April 2007

299 W07

Porridge

The n-fold rotational symmetry
(or the $z_k = z_{k+n}$ in Matt's solution

or that (for Robert) there are n-pts we take successive midpoints of in a "fixed" order)
means there is underlying representation theory.

Here, it feels more like finding an eigenbasis.

1) For a fixed $0 \le \Theta < 2\pi$, what are the possible limits of $(\cos\Theta)^k$ as $k \to \infty$?

2) For $(e^{i\Theta} + e^{-i\Theta})^k$ as $k \to \infty$?

3) If $A, B \in M_n\mathbb{C}$ and $AB = BA$ and $\mathbb{C}^n$ has a basis consisting of eigenvectors of A, that have <u>distinct</u> eigenvalues, show then any eigenvector of A is also an eigenvector of B.

aka "eigenbasis" of A

(This is related to Schur's Lemma. Think how/why)

(4) Consider the $n \times n$ permutation matrix A corresponding to the n-cycle $\sigma = (1\,2\,3 \cdots n)$:

$$A = \begin{bmatrix} 0 & 0 & & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & & 0 & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Find A's eigenvalues and corresponding eigenvectors.

[This has something to do with decomposing the regular representation of $G = C_n \cong \mathbb{Z}/n\mathbb{Z}$. It also has something to do with understanding irreducible representations of $G = S^1$.]

(5) If ~~all~~ x_k = grams of porridge the k^{th} knight has in his bowl

Consider the vector $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$.

After one step, what is the new vector $\underline{x}'$?

(6) Writing $\underline{x}' = B\underline{x}$ find the matrix B.

(i.e. the porridge stealing is linear. Or you can think of Markov processes).

(7) Explain why $AB = BA$.

At worst, multiply them and verify.
Better, explain why ~~the~~ one expects it from the symmetry of the problem.

(8) Find an eigenbasis for B.

At worst, start from scratch. Better use #3 and realize this problem has numerous parts by design.

(9) Rewrite B in this basis. Call it B'

(10) What is the limit $(B')^k$ as $k \to \infty$?
(11) What is the limit B^k as $k \to \infty$?

(12) What about $(B')^k \underline{x}$ and $B^k \underline{x}$ as $k \to \infty$?

(13) When n is odd notice

$$B^k \underline{x} \longmapsto \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{where } \alpha = \tfrac{1}{n} \textstyle\sum x_i$$

(14) When n is even, note $B^k \underline{x} \mapsto \text{Const}_1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \text{Const}_2 \begin{bmatrix} 1 \\ -1 \\ \vdots \\ 1 \end{bmatrix}$

(15) With this viewpoint one can see it representation theoretically as saying the porridge stealing has C_n-rotational symmetry. i.e. commutes with the action of C_n. So if we break down space into the irreps of C_n, the porridge stealing must respect that decomp & we can look at its process one irrep at a time.

This gives a hint for the cube. S_4 acts as rotations of the cube & Martian porridge stealing commutes with that, so also preserves the irreps of S_4 (of how it acts on $\mathbb{C}^6 \approx \{faces\}$).

You can try to redo it this way.

The fact that there is a homomorphism

$$S_4 \twoheadrightarrow S_3$$

and $C_3 \hookrightarrow S_3 \to C_2$ explains some of Matt's & Robert's reductions.