Simple KLR modules and their characters

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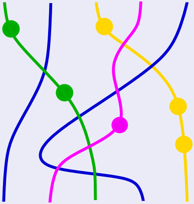
Definition of the algebra $R(\nu)$

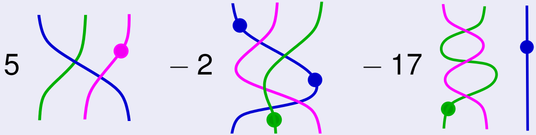
Associated to graph Γ consider braid-like diagrams with dots whose strands are labelled by the vertices $i \in I$ of the graph Γ.



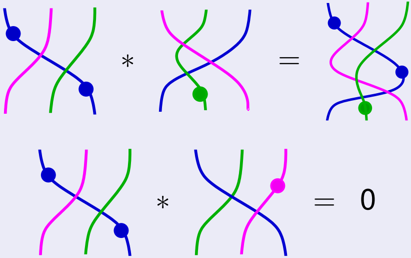
Let $\nu = \sum_{i \in I} \nu_i \cdot \alpha_i$, for $\nu_i = 0, 1, 2, \ldots$

ν keeps track of how many strands of each color occur in a diagram

Form an abelian group by taking $\mathbb{Z}$-linear (or $\Bbbk$-linear) combinations of diagrams:



Multiplication is given by stacking diagrams on top of each other when the colors match:



Definition

Given $\nu \in \mathbb{N}[I] = Q^+$ define the ring $R(\nu)$ as the set of planar diagrams colored by ν, modulo planar braid-like isotopies and the following local relations:

There are induction and restriction functors corresponding to inclusions $R(\nu) \otimes R(\nu') \subset R(\nu + \nu')$

$$\mathrm{Ind}_{\nu,\nu'}^{\nu+\nu'} : R(\nu) \otimes R(\nu') - \mathrm{mod} \to R(\nu + \nu') {-} \mathrm{mod}$$

$$\mathrm{Res}_{\nu,\nu'}^{\nu+\nu'} : R(\nu + \nu') {-} \mathrm{mod} \to R(\nu) \otimes R(\nu') {-} \mathrm{mod}$$

Shuffles

Induction

We saw yesterday afternoon that to generate

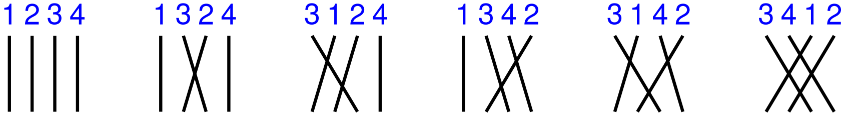
$$\mathrm{Ind}_{R(m\alpha_i)\otimes R(n\alpha_i)}^{R((m+n)\alpha_i)} M \boxtimes N$$

we need the minimal length (left) coset representatives

$$\mathcal{S}_{m+n}/\mathcal{S}_m \times \mathcal{S}_n$$

Example: $\mathrm{Ind}_{2,2}^{4}$

$12 \sqcup\!\sqcup 34 =$



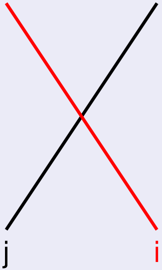
If $u \in M$, $v \in N$ are such that $1_{i_1 i_2} u = u$ and $1_{i_3 i_4} v = v$, then $1_{i_1 i_2 i_3 i_4} u \otimes v = u \otimes v$.

$$\begin{aligned}\psi_1 \psi_2 1_{\underline{\boldsymbol{i}}} u \otimes v =& \psi_1 \psi_2 1_{i_1 i_2 i_3 i_4} u \otimes v \\ =& 1_{i_3 i_1 i_2 i_4} \psi_1 \psi_2 u \otimes v \\ =& 1_{s_1 s_2 (\underline{\boldsymbol{i}})} \psi_1 \psi_2 u \otimes v\end{aligned}$$

So $\psi_{\widehat{w}} 1_{\underline{\boldsymbol{i}}} = 1_{w(\underline{\boldsymbol{i}})} \psi_{\widehat{w}}$ where $w(i_1 \cdots i_m) = (i_{w^{-1}(1)} \cdots i_{w^{-1}(m)})$.

Recall the grading

deg

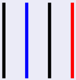


$= -(\alpha_i, \alpha_j)$

quantum shuffle

$$i_1 i_2 \sqcup\!\sqcup_q i_3 i_4 =$$

$i_1\ i_2\ i_3\ i_4$



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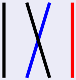
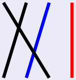
$i_1\ i_3\ i_2\ i_4$



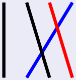
$-(\alpha_{i_2}, \alpha_{i_3})$

$i_3\ i_1\ i_2\ i_4$



$-(\alpha_{i_1} + \alpha_{i_2}, \alpha_{i_3})$

$i_1\ i_3\ i_4\ i_2$



$-(\alpha_{i_2}, \alpha_{i_3} + \alpha_{i_4})$

$i_3\ i_1\ i_4\ i_2$

$$-(\alpha_{i_1}, \alpha_{i_3}) - (\alpha_{i_2}, \alpha_{i_3} + \alpha_{i_4})$$

$i_3\ i_4\ i_1\ i_2$

$-(\alpha_{i_1} + \alpha_{i_2}, \alpha_{i_3} + \alpha_{i_4})$

quantum shuffle

$$
\begin{aligned}
ik \mathbin{⧢}_q ij = & q^0 ikij + q^{-(\alpha_k,\alpha_i)} iikj \\
& + q^{-(\alpha_i+\alpha_k,\alpha_i)} iikj + q^{-(\alpha_k,\alpha_i+\alpha_{i_4})} iikj \\
& + q^{-(\alpha_i,\alpha_i)-(\alpha_j,\alpha_i+\alpha_k)} iijk + q^{-(\alpha_i+\alpha_j,\alpha_i+\alpha_k)} ijik
\end{aligned}
$$

Characters

$$\mathrm{gdim} L(i^m) = [m]_i!$$

$$\mathrm{ch} M = \sum_{\underline{\boldsymbol{i}}} \mathrm{gdim}(1_{\underline{\boldsymbol{i}}} M)(\underline{\boldsymbol{i}})$$

$$\mathrm{ch} L(i^m) = [m]_i!(ii\cdots i) = [m]_i! i^m$$

$$\mathrm{ch} M \circ N = \mathrm{ch}\,\mathrm{Ind}\, M \boxtimes N = \mathrm{ch} M \sqcup\!\sqcup_q \mathrm{ch} N$$

$$\mathrm{ch}\,\mathrm{Ind}\, L(i) \boxtimes \cdots \boxtimes L(i) = q_i^{-\binom{m}{2}} [m]_i!$$

Recall

$$\begin{aligned}\sum_{w \in \mathcal{S}_3} q^{-2\ell(w)} &= (q^{-4} + q^{-2} + 1)(q^{-2} + 1)(1) \\ &= \frac{q^{-6} - 1}{q^{-2} - 1} \frac{q^{-4} - 1}{q^{-2} - 1} \frac{q^{-2} - 1}{q^{-2} - 1} \\ &= q^{-3} \frac{q^3 - q^{-3}}{q - q^{-1}} \frac{q^2 - q^{-2}}{q - q^{-1}} \frac{q - q^{-1}}{q - q^{-1}} \\ &= q^{-\binom{m}{2}} [m]!.\end{aligned}$$

E_j and ε_j

Just as E_j descends to the Grothendieck ring, we can define it on characters. $E_j \mid_{\mathbb{Z}[q,q^{-1}]} = \text{id}$.

$$E_j(i_1 \cdots i_m) = \begin{cases} (i_1 \cdots i_{m-1}) & \text{if } i_m = j \\ 0 & \text{else.} \end{cases}$$

$$E_j \mathrm{ch} M = \mathrm{ch} E_j M = \sum_{\underline{i}} \mathrm{gdim}(1_{\underline{i}} M) E_j(\underline{i})$$

ε_j^* and ε_j

$$\varepsilon_j^*(i_1 \cdots i_m) = k \text{ if } k \text{ is maximal such that } i_1 = i_2 = \cdots = i_k = j$$

$$\varepsilon_j(i_1 \cdots i_m) = k \text{ if } k \text{ is maximal such that } i_m = i_{m-1} = \cdots = i_{m-(k-1)} = j$$

For M simple

$$\varepsilon_j^*(M) = \max\{\varepsilon_j^*(\underline{\boldsymbol{i}}) \mid 1_{\underline{\boldsymbol{i}}} M \neq 0\} = \max\{k \geq 0 \mid (E_j^*)^k M \neq \mathbf{0}\}$$

$$\varepsilon_j(M) = \max\{\varepsilon_j(\underline{\boldsymbol{i}}) \mid 1_{\underline{\boldsymbol{i}}} M \neq 0\} = \max\{k \geq 0 \mid E_j^k M \neq \mathbf{0}\}$$

Example

$\varepsilon_i(L(i^m)) = m = \varepsilon_i^*(L(i^m))$

Computations on the whiteboard

Functors

Let M be a simple $R(\nu)$-module and $i \in I$. We set

$$\widetilde{f}_i M := \operatorname{cosoc}\ \mathrm{Ind}_{\nu,i}^{\nu+i} M \boxtimes L(i),$$
$$\widetilde{e}_i M := \operatorname{soc} E_i M \quad \text{where}$$
$$E_i M := \mathrm{Res}_{\nu-i}^{\nu-i,i} \circ \mathrm{Res}_{\nu-i,i}^{\nu} M.$$

Likewise we can define $\widetilde{e}_i^{\,*}$ where $e_i^* := \mathrm{Res}_{\nu-i}^{i,\nu-i} \circ \mathrm{Res}_{i,\nu-i}^{\nu} M$.

$\widetilde{e}_i M$ and $\widetilde{e}_i^{\,*} M$ are simple or zero.
Their characters are hard to find.

$$\widetilde{f}_i(L(i^m)) = L(i^{m+1})\{k\}$$

For cyclotomic quotients, the fact $x_1^m L(i^m) = 0$ but $x_1^{m-1} L(i^m) \neq 0$ yields that simple modules M are R^Λ-modules iff $\varepsilon_i^*(M) \leq \langle h_i, \Lambda\rangle$.

Jump

Set $\mathrm{jump}_i(M) = \varepsilon_i(M) + \varepsilon_i^*(M) + \langle h_i, -\nu\rangle$ if M is a simple $R(\nu)$-module.

- $\mathrm{jump}_i(M) \geq 0$
- If $\mathrm{jump}_i(M) \neq 0$ then $\mathrm{jump}_i(\widetilde{f}_i M) = \mathrm{jump}_i(M) - 1$
- $\mathrm{jump}_i(M) = 0$ iff $\widetilde{f}_i M = \mathrm{Ind}\, M \boxtimes L(i) = \mathrm{Ind}\, L(i) \boxtimes M$ is irreducible. In this case $\mathrm{ch}\widetilde{f}_i M = \mathrm{ch} M \sqcup\!\sqcup_q i$.

Computations on the whiteboard

Crystal Graphs

A *crystal* is a set B together with maps

- $\mathrm{wt}\colon B \longrightarrow P$,
- $\varepsilon_i, \varphi_i\colon B \longrightarrow \mathbb{Z} \sqcup \{\infty\}$ for $i \in I$,
- $\tilde{e}_i, \tilde{f}_i\colon B \longrightarrow B \sqcup \{0\}$ for $i \in I$,

satisfying certain properties, such as:

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle \quad \text{for any } i.$$
$$\text{When } \tilde{e}_i b \neq 0, \quad \mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$$
$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1.$$

$a = \tilde{e}_i b$ if and only if $\tilde{f}_i a = b$, and in this case we draw

$$a \xrightarrow{i} b$$

Example (B_i ($i \in I$))

$$\begin{array}{llllllllll}
 & \cdots & \xrightarrow{i} & b_i[-1] & \xrightarrow{i} & b_i[0] & \xrightarrow{i} & b_i[1] & \xrightarrow{i} & b_i[2] & \xrightarrow{i} \cdots \\
\mathrm{wt} & & & \alpha_i & & 0 & & -\alpha_i & & -2\alpha_i & \\
\varepsilon_i & & & -1 & & 0 & & 1 & & 2 & \\
\varphi_i & & & 1 & & 0 & & -1 & & -2 &
\end{array}$$

$\varepsilon_j(b_i[n]) = \varphi_j(b_i[n]) = -\infty$ if $j \neq i$.

Example

Set $\mathrm{wt}(M) = -\nu$ if M is an $R(\nu)$-module. Then the set of simple $R(\nu)$-modules, up to grading shift and isomorphism, for all $\nu \in \mathbb{N}[I]$ with data $\mathrm{wt}, \widetilde{e}_i, \widetilde{f}_i, \varepsilon_i, \varphi_i = \varepsilon_i + \langle h_i, wt \rangle$ forms a crystal graph $\mathcal{B}$.

Theorem (Lauda-V)

$\mathcal{B} \simeq B(\infty)$

quantum Serre relations

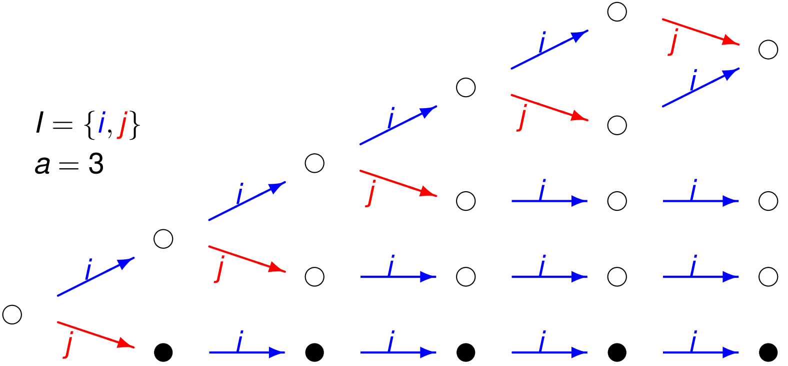
$$\sum_{n=0}^{a+1}(-1)^n E_i^{(a+1-n)} E_j E_i^{(n)} = 0$$

where $a = a_{ij} := -\langle h_i, \alpha_j \rangle = -2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$

Some of $B(\infty)$

$I = \{i, j\}$

$a = 3$



$\varepsilon_i = 0$ $\varepsilon_i^* = 0$

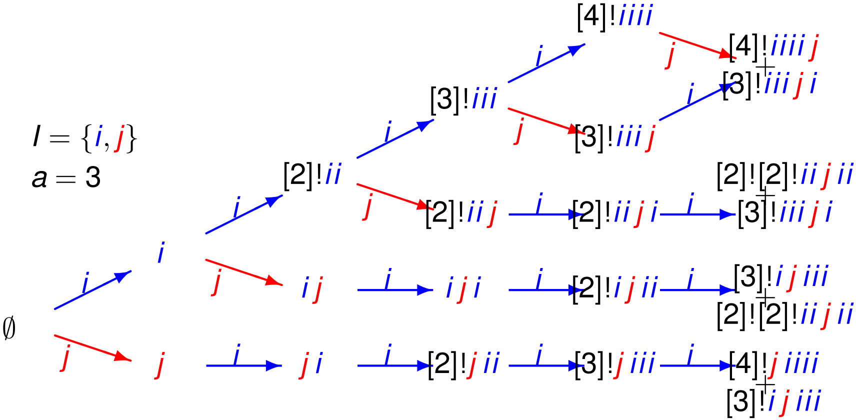
$\varepsilon_i = 1$ $\varepsilon_i^* = 0$

$\varepsilon_i = 2$ $\varepsilon_i^* = 0$

$\varepsilon_i = 3$ $\varepsilon_i^* = 0$

$\varepsilon_i = 4$ $\varepsilon_i^* = 1$

$I = \{i, j\}$
$a = 3$



$\emptyset$

i

j

$[2]!\,ii$

$i\,j$

$j\,i$

$[3]!\,iii$

$[2]!\,ii\,j$

$i\,j\,i$

$[2]!\,j\,ii$

$[4]!\,iiii$

$[3]!\,iii\,j$

$[2]!\,ii\,j\,i$

$[2]!\,i\,j\,ii$

$[3]!\,j\,iii$

$[4]!\,iiii\,j + [3]!\,iii\,j\,i$

$[2]![2]!\,ii\,j\,ii + [3]!\,iii\,j\,i$

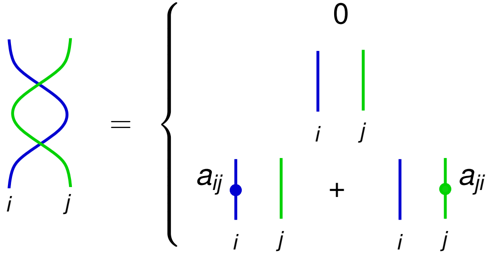
$[3]!\,i\,j\,iii + [2]![2]!\,ii\,j\,ii$

$[4]!\,j\,iiii + [3]!\,i\,j\,iii$

$\varepsilon_i = 0$	$\varepsilon_i = 1$	$\varepsilon_i = 2$	$\varepsilon_i = 3$	$\varepsilon_i = 4$
$\varepsilon_i^* = 0$	$\varepsilon_i^* = 0$	$\varepsilon_i^* = 0$	$\varepsilon_i^* = 0$	$\varepsilon_i^* = 1$

Local Relation (reminder)

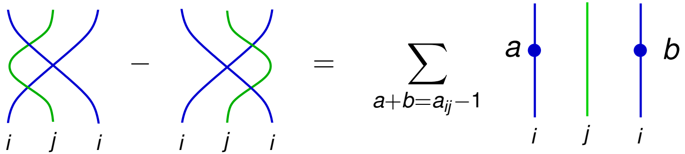
for $i \neq j$



if $i = j$

if $i \cdot j = 0$

if $i \cdot j \neq 0$



if