April 2007

299 W07

Porridge

The n-fold rotational symmetry
(or the $z_k = z_{k+n}$ in Matt's solution

or that for Robert there are n-pts we take
successive midpoints of in a "fixed" order)
means there is underlying representation
theory.

Here, it feels more like finding
an eigenbasis.

(1) For a fixed $0 \leq \Theta < 2\pi$, what
are the possible limits of
$(\cos\Theta)^k$ as $k \to \infty$?

(2) For $(e^{i\theta} + e^{-i\theta})^k$ as $k \to \infty$?

(3) If $A, B \in M_n\mathbb{C}$ and $AB = BA$
and $\mathbb{C}^n$ has a basis consisting
of eigenvectors of A, that have distinct
eigenvalues, show then any eigenvector
of A is also an eigenvector of B.

aka "eigenbasis" of A

(This is related to Schur's Lemma. Think how/why)

(4) Consider the $n\times n$ permutation matrix A corresponding to the n-cycle $\sigma = (123 \cdots n)$:

$$A = \begin{bmatrix} 0 & 0 & & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Find A's eigenvalues and corresponding eigenvectors.

[This has something to do with decomposing the regular representation of $G = C_n \cong \mathbb{Z}/n\mathbb{Z}$. It also has something to do with understanding irreducible representations of $G = S^1$.]

(5) If x_k = grams of porridge the k^{th} knight has in his bowl

Consider the vector $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$.

After one step, what is the new vector $\underline{x}'$?

(6) Writing $\underline{x}' = B\underline{x}$ find the matrix B.

(i.e. the porridge stealing is linear.
Or you can think of Markov processes).

(7) Explain why $AB = BA$.

At worst, multiply them and verify.
Better, explain why ~~the~~ one expects it from the symmetry of the problem.

(8) Find an eigenbasis for B.

At worst, start from scratch. Better use #3 and realize this problem has numerous parts by design.

(9) Rewrite B in *this* basis. Call it B'

(10) What is the limit $(B')^k$ as $k \to \infty$?
(11) What is the limit B^k as $k \to \infty$?

(12) What about $(B')^k \underline{x}$ and $B^k \underline{x}$ as $k \to \infty$?

(13) When n is odd notice
$$B^k \underline{x} \longmapsto \alpha \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{where } \alpha = \tfrac{1}{n} \textstyle\sum x_i$$

(14) When n is even, note $B^k \underline{x} \mapsto \text{Const}_1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \text{Const}_2 \begin{bmatrix} 1 \\ -1 \\ \vdots \end{bmatrix}$

(15) With this viewpoint
one can see it representation theoretically
as saying the porridge stealing
has C_n-rotational symmetry.
i.e. commutes with the action of C_n.
So if we break down space into the
irreps of C_n, the porridge stealing must
respect that decomp & we can look at
its process one irrep at a time.

This gives a hint for the cube.
S_4 acts as rotations of the cube
& Martian porridge stealing commutes
with that, so also preserves the
Irreps of S_4 (of how it acts on $\mathbb{C}^6 \approx \{faces\}$).

You can try to redo it this way.

The fact that there is a homomorphism
$S_4 \twoheadrightarrow S_3$
and $C_3 \hookrightarrow S_3 \to C_2$ explains some of
Matt's + Robert's reductions.