,t^2)C^-_\mu(qt;q,t^2)}{C^0_\mu(qT^2/t;q,t^2)C^-_\mu(t^2;q,t^2)} & \lambda=\mu^2\\ 0 & \text{otherwise.}\end{cases}$$

The action of the Macdonald involution on lifted Koornwinder polynomials dualizes this to

$$I_K(P_\lambda(;q,t);q,t,T;\pm 1,\pm\sqrt{t})=\begin{cases}\frac{C^0_\mu(T^2;q^2,t)C^-_\mu(q;q^2,t)}{C^0_\mu(qT^2/t;q^2,t)C^-_\mu(t;q^2,t)} & \lambda=\mu^2\\ 0 & \text{otherwise.}\end{cases}$$

Taking $T\in\{t^n,t^{n+1/2}\}$ gives that the four integrals

$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1};q,t);q,t;\pm 1,\pm\sqrt{t})$$
$$I_K^{(n-1)}(P_\lambda(x_1^{\pm 1},\dots,x_{n-1}^{\pm 1},1,-1;q,t);q,t;\pm t,\pm\sqrt{t})$$
$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},1;q,t);q,t;t,-1,\pm\sqrt{t})$$
$$I_K^{(n)}(P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},-1;q,t);q,t;1,-t,\pm\sqrt{t})$$

vanish unless all ($2n$ or $2n+1$, as appropriate) parts of λ have the same parity.

We take
$$S=\langle U_0,U_n,U_iU_{2n-i}^{-1}\rangle\subseteq B(\widetilde{S}_{2n})$$

The relevant chambers are such that r and r^ω have the same sign, where ω is the longest element of S_n, and r is a root such that $r+r^\omega\ne 0$. Invariant functionals ($L(\sigma f)=0$, $\sigma\in S$) vanish on E^C_λ unless $\lambda_i=\lambda_{2n+1-i}$, $1\le i\le n$.

The associated right ideal of H_{2n} is given by $I_1^S=\langle T_0-t,T_n-t,T_i-T_{2n-i}\rangle$.

The functional, which obeys $LI_1^S=0$ (equivalently $L(\sigma f)=0$, $\forall\sigma\in S$) is

$$L(p)=\int p(z_1,z_2,\dots z_n,1/z_n,\dots 1/z_2,1/z_1)\prod_{1\le i\le n}\frac{(z_i^2,qz_i^{-2};q)}{(tz_i^2,qtz_i^{-2};q)}\prod_{1\le i<j\le n}\frac{(z_iz_j^{\pm 1},qz_i^{-1}z_j^{\pm 1};q)}{(tz_iz_j^{\pm 1},qtz_i^{-1}z_j^{\pm 1};q)}\,dT.$$

Case 2.

Theorem 4.2. *Let μ, ν be partitions. Then*

$$\int P^{(2n)}_{\mu\bar{\nu}}(\cdots \pm \sqrt{z_i} \ldots ; q, t) \prod_{1\le i\le n} \frac{((z_i/z_j)^{\pm 1}; q^2)}{(t^2(z_i/z_j)^{\pm 1}; q^2)}\, dT = \int P^{(2n)}_{\mu\bar{\nu}}(\cdots \pm \sqrt{z_i} \ldots ; q, t) \widetilde{\Delta}^{(n)}_S(q^2, t^2)\, dT = 0$$

unless $\mu = \nu$, when (suitably normalized) the integral is

$$\frac{(-1)^{|\mu|} C^-_\mu(q; q, t) C^+_\mu(t^{2n-2}q; q, t) C^0_\mu(t^n, -t^n; q, t)}{C^-_\mu(t; q, t) C^+_\mu(t^{2n-2}t; q, t) C^0_\mu(qt^{n-1}, -qt^{n-1}; q, t)}.$$

Remark. The nonzero values can be computed using the Pieri identity as in the proof of Corollary 3.2. The same nonzero values (apart from the factor $(-1)^{|\mu|}$) appear in Conjecture 5 of [10], which remains open.

Note that this is well-defined because $P^{(2n)}_{\mu\bar{\nu}}(\cdots \pm \sqrt{z_i} \ldots ; q, t)$ is invariant under $\sqrt{z_i} \mapsto -\sqrt{z_i}$ and it is therefore in $\mathbb{C}(q, t)[z_1{}^{\pm 1}, \ldots, z_n{}^{\pm 1}]$.

Since multiplying a Macdonald polynomial by $(z_1 z_2 \cdots z_n)^m$ has the effect of adding m to each part (which works for all $m \in \frac{1}{n}\mathbb{Z}$) we can restate this in terms of ordinary Macdonald polynomials as follows:

Corollary 4.3.

$$[P_{m^n}(; q^2, t^2)] P_\lambda([2p_{k/2}]; q, t) = 0$$

unless $\lambda = (2m)^{2n} - \lambda$.

This statement is self-dual, i.e., applying Macdonald's involution gives the same identity.

We take

$$S = \langle U_{2i-1}, U_{2i}U_{2i-1}U^{-1}_{2i+1}U^{-1}_{2i}, U^{-1}_{2i}U_{2i-1}U^{-1}_{2i+1}U_{2i}, \pi^2 \rangle.$$

Chambers are such that r and r^ι have opposite signs, where $\iota(2i-1) = 2i$, $\iota(2i) = 2i-1$.

$$I^S_2 = \langle T_{2i-1} - t\,, T_{2i}(T_{2i-1} - T_{2i+1}),\ \pi^2 - 1 \rangle$$

I^S_2-invariant functionals vanish on E^C_λ unless $\lambda_{2i-1} + \lambda_{2i} = 0$, $1 \le i \le n$.

The functional, which obeys $LI^S_2 = 0$ is

$$L(p) = \int p(z_1, -z_1, z_2, -z_2, \ldots z_n, -z_n) \prod_{1\le i<j\le n} \frac{(z_i^2/z_j^2, q^2 z_j^2/z_i^2; q^2)}{(t^2 z_i^2/z_j^2, q^2 t^2 z_j^2/z_i^2; q^2)}\, dT$$

Theorem 4.4. *Let λ be a weight of the double cover of GL_{2n}, i.e., a half-integer vector such that $\lambda_i - \lambda_j \in \mathbb{Z}\, \forall i, j$.*

$$\int P^{(2n)}_\lambda(\ldots t^{\pm 1/2} z_i \ldots ; q, t) \prod_{1\le i<j\le n} \frac{((z_i/z_j)^{\pm 1}; q)}{(t^2(z_i/z_j)^{\pm 1}; q)}\, dT = 0$$

unless $\lambda_i = -\lambda_{2n+1-i}$.

We allow half-integral λ here in order to allow m odd in the symmetric function analogue:

Corollary 4.5.
$$[P_{m^n}(;q,t^2)]P_\lambda([p_k(t^{k/2}+t^{-k/2})];q,t)=0$$
unless $\lambda = m^{2n}-\lambda$. *Dually,*
$$[P_{m^n}(;q^2,t)]P_\lambda(;q,t)=0$$
unless $\lambda=(2m)^n-\lambda$.

Remark. Experimentally, the nonzero values appear to be nice, but the kernel of the specialization $f\mapsto f(\cdots t^{\pm 1/2}z_i\cdots)$ is too complicated for us to obtain recurrences from the Pieri identity.

For this vanishing integral, we can take the same S_2 and I_2^S, but use a different functional:
$$L(p)=\int p(z_1,tz_1,z_2,tz_2,\dots z_n,tz_n)\prod_{1\le i<j\le n}\frac{(z_i/z_j,qz_j/z_i;q)}{(t^2z_i/z_j,qt^2z_j/z_i;q)}\,dT$$
$$=\int p(z_1,tz_1,z_2,tz_2,\dots z_n,tz_n)\widetilde{\Delta}_S^{(n)}(q,t^2)\,dT \quad (4.2)$$

Case 3.

Theorem 4.6. *$(q\mapsto q^2)$*
$$\int P_{\mu\bar\nu}^{(2n)}(\dots q^{\pm 1/4}z_i\dots;q,t)\prod_{1\le i<j\le n}\frac{((z_i/z_j)^{\pm1};q^{1/2})}{(t(z_i/z_j)^{\pm1};q^{1/2})}\,dT=0$$
unless $\mu=\nu$.

Corollary 4.7. *For any partition* λ,
$$[P_{m^n}(;q,t)]P_\lambda([p_k(q^{k/2}+q^{-k/2})];q^2,t)=0 \quad (4.3)$$
unless $\lambda=m^{2n}-\lambda$. *Dually,*
$$[P_{m^n}(;q,t)]P_\lambda(;q,t^2)=0$$
unless $\lambda=(2m)^n-\lambda$.

Remark. That (4.3) holds when $\ell(\lambda)>2n$ follows immediately from the fact that Macdonald polynomials are triangular with respect to the dominance order and the way the specialization acts on monomials.

Again, the nonzero values appear nice, but the Pieri trick fails.

We take
$$S=\langle U_iU_{i+n}^{-1},\pi\rangle$$
Chambers are such that r and r^ι have opposite signs, where $\iota(i)=i+n$, $\iota(i+n)=\iota(i)$. The associated right ideal is
$$I_3^S=\langle T_i-T_{i+n},\ \pi-1\rangle.$$
I_3^S-invariant functionals vanish on E_λ^C unless $\lambda_i+\lambda_{i+n}=0$.

The functional is:
$$L(p)=\int p(q^{1/2}z_1,q^{1/2}z_2,\dots q^{1/2}z_n,z_1,z_2,\dots z_n)\prod_{1\le i<j\le n}\frac{(z_i/z_j,q^{1/2}z_j/z_i;q^{1/2})}{(tz_i/z_j,q^{1/2}tz_j/z_i;q^{1/2})}$$
$$=\int p(q^{1/2}z_1,q^{1/2}z_2,\dots q^{1/2}z_n,z_1,z_2,\dots z_n)\widetilde{\Delta}_S^{(n)}(\sqrt{q},t). \quad (4.4)$$

4.2. Koornwinder polynomial results: $\widetilde{C}_{2n}$

Case 1.

Theorem 4.8. *In symmetric function terms,*

$$I_K(\tilde{K}_\lambda([2p_{k/2}];q,t,T;a,-a,c,-c);q^2,t^2,T;-t,-qt,a^2,c^2)=0$$

unless $\lambda=\mu^2$*, when it is*

$$\frac{(-1)^{|\mu|}C^-_\mu(qt;q,t^2)C^+_\mu(a^2c^2T^2/t^4;q,t^2)C^0_\mu(T,-a^2T/t,-c^2T/t,a^2c^2T/t^2;q,t^2)}{C^+_\mu(a^2c^2T^2/qt^3;q,t^2)C^-_\mu(t^2;q,t^2)C^0_{\mu^2}(a^2c^2T^2q/t^2;q^2,t^2)}.$$

Dually,

$$I_K(\tilde{K}_\lambda([2p_{k/2}];q,t,T;a,-a,c,-c);q^2,t^2,T;-1,-t,a^2,c^2)=0$$

unless $\lambda=2\mu$*, when it is*

$$\frac{(-1)^{|\mu|}C^-_\mu(q;q^2,t)C^+_\mu(a^2c^2T^2/t^3;q^2,t)C^0_\mu(T,-a^2T/t,-c^2T/t,a^2c^2T/t^2;q^2,t)}{C^-_\mu(t;q^2,t)C^+_\mu(a^2c^2T^2/qt^2;q^2,t)C^0_{2\mu}(a^2c^2T^2/t^3;q^2,t^2)}.$$

The nonzero values are computed via the Pieri identities for Koornwinder polynomials [3]. For $T=t^{2n}$, both formal integrals become actual integrals; similarly, for $T=t^{2n+1}$, the second formal integral becomes:

$$I^{(n)}_K(K^{(2n+1)}_\lambda(\dots,\pm z_i,\dots,\sqrt{-1};q,t;a,-a,c,-c);q^2,t^2;-t,-t^2,a^2,c^2).$$

For this identity, we work with the case $b=-a,d=-c$ of H^C_n and its polynomial representation, and take

$$S=\langle U_{2i-1},U^{\pm1}_{2i}U_{2i-1}U^{-1}_{2i+1}U^{\mp1}_{2i},U^{\pm1}_0U_1U^{\mp1}_0,U^{\pm1}_{2n}U_{2n-1}U^{\mp1}_{2n}\rangle\subseteq B(\widetilde{C}_{2n})$$

with associated right ideal

$$I^K_1=\langle T_{2i-1}-t,T_{2i}(T_{2i-1}-T_{2i+1}),T_0(T_1-t),T_{2n}(T_{2n-1}-t)\rangle$$

The functional is:

$$L(p)=\int p(z_1^{1/2},-z_1^{1/2},z_2^{1/2},-z_2^{1/2},\dots z_n^{1/2},-z_n^{1/2})\Delta^{(n)}_K(a^2,-t,c^2,-qt;q^2,t^2)$$

Theorem 4.9. *In symmetric function terms,*

$$I_K(\tilde{K}_\lambda([p_k(t^{k/2}+t^{-k/2})];q,t,T;a,b,c,d);q,t^2,T;t^{1/2}a,t^{1/2}b,t^{1/2}c,t^{1/2}d)=0 \quad (4.5)$$

unless $\lambda=\mu^2$*. The dual statement is:*

$$I_K(\tilde{K}_\lambda(;q,t;T;a,b,c,d);q^2,t,T;a,b,c,d)=0 \quad (4.6)$$

unless $\lambda = 2\mu$.

In the case $n = 1$ the above identity becomes an identity of Askey-Wilson polynomials and admits a direct hypergeometric proof (Rahman, personal communication).

Once again, the Pieri trick fails, but in fact the nonzero values

$$\frac{t^{-|\mu|}C^0_\mu(T, T\frac{ab}{t}, T\frac{ac}{t}, T\frac{ad}{t}, T\frac{bc}{t}, T\frac{bd}{t}, T\frac{cd}{t}, T\frac{abcd}{t^2}; q, t^2)C^+_\mu(T^2\frac{abcd}{t^4}; q, t^2)C^-_\mu(qt; q, t^2)}{C^0_{2\mu^2}(T^2\frac{abcd}{t^2}; q, t^2)C^+_\mu(T^2\frac{abcd}{qt^3}; q, t^2)C^-_\mu(t^2; q, t^2)}$$

for the first integral and

$$\frac{q^{|\mu|}C^0_\mu(T, T\frac{ab}{t}, T\frac{ac}{t}, T\frac{ad}{t}, T\frac{bc}{t}, T\frac{bd}{t}, T\frac{cd}{t}, T\frac{abcd}{t^2}; q^2, t)C^+_\mu(T^2\frac{abcd}{t^3}; q^2, t)C^-_\mu(q; q^2, t)}{C^0_{2\mu^2}(T^2\frac{abcd}{t^2}; q^2, t)C^+_\mu(T^2\frac{abcd}{t^2q}; q^2, t)C^-_\mu(t; q^2, t)}.$$

for the second can be obtained as a limit of the elliptic version derived in [9].

We take S and I^K_1 as above, but now with generic a, b, c, d and the functional we need is:

$$L(p) = \int p(t^{-\frac{1}{2}}z_1, t^{\frac{1}{2}}z_1, t^{-\frac{1}{2}}z_2, t^{\frac{1}{2}}z_2, \dots, t^{-\frac{1}{2}}z_n, t^{\frac{1}{2}}z_n)\Delta^{(n)}_K(t^{\frac{1}{2}}a, t^{\frac{1}{2}}b, t^{\frac{1}{2}}c, t^{\frac{1}{2}}d; q, t^2)$$

Case 2.

Theorem 4.10. *In symmetric function terms,*

$$I_K(\tilde{K}_\lambda([p_k(q^{k/2} + q^{-k/2})]; q^2, t, T^2; a, b, qa, qb); q, t, T; \pm\sqrt{t}, q^{1/2}a, q^{1/2}b) = 0 \qquad (4.7)$$

unless $\lambda = \mu^2$.

The dual statement is:

$$I_K(\tilde{K}_\lambda(\,; q, t^2; T^2; a, b, ta, tb); q, t, T; \pm\sqrt{t}, a, b) = 0$$

unless $\lambda = 2\mu$.

We take $c = q^{1/2}a$, $d = q^{1/2}b$ (so consider the case $q \mapsto \sqrt{q}$ above)

$$S = \langle U_0U^{-1}_{2n}, U_iU^{-1}_{2n-i}, U_n\rangle \subseteq B(\widetilde{C}_{2n})$$

and associated right ideal

$$S = \langle T_0 - T_{2n}, T_i - T_{2n-i}, T_n - t\rangle$$

The functional is:

$$L(p) := \int p(q^{1/4}z_1, \dots q^{1/4}z_n, q^{1/4}/z_n, \dots q^{1/4}/z_1)\Delta^{(n)}_K(\sqrt{t}, -\sqrt{t}, q^{1/4}a, q^{1/4}b; q^{1/2}, t).$$

5. A construction using the Hecke algebra

In this section, we give another proof of the existence of nonzero functional L in the non-vanishing case (another proof of the vanishing condition can also be deduced) along with an explicit construction of $L \in \mathcal{L}_\lambda$ (up to scalar). We only do this for the $\widetilde{S}_{2n}$ case, leaving the Koornwinder case to the reader. We do not explicitly compute the scalar that relates the L constructed in this section to the integral given in section 3. We also warn the reader that since we are computing in $(V_{\leq\lambda}/V_{<\lambda})^*$ versus $V^*_{\leq\lambda}$, we do not give information about $L(E_\mu)/L(E_\nu)$ except when $\mu^+ = \nu^+$.

In what follows we will use the presentation of H_{2n} as generated by $T_1, T_2, \dots, T_{2n-1}$, $Y_1^{\pm 1}, \dots, Y_{2n}^{\pm 1}$ because it allows us to work more explicitly with a basis of $\mathcal{L}_\lambda$ given by simultaneous Y_i-eigenfunctionals.

This presentation also gives us another description of $V_{\leq\lambda}/V_{<\lambda}$ and of its dual $\mathcal{L}_\lambda = (V_{\leq\lambda}/V_{<\lambda})^*$. Given $\lambda \vdash 2n$, let $J = J_\lambda = \{j \mid s_j\lambda = \lambda\}$, let $H(\lambda)$ be the parabolic subalgebra generated by $\{T_j \mid j \in J\}$ and all the $Y_1^{\pm 1}, \dots, Y_{2n}^{\pm 1}$, and let $\mathbb{C}(q,t)_\lambda$ be the one-dimensional $H(\lambda)$ module on which $Y_i - q^{\lambda_i}t^{2n-i} = 0$, $T_j - t = 0\, \forall j \in J$. Then we have

$$\mathcal{L}_\lambda \simeq \mathbb{C}(q,t)_\lambda \otimes_{H(\lambda)} H_{2n}.$$

(Note that $V_{\leq\lambda}/V_{<\lambda}$ is isomorphic to $H_{2n} \otimes_{H(w_0\lambda)} \mathbb{C}(q,t)_{w_0\lambda}$ (which is isomorphic to $H_{2n} \otimes_{H(\lambda)} \mathbb{C}(q,t)_\lambda$ when q, t are generic) and thus is in this sense self dual. This can be seen directly or follows from the Mackey decomposition of V.)

For ease of notation, we introduce the standard invariant form $\langle\ ,\ \rangle$, and let $\delta = \delta_{2n} = (2n-1, \dots, 2, 1, 0)$, $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0), \alpha_i = \varepsilon_i - \varepsilon_{i+1}$. We can then write $\lambda_i = \langle\lambda, \varepsilon_i\rangle$. We also have $\langle\lambda,\mu\rangle = \langle w\lambda, w\mu\rangle$, where $w \in S_{2n}$ acts as $w(\lambda_1, \dots, \lambda_{2n}) = (\lambda_{w^{-1}(1)}, \dots \lambda_{w^{-1}(2n)})$.

We observe that the center $Z(H_{2n})$ is given by symmetric Laurent polynomials in $Y_1, \dots, Y_{2n}$ and each $\mathcal{L}_\lambda$ has distinct central character. Further, the Y-weight spaces of $\mathcal{L}_\lambda$ are all one-dimensional and hence give a distinguished basis of the module, up to scalars. From the above description of $\mathcal{L}_\lambda$, it is easy to see that basis of simultaneous Y_i-eigenvectors is $\{v_w \mid w \in W^J\}$ with

$$v_w(Y_i - q^{\langle w^{-1}\lambda, \varepsilon_i\rangle}t^{\langle w^{-1}\delta, \varepsilon_i\rangle}) = 0,$$

where W^J is the set of minimal length right coset representatives for $\langle s_j \mid j \in J\rangle \subseteq S_{2n}$. We normalize this basis so that the *right* action of the $T_i, 1 \leq i < n$ is given by

$$v_w T_i = \frac{t-1}{1 - q^{-\langle w^{-1}\lambda, \alpha_i\rangle}t^{-\langle w^{-1}\delta, \alpha_i\rangle}} v_w + \frac{1 - q^{\langle w^{-1}\lambda, \alpha_i\rangle}t^{1+\langle w^{-1}\delta, \alpha_i\rangle}}{1 - q^{-\langle w^{-1}\lambda, \alpha_i\rangle}t^{-\langle w^{-1}\delta, \alpha_i\rangle}} v_{ws_i}$$

with the convention $v_{ws_i} = 0$ if $ws_i \notin W^J$. In that case, notice $v_w T_i = t v_w$, $v_w Y_i Y_{i+1}^{-1} = t v_w$, and in particular $\lambda_{w(i)} = \lambda_{w(i+1)}$. Observe the above action does not depend on the relative lengths $\ell(w)$ and $\ell(ws_i)$, which is why this particular normalization is preferred in this setting.

We note that this basis is dual to the one given by the nonsymmetric Macdonald polynomials up to proportionality. (We leave it to the reader to rescale as necessary, possibly also rescaling the T_i, to get exactly the dual basis.)

We want to express our functional L (given by integrating a specialized polynomial against a given density) in terms of this basis $\{v_w\}$. To do this, we conjugate the right ideal I such that $L \cdot I = 0$ to a related right ideal $T_u I T_u^{-1}$ which has a nicer presentation in terms of the Y_i. This corresponds to working with the twisted module $\mathcal{L}_\lambda^u \simeq \mathcal{L}_\lambda$. Hence we explicitly describe the functional LT_u^{-1} in terms of the dual basis to nonsymmetric Macdonald polynomials, but not L itself. The vanishing result stated above will hold for suitable nonsymmetric Macdonald polynomials with E_ν replaced by $T_u E_\nu$. This twist by u is motivated by the viewpoint of section 7 regarding nonstandard large commutative subalgebras.

For each functional L and corresponding ideal I such that $LI = 0$, we describe $T_u I T_u^{-1} = I'$, determine all λ such that there exists nonzero $v \in \mathcal{L}_\lambda$ with $vI' = 0$, and show this v is unique up to scalar.

It will follow from our explicitly computed generators of $T_u I T_u^{-1}$ that it contains a large binomial ideal in the commutative subalgebra $\mathbb{C}[Y_i^{\pm 1}]$. This translates directly to conditions under which $LT_u^{-1}(E_\nu)$ is forced to vanish. In particular this implies for any L such that $LT_u^{-1}I = 0$ that $LT_u^{-1}(E_\nu)$ vanishes. In each case this will immediately give the desired vanishing result for P_{ν^+}. However, one can ask for something stronger, namely that for each partition λ either the stated vanishing condition holds or there exists a *unique* I-killed functional.

In what follows, all ideals are right ideals.

5.1. Second proof of Theorem 4.1

Recall $I_1^S = \langle T_0 - t,\ T_n - t,\ T_i - T_{2n-i}\,(1 \le i < n)\rangle$ is the right ideal with given generators. Let u be the permutation defined by

$$u(i) = \begin{cases} 2(n-i)+1 & i \le n \\ 2(i-n) & i > n \end{cases}$$

and set $I' = {I'_1}^A = T_u I_1^S T_u^{-1}$. Observe $\ell(u) = 2\binom{n}{2}$. Then

$$I' = \langle T_{2i-1} - t,\ T_{2i}(T_{2i+1} - T_{2i-1}),\ tY_{2i} - Y_{2i-1}\,(1 \le i < n)\rangle,$$

One can verify

$$\begin{aligned} T_u(T_i - T_{2n-i})T_u^{-1} &= T_{u(2n-i)}(T_{u(i)} - T_{u(i)-2})T_{u(2n-i)}^{-1} \\ &= T_{2(n-i)}(T_{2(n-i)+1} - T_{2(n-i)-1})T_{2(n-i)}^{-1}, \\ T_u(T_n - t)T_u^{-1} &= T_1 - t, \\ T_u(tT_0^{-1} - 1)T_u^{-1} &= Y_{2n-1}^{-1} Y_{2n} T_{2n-1} - 1. \end{aligned}$$

From the first two equations, we can show $T_{2i-1} - t \in I'$, inductively as $T_{2i+1} - t = ((T_{2i-1} - t)T_{2i}T_{2i-1} + T_{2i}(T_{2i+1} - T_{2i-1})T_{2i}^{-1}(T_{2i}^2 - tT_{2i}))T_{2i+1}^{-1}T_{2i}^{-1} \in I'$. Then $tY_{2n-1}^{-1}Y_{2n} - 1 = Y_{2n-1}^{-1}Y_{2n}t - \overline{T}_{2n-1} + T_{2n-1} - t = (Y_{2n-1}^{-1}Y_{2n}T_{2n-1} - 1)\overline{T}_{2n-1} + (T_{2n-1} - t) \in I'$.

To show $tY_{2i} - Y_{2i-1} \in I'$, it suffices to show $Y_{2i}(T_{2i-1} - t) \in I'$ as $Y_{2i}(T_{2i-1} - t) = (T_{2i-1} - t)Y_{2i-1} - (tY_{2i} - Y_{2i-1})$. Note

$$\begin{aligned} Y_{2i}(T_{2i-1} - t) &= t^{-2}T_{2i}T_{2i+1}Y_{2i+2}T_{2i+1}T_{2i}(T_{2i-1} - t) \\ &= t^{-2}T_{2i}T_{2i-1}Y_{2i+2}T_{2i+1}T_{2i}(T_{2i-1} - t) \\ &\quad + t^{-2}(T_{2i}T_{2i+1} - T_{2i}T_{2i-1})Y_{2i+2}T_{2i+1}T_{2i}(T_{2i-1} - t) \\ &\in t^{-2}Y_{2i+2}T_{2i}T_{2i-1}T_{2i+1}T_{2i}(T_{2i-1} - t) + I' \\ &= t^{-2}Y_{2i+2}(T_{2i+1} - t)T_{2i}T_{2i-1}T_{2i+1}T_{2i} + I' \end{aligned}$$

Because $tY_{2i} - Y_{2i-1} \in I'$ it follows that if a functional L' is annihilated by I' then $L'(E_\mu) = 0$ unless $\mu_1 = \mu_2$, $\mu_3 = \mu_4$ and so on, thus directly proving the vanishing result, Theorem 3.1.

We may thus restrict our attention to partitions of the form $\lambda = \mu^2$. We wish to show that in this case, $\mathcal{L}_\lambda$ contains a unique I'-killed functional and give an explicit expression for that functional in terms of the basis $\{v_w\}$. Of course, it in only possible to determine the functional up to an overall scalar (and in fact because we are only considering this one partition at a time, we have such a scalar for every valid partition). What this does determine is the relative values of an I'-killed functional on nonsymmetric Macdonald polynomials. The actual values of such a functional are at least in principle determined by its values on symmetric Macdonald polynomials (since for $t = 1$ we can exhibit a functional for which those values are nonzero). Moreover, experimentally, the resulting scale factors are still nice. However, it appears somewhat nontrivial to prove a closed form.

Next we will determine under what conditions $\mathcal{L}_\lambda$ contains a functional annihilated by I', and show that it is unique up to scalar. We will give an explicit expression for this functional in terms of the v_w.

We will need some more notation.

For $w \in W$ let $R(w) = \{\alpha > 0 \mid w\alpha < 0\}$. Notice for $w \in S_{2n}$ we have $R(w) = \{\varepsilon_i - \varepsilon_j \mid i < j, w(i) > w(j)\}$, and $|R(w)| = \ell(w)$. For ι an involution acting on the weight lattice, let $R^\iota(w) = \{\frac{1}{2}(\alpha + \iota(\alpha)) \mid \alpha \in R(w)\}$. Since ι is an involution, the sizes of its orbits are either one or two. When it is necessary to differentiate, we set $R_1^\iota = \{\alpha \in R(w) \mid \iota(\alpha) = \alpha\}$, $R_2^\iota = \{\frac{1}{2}(\alpha + \iota(\alpha)) \mid \alpha \in R(w), \iota(\alpha) \neq \alpha\}$.

Proposition 5.1. *Suppose $\lambda = \mu^2$. Then any $v \in \mathcal{L}_\lambda$ with $vI' = 0$ is proportional to the nonzero I'-killed functional*

$$\sum_{\substack{w \in W^J \\ \iota(w^{-1}\lambda) = w^{-1}\lambda}} \Big(\prod_{\beta \in R^\iota(w^{-1})} q^{-\langle \lambda, \beta \rangle} t^{-\langle \delta, \beta \rangle} \frac{1 - q^{-\langle \lambda, \beta \rangle} t^{1 - \langle \delta, \beta \rangle}}{1 - q^{\langle \lambda, \beta \rangle} t^{1 + \langle \delta, \beta \rangle}} \Big) v_w, \tag{5.1}$$

where ι is the involution on the weight lattice with $\iota(\varepsilon_{2i-1}) = \varepsilon_{2i}$.

Proof. Write $v = \sum_{w \in W^J} c_w v_w$ and suppose $vI' = 0$.

That $v(tY_{2i} - Y_{2i-1}) = 0$ forces $\langle w^{-1}\lambda, \alpha_i \rangle = 0$ and $\langle w^{-1}\delta, \alpha_i \rangle - 1 = 0$ whenever $c_w \neq 0$. In particular $v \neq 0$ implies $\iota\lambda = \lambda$, which we have already included in our hypotheses as $\lambda = \mu^2$. Also, it automatically follows for such an expression that $v(T_{2i-1} - t) = 0$.

That $vT_{2i}(T_{2i+1} - T_{2i-1}) = 0$ forces a relation on c_w and $c_{ws_{2i}s_{2i+1}s_{2i-1}s_{2i}}$, and the resulting relation between nonzero c_w and c_{id} is independent of reduced expression for w and given by (5.1).

□

5.2. Second proof of Theorems 4.2, 4.4

In order to accommodate Theorem 4.4, we must allow half-integral weights, i.e., include $(z_1z_2\cdots z_{2n})^{-1/2}$ in the algebra of polynomials on which we act. Recall $I_2^S = \langle \pi^2 - 1,\ T_0(T_{2n-1}-T_1),\ T_{2i-1}-t, T_{2i}(T_{2i+1}-T_{2i-1}),\ (1\le i<n)\rangle$. Let v be the permutation defined by

$$v(2i+1) = n-i$$
$$v(2i) = n+i$$

and set $I' = T_vIT_v^{-1}$. Notice that $v = u^{-1}$ with u the permutation for the ideal in section 5.1, so that $I' = T_{u^{-1}}IT_{u^{-1}}^{-1}$. Then

$$I' = \langle T_n - t,\ T_i - T_{2n-i},\ Y_iY_{2n-i+1} - t^{2n-1}, (1\le i<n)\rangle,$$

We can use the same computations as with the first ideal, using the fact there is an anti-involution $*$ on the Hecke algebra sending $T_w \mapsto T_{w^{-1}}$, i.e., if $T_uaT_u^{-1} = b$ then $T_vb^*T_v^{-1} = a^*$. Hence we get $T_n - t, T_i - T_{2n-i} \in I'$. To be more precise,

$$\begin{aligned} T_i - T_{2n-i} &= T_vT_{u(2n-i)}(T_{u(i)} - T_{u(i)-2})T_{u(2n-i)}^{-1}T_v^{-1}\\ &= T_vT_{2(n-i)}(T_{2(n-i)+1} - T_{2(n-i)-1})T_{2(n-i)}^{-1}T_v^{-1}\\ T_n - t &= T_v(T_1-t)T_v^{-1}. \end{aligned}$$

One can verify

$$\begin{aligned} T_{u^{-1}}(\pi^{-2}-1)T_{u^{-1}}^{-1} &= T_{n-1}\cdots T_2T_1T_nT_{n-1}\cdots T_2\pi^{-2}T_{2n-2}^{-1}\cdots T_1^{-1} - 1\\ &\in T_{2n-1}\cdots T_{n+1}T_nT_{n-1}\cdots T_2\pi^{-2}T_{2n-2}^{-1}\cdots T_1^{-1} - 1\ + I'\\ &= t^{2n-1}Y_{2n}^{-1}Y_1^{-1} - 1 + I'. \end{aligned}$$

The second step comes from the fact that $T_{2n-1}\cdots T_{n+1} - T_{n-1}\cdots T_1 = \sum_{i=1}^{n-1}(T_{2n-i} - T_i)T_{2n-i-1}T_{2n-i-2}\cdots T_{n+1}T_{i-1}\cdots T_2T_1 \in I'$.

Proposition 5.2. *Suppose λ satisfies $\lambda_i = -\lambda_{2n-i+1}, (1\le i<n)$, i.e., if we set $\nu = (\frac{m}{2}^{2n}) + \lambda$, then $\nu = (m^{2n}) - \nu$ (even for $m=0$). Then any $v\in\mathcal{L}_\lambda$ with $vI' = 0$ is proportional to the nonzero I'-killed functional*

$$v = \sum_{\substack{w\in W^J\\ \iota(w^{-1}\lambda)=w^{-1}\lambda}} \Big(\prod_{\beta\in R_2^\iota(w^{-1})} \frac{-1}{q^{\langle\lambda,\beta\rangle}t^{\langle\delta,\beta\rangle}}\frac{1-q^{-\langle\lambda,\beta\rangle}t^{1-\langle\delta,\beta\rangle}}{1-q^{\langle\lambda,\beta\rangle}t^{1+\langle\delta,\beta\rangle}}\prod_{\beta\in R_1^\iota(w^{-1})}\frac{-1}{q^{\langle\lambda,\beta\rangle}t^{\langle\delta,\beta\rangle}}\Big)v_w. \tag{5.2}$$

Here ι is the involution on the weight lattice with $\iota(\varepsilon_i) = -\varepsilon_{2n-i+1}$.

Proof. In the above expression $R_1^\iota(w^{-1})$ represents the ι-orbit sums on $R(w^{-1})$ where the orbit has size 1, and $R_2^\iota(w^{-1})$ represents the ι-orbit sums on $R(w^{-1})$ where the orbit has size 2.

Write $v = \sum_{w\in W^J} d_w v_w$ and suppose $vI' = 0$.

That $v(Y_iY_{2n-i+1} - t^{2n-1}) = 0$ forces $\langle w^{-1}\lambda, \varepsilon_i + \varepsilon_{2n-1}\rangle = 0$, $\langle w^{-1}\delta, \varepsilon_i + \varepsilon_{2n-1}\rangle - 2n + 1 = 0$ whenever $d_w \neq 0$. In particular $v \neq 0$ implies $\iota\lambda = \lambda$, which is in the hypotheses of our proposition.

That $v(T_i - T_{2n-i}) = 0$ forces a relation on d_w and $d_{ws_is_{2n-i}}$ corresponding to the first term in the above product. (Note that $\iota(\alpha_i) = \alpha_{2n-i}$.) That $v(T_n - t) = 0$ forces a relation on d_w and d_{ws_n} corresponding to the second term in the above product. (Note that $\iota(\alpha_n) = \alpha_n$.) The resulting relation between nonzero d_w and d_{id} is independent of reduced expression for w and given by (5.2).

□

5.3. Second proof of Theorem 4.6

Again we must allow half-integral weights.

Recall $I = \langle \pi - 1,\ T_i - T_{i+n},\ (0 \le i < n)\rangle$. Let u be the permutation defined by

$$u(i) = \begin{cases} i & i \le n \\ 3n - i + 1 & i > n \end{cases}$$

and set $I' = T_u I T_u^{-1}$. Note that u is the longest element of $\underbrace{S_1 \times \cdots \times S_1}_{n} \times S_n$.

Then

$$I' = \langle T_n + 1 - t - t^{1-n}Y_{n+1},\ T_i - T_{2n-i},\ Y_iY_{2n-i+1} - t^{2n-1}, (1 \le i < n)\rangle.$$

One can verify

$$T_u(T_i - T_{i+n})T_u^{-1} = T_i - T_{2n-i}$$
$$T_u(\pi - 1)T_u^{-1} = T_{n+1}T_{n+2}\cdots T_{2n-1}\pi T_{n+1}^{-1}\cdots T_{2n-1}^{-1} - 1.$$

Hence $I' \ni T_{n+1}T_{n+2}\cdots T_{2n-1}\pi - T_{2n-1}\cdots T_{n+1} \equiv T_{n+1}T_{n+2}\cdots T_{2n-1}\pi - T_{n-1}\cdots T_1$. And so $I' \ni T_{n+1}\cdots T_{2n-1}\pi T_1^{-1}\cdots T_{n-1}^{-1}T_n^{-1} - T_n^{-1} = t^{-n}Y_{n+1} - T_n^{-1} = t^{-1}(t^{1-n}Y_{n+1} - (T_n + 1 - t))$.

Then also $I' \ni (t^{-n}Y_{n+1} - T_n^{-1})(t^nY_n + t^{2n-1}T_n) = Y_{n+1}Y_n - t^{2n-1}$. From this it is easy to show $Y_iY_{2n-i+1} - t^{2n-1} \in I'$.

Proposition 5.3. *Suppose λ satisfies $\lambda_i = -\lambda_{2n-i+1}, (1 \le i < n)$, i.e., if we set $\nu = (\frac{m}{2}^{2n}) + \lambda$, then $\nu = (m^{2n}) - \nu$ (even for $m = 0$, sorting parts in the latter expression so it is a partition). Then any $v \in \mathcal{L}_\lambda$ with $vI' = 0$ is proportional to the nonzero I'-killed functional*

$$v = \sum_{\substack{w\in W^J \\ \iota(w^{-1}\lambda) = w^{-1}\lambda}} \Bigg(\prod_{\beta\in R_2^\iota(w^{-1})} \frac{-1}{q^{\langle\lambda,\beta\rangle}t^{\langle\delta,\beta\rangle}} \frac{1 - q^{-\langle\lambda,\beta\rangle}t^{1-\langle\delta,\beta\rangle}}{1 - q^{\langle\lambda,\beta\rangle}t^{1+\langle\delta,\beta\rangle}} \times$$
$$\prod_{\beta\in R_1^\iota(w^{-1})} (-1)\frac{1 - q^{-\frac{\langle\lambda,\beta\rangle}{2}}t^{\frac{1-\langle\delta,\beta\rangle}{2}}}{1 - q^{\frac{\langle\lambda,\beta\rangle}{2}}t^{\frac{1+\langle\delta,\beta\rangle}{2}}}\Bigg) v_w \quad (5.3)$$

Here again ι is the involution on the weight lattice with $\iota(\varepsilon_i) = -\varepsilon_{2n-i+1}$.

Proof. In the above expression $R_1^\iota(w^{-1})$ represents the ι-orbit sums on $R(w^{-1})$ where the orbit has size 1, and $R_2^\iota(w^{-1})$ represents the ι-orbit sums on $R(w^{-1})$ where the orbit has size 2.

Write $v = \sum_{w\in W^J} b_w v_w$ and suppose $vI' = 0$.

That $v(Y_iY_{2n-i+1} - t^{2n-1}) = 0$ forces $\langle w^{-1}\lambda, \varepsilon_i + \varepsilon_{2n-1}\rangle = 0$, $\langle w^{-1}\delta, \varepsilon_i + \varepsilon_{2n-1}\rangle - 2n+1 = 0$ whenever $b_w \neq 0$. In particular $v \neq 0$ implies $\iota\lambda = \lambda$, which we have already included in our hypotheses.

That $v(T_i - T_{2n-i}) = 0$ forces a relation on b_w and $b_{ws_is_{2n-i}}$ corresponding to the first term in the above product. (Note that $\iota(\alpha_i) = \alpha_{2n-i}$.) That $v(T_n + 1 - t - t^{1-n}Y_{n+1}) = 0$ forces a relation on b_w and b_{ws_n} corresponding to the second term in the above product. (Note that $\iota(\alpha_n) = \alpha_n$.) The resulting relation between nonzero b_w and b_{id} is independent of reduced expression for w and given by (5.3).

□

6. Extended affine Weyl groups

Let W be a finite Weyl group acting on a Euclidean space $\mathbb{R}^n$, with associated root lattice Λ_0, not assumed to span $\mathbb{R}^n$. A *generalized weight lattice* for W is a lattice Λ (spanning $\mathbb{R}^n$) containing Λ_0 such that

$$\frac{2\langle r,\nu\rangle}{\langle r,r\rangle} \in \mathbb{Z}$$

for all roots r and vectors $\nu \in \Lambda$. (The specification of a pair (W,Λ) is equivalent to the classical notion of a root datum. In particular, there is a one-to-one correspondence between isomorphism classes of pairs (W,Λ) and isomorphism classes of connected compact Lie groups; here Λ is the inverse image of the identity element under the exponential map.)

An *extended affine Weyl group* is then a group of the form $\widetilde{W} = G \ltimes \Lambda$, where Λ is a generalized weight lattice for a finite Weyl group W, and $G \subset \mathrm{Aut}(\Lambda)$ contains W as a normal subgroup. Given $\nu \in \Lambda$, we denote the corresponding element of $\widetilde{W}$ by τ_ν to avoid confusion.

An alcove is the closure of a fundamental region for the normal subgroup $W \ltimes \Lambda_0$; the *standard alcove* is the unique alcove containing the origin contained in the fundamental chamber of W. The union of the boundaries of the alcoves is a union of hyperplanes; the distance between two alcoves is the number of such hyperplanes that separate their interiors. Given $w \in \widetilde{W}$, the length of w is the distance between the standard alcove and its image under w. In particular, the elements of length 0 are those that preserve the standard alcove, and there is a natural map from $\widetilde{W}$ to the length 0 subgroup with kernel $W \ltimes \Lambda_0$.

The braid group $B(\widetilde{W})$ is generated by elements $U(w)$ for $w \in \widetilde{W}$, subject to the relations $U(w_1w_2) = U(w_1)U(w_2)$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$; thus $B(\widetilde{W})$ contains a subgroup identified with the length 0 subgroup of $\widetilde{W}$, and is generated over this subgroup by $U(s)$ for s of length 1.

The Hecke algebra $H(\widetilde{W})$ is obtained from the group algebra of $B(\widetilde{W})$ by adding further quadratic relations $(U(s) - t_s^{\frac{1}{2}})(U(s) + t_s^{-\frac{1}{2}}) = 0$. We require $t_s = t_{s'}$ if s and s'

are conjugate, since then $U(s)$ and $U(s')$ are conjugate. More generally, if $\sigma \in \widetilde{W}$ is of length 0 (and so acts on the affine Dynkin diagram) we have σ act on scalars by

$$\sigma t_s \sigma^{-1} = t_{\sigma s \sigma^{-1}}.$$

If there are simple reflections which are conjugate in $\widetilde{W}$ but not in $W \ltimes \Lambda_0$ then this action on scalars is nontrivial, and therefore the resulting extended affine Hecke algebra is no longer a central algebra over $\mathbb{C}[t_s^{\pm 1/2}]$. However if we specialize the t_s appropriately, one can indeed obtain a central algebra over $\mathbb{C}[t_s^{\pm 1/2}]$. Alternatively, we can view such σ as giving an intertwining map between two different Hecke algebras.

For instance, in the case of H_n^C the outer involution σ in general gives an intertwining map between two different instances of H_n^C. In particular it takes nonsymmetric Koornwinder polynomials for one set of parameters to nonsymmetric Koornwinder polynomials with modified parameters. This becomes significant because the construction of Y operators given in the next section includes such intertwiners and this explains for instance the difference operator of [10].

For the cases $\widetilde{S}_n$, $\widetilde{C}_n$ which are of particular interest to us, we can represent elements of the corresponding braid groups pictorially as periodic braids. We follow (American) book-spine conventions; that is, the leftmost symbol in a word corresponds to the topmost move in the corresponding braid picture. To save space, commuting symbols may be drawn as occurring at the same time.

The generators of the braid group are denoted U_i; in the Hecke algebra, they satisfy $U_i - U_i^{-1} = t_i^{1/2} - t_i^{-1/2}$, and we define $T_i = \sqrt{t_i} U_i$.

In $\widetilde{S}_n$, U_i corresponds to a picture in which (reading down) the jth strand (from the left) crosses under the $j+1$st strand for all $j \equiv i \mod n$. Similarly π corresponds to

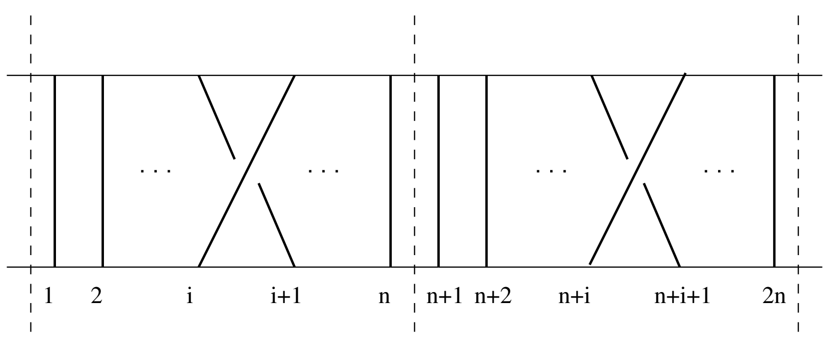

FIGURE 1. $U_i \in B(\widetilde{S}_n)$

the operation that simply moves each strand one step to the right.

The elements $Y_i \in B(\widetilde{S}_n)$ (i.e., the elements of the braid group given by replacing T_i by U_i and $\overline{T}_i$ by U_i^{-1} in (2.3), (2.4), (2.5)) moves the strands congruent to $i \mod n$ n steps to the right, underneath the adjacent strands congruent to $1 \ldots i-1$ and over the remaining strands. See figures (6), (6).

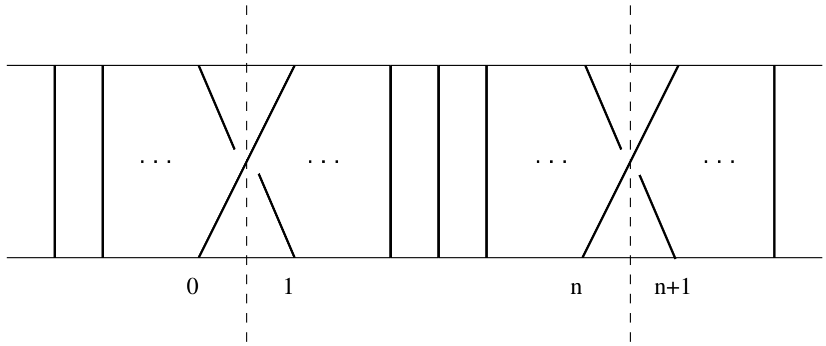

FIGURE 2. $U_0 \in B(\widetilde{S}_n)$

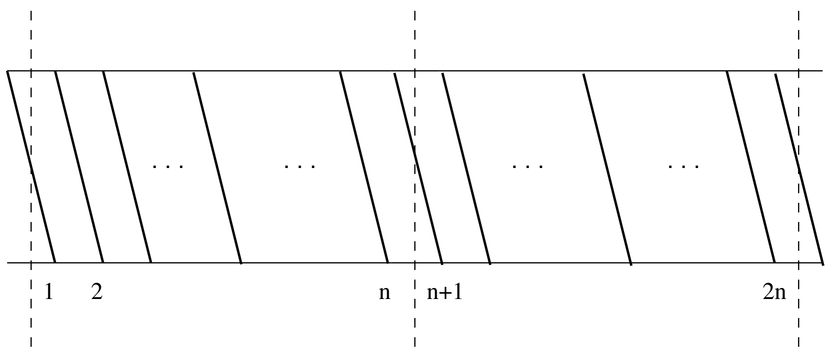

FIGURE 3. $\pi \in B(\widetilde{S}_n)$

Similarly $B(\widetilde{C}_n)$ corresponds to braids which are symmetric with respect to rotations about a vertical line between i and $i+1$ for $i \equiv 0 \mod n$. Note that the two rotation symmetries generate a translation, and thus $B(\widetilde{C}_n)$ is naturally a subgroup of $B(\widetilde{S}_{2n})$

7. Commutative subgroups of affine braid groups

The cleanest proof of our quadratic transformations requires the construction of nonstandard commutative subalgebras of affine Hecke algebras. It turns out that there is a natural construction that associates a commutative subgroup of an extended affine braid group to each chamber of the associated finite Weyl group.

More precisely, to each chamber we may associate an injective homomorphism $\Lambda \to B(\widetilde{W})$. We first consider a related construction which associates a map $\widetilde{W} \to B(\widetilde{W})$ to each alcove of $\widetilde{W}$. For the standard alcove this is just the map

$$w \mapsto U(w)$$

used to define $B(\widetilde{W})$. More generally we define

$$U_{w_1}(w_2) := U(w_1)^{-1}U(w_1w_2).$$

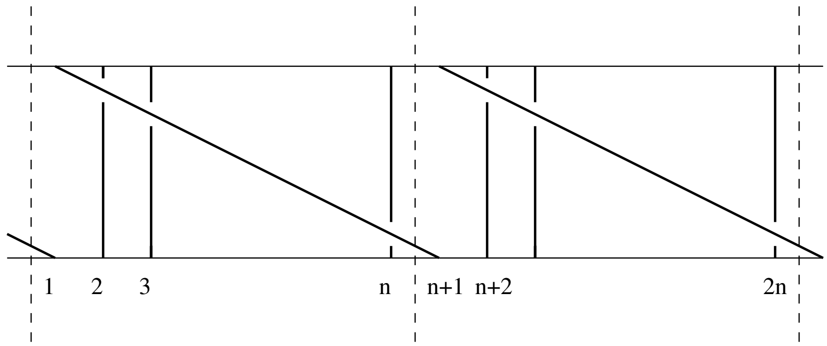

FIGURE 4. $Y_1 \in B(\widetilde{S}_n)$

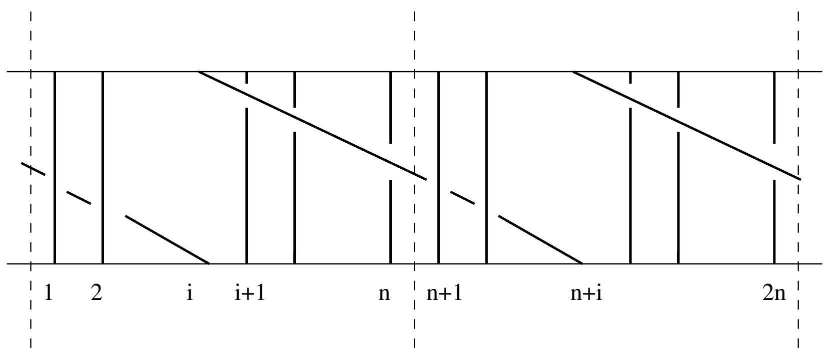

FIGURE 5. $Y_i \in B(\widetilde{S}_n)$

Note that if we multiply w_1 on the right by an element of length 0 that this has no effect on $U(w_1)$, which is therefore a function only depending on the associated alcove. More precisely we have the following.

Lemma 7.1. *Let A_0 denote the standard alcove of the extended affine Weyl group $\widetilde{W}$. Then for any simple reflection s of $\widetilde{W}$ and any element $w \in \widetilde{W}$,*

$$U_w(s) = U(s)^{\pm 1}$$

The sign is positive if and only if the simple root corresponding to s is positive for the alcove A_0^w.For any length 0 element σ

$$U_w(\sigma) = \sigma.$$

Proof. By definition, we have

$$U_w(s) = U(w)^{-1}U(ws).$$

If $\ell(ws) > \ell(w)$, then $U(ws) = U(w)U(s)$, and thus $U_w(s) = U(s)$; otherwise, $U(w) = U(ws)U(s)$, and $U_w(s) = U(s)^{-1}$. Since $\ell(ws) > \ell(w)$ if and only if s is positive for the alcove A_0^w, the claim follows.

For length 0 elements, we find that $\ell(w\sigma) = \ell(\sigma w) = \ell(w)$, and the claim follows. $\square$

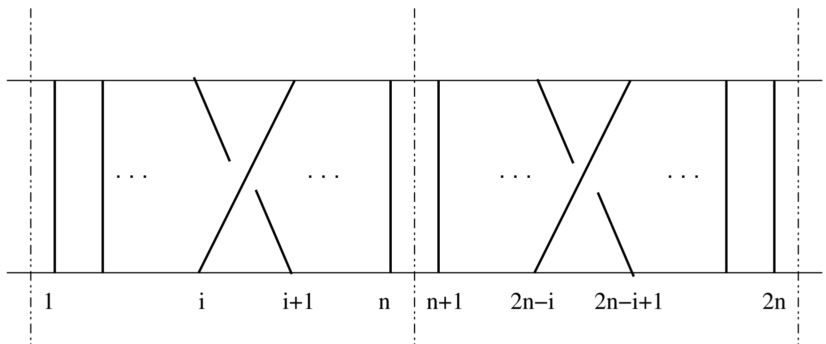

FIGURE 6. $U_i \in B(\widetilde{C}_n)$

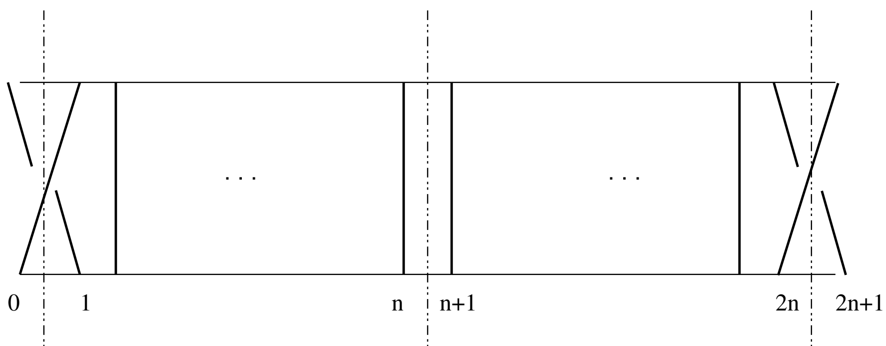

FIGURE 7. $U_0 \in B(\widetilde{C}_n)$

With this in mind, we will also write

$$U_A(w) = U_{w_A}(w),$$

where A is the alcove $A_0^{w_A}$.

We also trivially have:

Lemma 7.2. *For any element $w \in \widetilde{W}$ and any alcove A,*

$$U_A(w)^{-1} = U_{A^w}(w^{-1}).$$

Similarly, for any elements w_1, $w_2 \in \widetilde{W}$ and any alcove A,

$$U_A(w_1 w_2) = U_A(w_1) U_{A^{w_1}}(w_2).$$

We can thus describe $U_A(w)$ as follows: Take any expression (reduced or not) for w in terms of simple reflections, say

$$w = s_1 s_2 \dots s_n \sigma$$

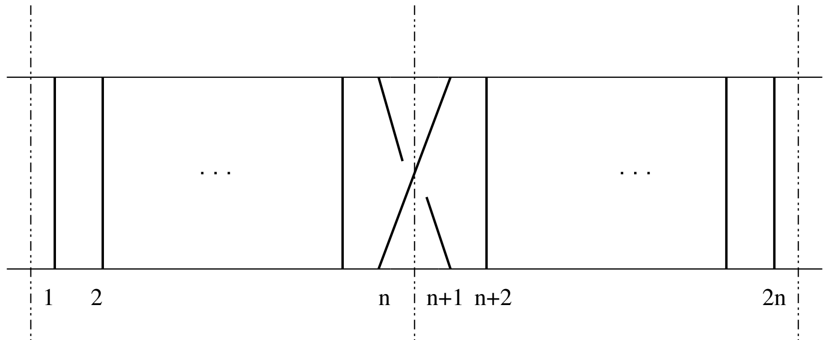

FIGURE 8. $U_n \in B(\widetilde{C}_n)$

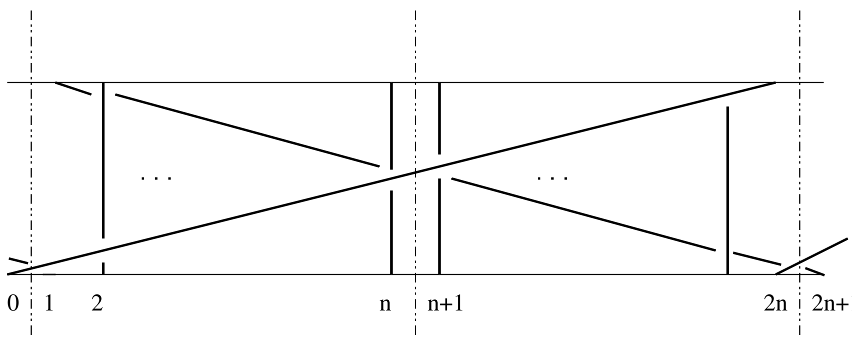

FIGURE 9. $Y_1 \in B(\widetilde{C}_n)$

Then by iterating the second lemma, we obtain

$$U_A(w) = U_A(s_1)U_{A^{s_1}}(s_2)U_{A^{s_1 s_2}}(s_3)\cdots U_{A^{s_1\cdots s_{n-1}}}(s_n)\sigma \tag{7.1}$$
$$= U(s_1)^{\pm 1}U(s_2)^{\pm 1}\cdots U(s_n)^{\pm 1}\sigma, \tag{7.2}$$

where each sign is given by the sign of the given simple root on the current choice of alcove.

So far everything we have been saying could apply just as well to any (extended) Coxeter group. In the case of an extended affine Weyl group, we have the additional structure of the associated finite Weyl group W. In particular, in addition to the alcoves of $\widetilde{W}$, we may consider the chambers of W.

Using the natural quotient map $\widetilde{W} \to G$ we may associate to each simple root of $\widetilde{W}$ a root of G and may thus sensibly talk about the sign of a root with respect to a chamber. Thus given a chamber C of the finite Weyl group W and a simple reflection of $\widetilde{W}$, we can define

$$U_C(s) = U(s)^{\pm 1},$$

with positive sign precisely when the corresponding root is positive for C; that is, when the corresponding halfspace contains C. Then for any word $w = s_n s_{n-1} \dots s_1 \sigma$ in the

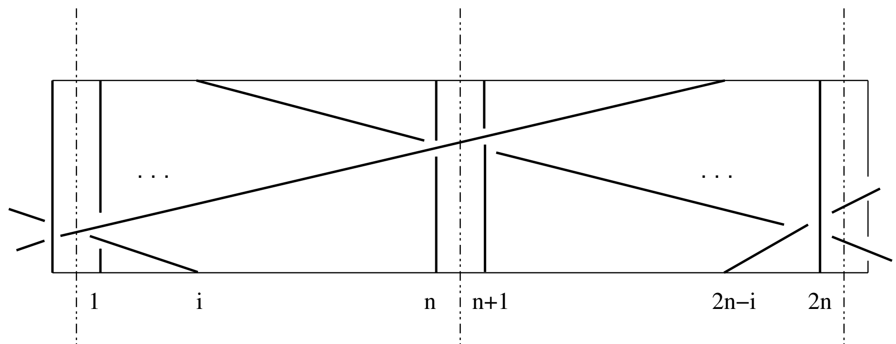

FIGURE 10. $Y_i \in B(\widetilde{C}_n)$

generators of $\widetilde{W}$, we define

$$U_C(w) = U_C(s_1)U_{C^{s_1}}(s_2)U_{C^{s_1s_2}}(s_3)\cdots U_{C^{s_1\cdots s_{n-1}}}(s_n)\sigma.$$

Theorem 7.3. *Let $\widetilde{W}$ be an extended affine Weyl group, and let C be a chamber of the associated finite Weyl group W. Then for any word w in the generators of $\widetilde{W}$, there exists a vector v_w such that for any alcove $A \subset v_w + C$,*

$$U_A(w) = U_C(w).$$

In particular, $U_C(w)$ depends on w only via its image in $\widetilde{W}$, and

$$U_C(w_1w_2) = U_C(w_1)U_{C^{w_1}}(w_2).$$

Proof. We restrict our attention to the case $\widetilde{W} = W \ltimes \Lambda_0$; the general case is analogous. Write $w = s_1 \dots s_n$, and consider

$$U_C(w) = U(s_1)^{\pm 1}\cdots U(s_n)^{\pm 1}.$$

For $1 \le i \le n$, let $H_i(w)$ denote either the half-space corresponding to s_i or its complement (the former precisely when $U(s_i)^{\pm 1}$ occurs with positive sign), and define a sequence of convex sets $D_i(w)$ by:

$$D_n(w) = H_n(w) \tag{7.3}$$
$$D_i(w) = D_{i+1}(w)^{s_i} \cap H_i(w). \tag{7.4}$$

We claim that the following is true for $1 \le i \le n$:

(a) The set $D_i(w)$ is nonempty, and satisfies

$$D_i(w) + C^{s_1\cdots s_{i-1}} = D_i(w).$$

(b) For any alcove $A \subset D_i(w)$,

$$U_A(s_i\cdots s_n) = U_{C^{s_1\cdots s_{i-1}}}(s_i\cdots s_n)$$

Indeed, a simple induction argument reduces to the case $n = 1$, in which case (a) and (b) are immediate.

Thus any choice $v_w \in D_1(w)$ proves the first claim of the theorem. The remaining claims follow from the corresponding results for alcoves. □

The point of using chambers rather than alcoves is that chambers are left invariant by translations. As a result, if Λ denotes the translation subgroup of $\widetilde{W}$, we find the following.

Corollary 7.4. *For any chamber C, U_C induces a homomorphism $U_C : \Lambda \to B(\widetilde{W})$. The homomorphisms associated to different choices of C are conjugate, in the sense that*

$$U_{C^w}(\tau_\nu) = U_C(w)^{-1} U_C(w\tau_\nu w^{-1}) U_C(w)$$

for arbitrary $w \in \widetilde{W}$.

Proof. The first claim is immediate. For the second claim, we write

$$U_C(w\tau_\nu w^{-1}) = U_C(w) U_{C^w}(\tau_\nu) U_{C^w}(w^{-1}).$$

□

Remark. Note more generally that for each chamber C we can extend this homomorphism to a homomorphism from the stabilizer of C to $B(\widetilde{W})$.

We will define $Y_\nu^C = U_C(\tau_\nu)$ accordingly, and write $Y_\nu = Y_\nu^{C_0}$. Note the Y_ν^C commute (as Λ is commutative).

It was observed by Cherednik [2, 1, page 265] such alternate Y_ν^C exist. Our application appears to be the first in which these alternate Y_ν^C play a major role.

In addition to the relevance of alternate chambers to our vanishing results, note also that with respect to our standard inner product for $\widetilde{S}_n$ it lets us express the adjoint to the standard Y_ν as $Y_{w_0\nu}^C$ where C is the opposite chamber to the standard one.

Theorem 7.5. *Suppose the weight λ is dominant for the chamber C, that is $\lambda \in C$. Then*

$$Y_\lambda^C = U(\tau_\lambda).$$

In general, if we write $\lambda = \lambda^+ - \lambda^-$ with $\lambda^\pm \in C$, then

$$Y_\lambda^C = U(\tau_{\lambda^+}) U(\tau_{\lambda^-})^{-1}.$$

Proof. Let w be a word expressing τ_λ in terms of the generators of $\widetilde{W}$, and choose v_w accordingly. In particular, we can choose v_w to be a dominant weight λ' for C. We thus find

$$Y_\lambda^C = U_{A_0+\lambda'}(\tau_\lambda) = U(\tau_\lambda\tau_{\lambda'}) U(\tau_{\lambda'})^{-1}$$

But since both λ and λ' are dominant for C, it follows that

$$\ell(\tau_\lambda) + \ell(\tau_{\lambda'}) = \ell(\tau_{\lambda+\lambda'}),$$

and thus

$$U(\tau_\lambda\tau_{\lambda'}) = U(\tau_\lambda) U(\tau_{\lambda'});$$

the result follows. □

In particular, we find that Y_λ agrees with the standard construction of a commutative subgroup of $B(\widetilde{W})$.

Theorem 7.6. *Let $H(\widetilde{W})$ be the Hecke algebra corresponding to $\widetilde{W}$, and let Y_ν^C denote the image in $H(\widetilde{W})$ of the corresponding element of $B(\widetilde{W})$. Then for any weight $\lambda \in \Lambda_0$, the sum*

$$\sum_{\mu \in \lambda^W} Y_\mu^C$$

over the W-orbit of λ is a central element of $H(\widetilde{W})$ independent of the choice of chamber C.

Proof. If we write $C = C_0^w$, then

$$\sum_{\mu \in \lambda^W} Y_\mu^C = U(w)^{-1} \sum_{\mu \in \lambda^W} Y_\mu U(w),$$

and thus the claim follows from the standard fact that

$$\sum_{\mu \in \lambda^W} Y_\mu$$

is central. □

Remark. For general $\lambda \in \Lambda$, this element commutes with all of the generators, but might act nontrivially on scalars.

For each λ in the root lattice of $\widetilde{W}$, we can thus define nonsymmetric Macdonald polynomials E_λ^C by

$$E_\lambda^C \propto U(w)^{-1} E_{w^{-1}\lambda},$$

where $C = C_0^w$, and the constant is chosen to make the coefficient of x^λ in E_λ^C equal to 1.

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