Patterns in Classified Data:
Tverberg-type Theorems for Data Science

By

THOMAS A. HOGAN

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DAVIS

Approved:

Professor Jesús A. De Loera, Chair

Professor Nina Amenta

Professor Luis Rademacher

Committee in Charge

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To my Oma and Opa.

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Abstract

Thomas A. Hogan
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Mathematics

This dissertation focuses on applications of geometric results inspired by Tverberg's theorem (so-called Tverberg-type problems). We study various aspects of partitioning data in such a manner that the convex hulls of the parts have specified combinatorial intersection patterns. For instance, Tverberg's theorem says that sufficiently many points in $\mathbb{R}^d$ can always be partitioned into m sets so that the convex hulls of the m parts all intersect. This dissertation contains new results and applications on three main variants of this theorem:

- *Tverberg theorems with altered nerves*: Given sufficiently many points in $\mathbb{R}^d$ and a desired intersection pattern (specified by a simplicial complex K with m vertices), can those points be partitioned into m sets in such a way that the convex hulls of the parts have the desired intersection pattern (so the nerve of the convex hulls of the parts is isomorphic to K)?
- *Stochastic Tverberg theorems*: Given sufficiently many points in $\mathbb{R}^d$, do "most" partitions of those points into m sets have the property that the convex hulls of the parts all intersect? In other words, if we randomly color each point with uniform probability $1/m$ are we likely to obtain a partition with the m convex hulls intersecting?
- *S-Tverberg problems*: Given a subset S of $\mathbb{R}^d$ and an integer $m \geq 2$, what is the smallest positive integer n with the following property: Any multiset of n points in S admits a partition into m subsets $A_1, A_2, \ldots, A_m$ with

$$\left(\bigcap_{i=1}^{m} \operatorname{conv}(A_i)\right) \cap S \neq \emptyset?$$

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We also introduce and explore applications of these new results in optimization, statistics and

computing. The work we present here is based on papers coauthored with Jesús A. De Loera, Frédéric Meunier, Nabil Mustafa, Déborah Oliveros, and Dominic Yang.

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CHAPTER 1

Introduction

This thesis is about partitioning data sets into classes in such a way that the convex hulls of the classes intersect in structured ways. The genesis of this field of study can be attributed to J. Radon, who proved the following remarkable result.

Theorem 1 (J. Radon 1923 [**Rad21**]). *Every set S with at least $d+2$ points in $\mathbb{R}^d$ can be partitioned into two subsets S_1 and S_2 such that the convex hulls of S_1 and S_2 share a point in common.*

This result, now known as Radon's lemma, is one of the most fundamental results in discrete geometry. It has a beautiful proof, and has cemented itself as a key lemma for important developments in discrete and computational geometry over the years. For proofs and more details about Radon's lemma and the introductory results that follow see [**Mat02**].

In 1959, B.J. Birch discovered that, in dimension two, Radon's lemma was a special case of an even more remarkable result: Any $3m-2$ points in the plane can be split into m groups in such a way that the m convex hulls share a point in common [**Bir59**]. He conjectured that this result could be extended further to any dimension. This beautiful conjecture, proved by H. Tverberg and known as Tverberg's theorem, serves as the central inspiration for the work in this thesis. However, we look at it from several new perspectives with applications to computing, optimization, and statistical learning in mind.

Figure 1.1. Two examples of Radon's theorem in dimension two.

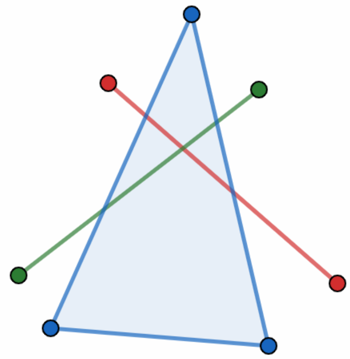


Figure 1.2. An example Tverberg partition where $d = 2$ and $m = 3$.

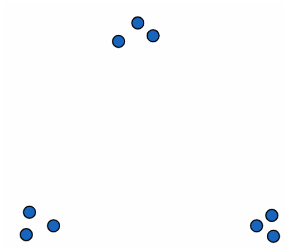


Figure 1.3. A set of nine points in dimension two with no Tverberg four-partition.

Theorem 2 (H. Tverberg 1966 [**Tve66**]). *Every set S with at least $(d+1)(m-1)+1$ points in Euclidean d-space $\mathbb{R}^d$ can be partitioned into m parts $\mathcal{P} = S_1, \ldots, S_m$ such that all the convex hulls of these parts have nonempty intersection.*

Such a partition, as depicted in Figure 1.2, is called a *Tverberg partition*. The number of points in this result is optimal, which is illustrated by the following example: Take a non-degenerate d-simplex embedded in $\mathbb{R}^d$, and consider a set S consisting of $m-1$ points very close to each vertex of the simplex. See Figure 1.3 for an example of such an S for $m = 4, d = 2$. Then S has a total of $(d+1)(m-1)$ points, but no Tverberg m-partition. Indeed, for each point $\boldsymbol{p}$ in $\mathbb{R}^d$ we can find a half-space containing $\boldsymbol{p}$ and at most $m-1$ points in S. Thus, for every point $\boldsymbol{p}$ in $\mathbb{R}^d$, and any m-partition $\mathcal{S}$ of S, we can find a separating hyperplane between $\boldsymbol{p}$ and at least one of the subsets of $\mathcal{S}$.

In fact, examples showing the optimality of Tverberg's theorem are much more common than one might expect, as shown by a "dimension-counting" argument [**BS18**]. Observe that for generic affine subspaces S_1, S_2 of $\mathbb{R}^d$, we have $\text{codimension}(S_1 \cap S_2) = \text{codimension}(S_1) + \text{codimension}(S_2)$. Inductively, we see that for generic affine subspaces $S_1, \ldots, S_m$, we have $\text{codimension}(\cap_{i\in[m]} S_i) =$

$\sum_{i\in[m]}(\text{codimension}(S_i))$. With this observation in mind, it is easy to show the following. If S is in general enough position, and in the partition $S = S_1 \cup \cdots \cup S_m$, we have $1 \leq |S_j| \leq (d+1)$ for every j, then the m-fold intersection of the affine hulls of the S_j is a single point if $|S| = (m-1)(d+1)+1$, and is empty if $|S| \leq (m-1)(d+1)$.

One consequence of this is that for $(m-1)(d+1)$ points randomly sampled from a continuous probability distribution, with probability one, we can expect there to be no Tverberg m-partition. But as soon as we add just one more point, Tverberg's theorem guarantees that there is always (not just with probability one!) a Tverberg m-partition. This strange phenomena motivates the new results presented in this thesis, as we will touch upon related concepts including probabilistic aspects and threshold phenomena of Tverberg partitions, as well as other induced intersection patterns with special constraints.

Overall our theoretical focus in this dissertation will be on *Tverberg-type problems*: exploring the partitions of data sets in $\mathbb{R}^d$ and the combinatorial properties exhibited by the convex hulls of their subsets. We prove a variety of new results contributing to the vast literature of generalizations (see also [**BS18**, **DLGMM19**]) around Tverberg's theorem. Here are three main avenues we explore:

- *Tverberg's theorem with altered nerves:* A generalization of Tverberg's theorem to different specified nerves- or intersection patterns of the convex hulls of the subset. See Figure 1.4 for an example of a nerve of a partitioned set.

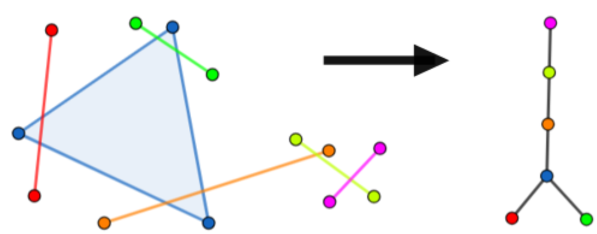


FIGURE 1.4. A partitioned point set on the left and its induced nerve on the right.

- *Stochastic Tverberg theorems:* versions of Tverberg's theorem where not just one- but most partitions of a large data set have that the convex hulls of each subset intersect. See Figure 1.5 for an example of a randomly partitioned data set.

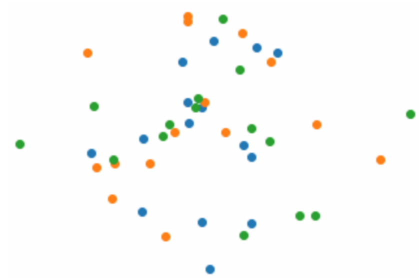


FIGURE 1.5. A three-color random set of 60 points from the Gaussian distribution.

- *Tverberg's theorem over discrete sets:* Versions of Tverberg's theorem where all points lie within a subset $S \subset \mathbb{R}^d$ and the intersection of convex hulls is required to have a non-empty intersection with S. See Figure 1.6 for an example of such a Tverberg partition in the case that $S = \mathbb{Z}^2$.

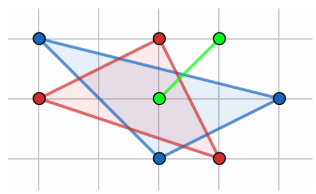


FIGURE 1.6. A partition of red, green, and blue lattice points such that the convex hulls of the three colors intersect at a lattice point

The first, natural as it may seem, was only introduced by De Loera et. al. [**DLHOY18**] in 2018. The story behind the second actually begins before Tverberg's theorem, when T.M. Cover (in 1964) [**Cov65**] and others started studying combinatorial properties of random bi-partitions of data. Finally, the third is somewhat classical, with its roots in J.P. Doignon's thesis [**Doi75**] from 1975.

We will also present applications of these results to optimization, statistics, and computing.

Before stating our contributions carefully, we introduce some preliminaries.

1.1. Convex geometry preliminaries

We begin with a short introduction to the topic of combinatorial convex geometry (more comprehensive surveys can be found in [**Mat02**] and [**Bal97**])- the study of combinatorial properties of convex geometric objects. Before stating the foundational combinatorial convex geometry results, we fix notation and terminology to be used throughout. An *affine subspace* of $\mathbb{R}^d$ has the form

$\boldsymbol{x} + L$, where $\boldsymbol{x} \in \mathbb{R}^d$ is some vector and L is a linear subspace of $\mathbb{R}^d$. An *affine combination* of points $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\} \subset \mathbb{R}^d$ is an expression of the form

$$a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + \cdots + a_n\boldsymbol{x}_n,$$

where $a_1, a_2, \ldots a_n \in \mathbb{R}$ and $a_1 + a_2 + \cdots + a_n = 1$. The *affine hull* of a set $X \subset \mathbb{R}^d$ is the intersection of all affine subspaces containing X, or equivalently the set of all affine combinations of X. Having defined "affine" subsaces, the other "affine" notions are constructed by imitating the "linear" notions. For example a set of points is *affinely independent* if no point in the set is an affine combination of the other points.

Given n points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in \mathbb{R}^d$, and $\boldsymbol{y}$ in $\mathbb{R}^d$, we say that $\boldsymbol{y}$ is a *convex combination* of the $\boldsymbol{x}_i$ if $\boldsymbol{y}$ can be written as a linear combination of the $\boldsymbol{x}_i$ using non-negative coefficients that sum to one. A set S is *convex* if every convex combination of elements in S is contained in S. Given a set $A \subset \mathbb{R}^d$, we define the *convex hull* of A, denoted by $\mathrm{conv}(A)$, as the set of all convex combinations of sets of points of A. A set of points $A \in \mathbb{R}^d$ is said to be in *convex position* if $\boldsymbol{x} \cap \mathrm{conv}(A \setminus \boldsymbol{x}) = \emptyset$ for any $\boldsymbol{x} \in A$.

A *hyperplane* is the set $\{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^T\boldsymbol{x} = b\}$, where $\boldsymbol{a} \in \mathbb{R}^d \setminus 0$ and $b \in \mathbb{R}$. A (closed) *half-space* is the set $\{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^T\boldsymbol{x} \geq b\}$, where $\boldsymbol{a} \in \mathbb{R}^d \setminus 0$ and $b \in \mathbb{R}$; the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^T\boldsymbol{x} = b\}$ is its boundary. The notion of *general position* will be important throughout this thesis. Intuitively general position means that no "unlikely coincidences" happen in the considered configuration. The precise definition is not fully standard, but here we mean that a set S of points in $\mathbb{R}^d$ is in general position as long as every subset $R \subset S$ of $d+1$ or fewer points is affinely independent.

A basic but important result about convex sets is the separability of disjoint convex sets by a hyperplane.

THEOREM 3 (separation theorem). *Let $C, D \subset \mathbb{R}^d$ be convex sets with $C \cap D = \emptyset$. Then there exists a hyperplane h such that C lies in one of the closed half-spaces determined by h, and D lies in the opposite closed half-space. In other words, there exist a unit vector $\boldsymbol{a} \in \mathbb{R}^d$ and a number $b \in \mathbb{R}$ such that for all $\boldsymbol{x} \in C$ we have $\boldsymbol{a}^T\boldsymbol{x} \geq b$ and for all $\boldsymbol{x} \in D$ we have $\boldsymbol{a}^T\boldsymbol{x} \leq b$. If C and D are closed and at least one of them is bounded, they can be separated strictly; in such a way that $C \cap h = D \cap h = \emptyset$.*

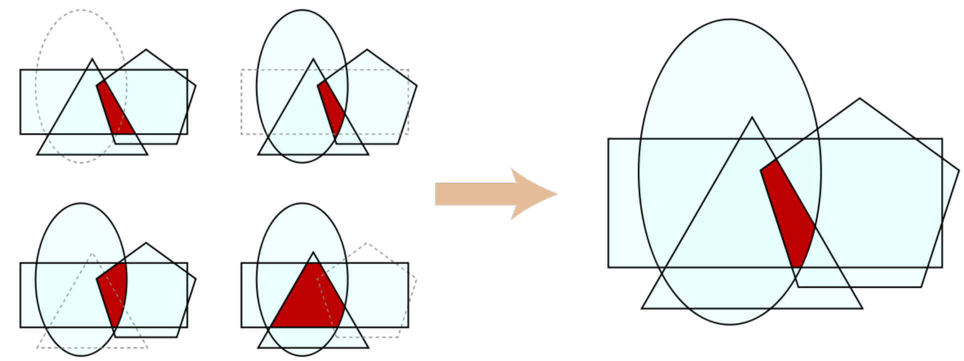


Figure 1.7. An example of Helly's theorem in two dimensions. Image credit: Wikipedia

Now we introduce some foundational theorems about convex sets that accompany Radon's lemma and Tverberg's theorem.

1.1.1. Helly's Theorem. Helly's theorem and its extensions are extremely important to computational and discrete geometry. See [**BO16**, **ADLS17**] for many interesting generalizations and applications.

Theorem 4 (E. Helly 1913 [**Hel23**]). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If $\bigcap \mathcal{K} \neq \emptyset$ for all $\mathcal{K} \subset \mathcal{F}$ of cardinality at most $d+1$, then $\bigcap \mathcal{F} \neq \emptyset$.*

For example, in two dimensions Helly's theorem implies that if every three convex sets in a larger collection intersect, then all the convex sets intersect. This scenario is depicted in Figure 1.7. In fact, Helly's theorem is sharp for all d. For example, consider the collection $\mathcal{G}$ of all of the facets of a non-degenerate d-simplex $S_d = \mathrm{conv}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1})$. Denote the $d+1$ members of $\mathcal{G}$ as $G_1, \ldots G_{d+1}$ where $G_i = \mathrm{conv}(\boldsymbol{x}_1, \ldots \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1} \ldots \boldsymbol{x}_{d+1})$. Any d members of $\mathcal{G}$ share a common vertex, but the $(d+1)$ members have empty intersection since no point in S_d is contained in every facet.

1.1.2. Carathéodory's Theorem. Carathéodory's theorem, proved by C. Carathéodory in 1911, is a classic result in combinatorial convex geometry. In essence, this useful theorem says that convex hull membership is a very finite property. For example given a set of blue points in the plane, and red point inside the convex hull of the blue points, Carathéodory's theorem says that there is always a set of at most three blue points whose convex hull contains the red point. This scenario is depicted in Figure 1.8. Here is the formal statement.

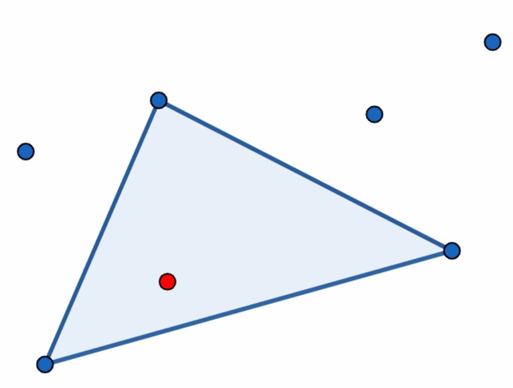


FIGURE 1.8. An example of Carathéodory's theorem in two dimensions.

THEOREM 5 (C. Carathéodory 1911 [**Car07**]). *Let S be any subset of $\mathbb{R}^d$. Then each point in the convex hull of S is a convex combination of at most $d+1$ elements of S.*

1.1.3. Centerpoint Theorem. The centerpoint theorem, originally due to R. Rado, has proved to be an indispensable consequence of Helly's theorem, and is crucial in many applications - as it guarantees that every data set has a higher dimensional analog of a median. See Chapter 2 for further discussion.

THEOREM 6 (R. Rado 1947 [**Rad47**]). *For every set S of n points in $\mathbb{R}^d$, there exists a point $\boldsymbol{p} \in \mathbb{R}^d$ such that every closed half space containing $\boldsymbol{p}$ contains at least $\frac{n}{d+1}$ points of S.*

With these preliminaries and related results in mind, we proceed to the main theoretical results of this dissertation.

1.2. Unavoidable patterns in big data classification

Tverberg's theorem says that sufficiently many points can always be partitioned into overlapping classes. Here "sufficiently many" means that there is a constant depending on the number of classes and the dimension of the points. *But what if we want a different intersection pattern?* For example, what if we want to partition a data set in such a way that there is one large data class whose convex hull intersects with all the other classes, but we want the convex hulls of the other classes to be pairwise disjoint. Does there exist a similar constant?

From this viewpoint, the intersection pattern (nerve) of the convex hulls in Tverberg's theorem is very specific, a simplex; In Chapter 3 we will investigate other possible intersection patterns.

Informally, the main results of Chapter 3 demonstrate that, given sufficiently many points, many other kinds of nerves can always be induced by a suitable partition of the point set. In particular, we show that any tree or cycle - special one-dimensional simplicial complexes - can be induced as the nerve.

Our geometric results are naturally motivated from Ramsey theory (see [**GRS90**]) where one studies how every sufficiently large system must contains a large well-organized subsystem. Here "sufficiently large" is governed by geometric Ramsey numbers.

Perhaps the most well known example of Ramsey theory is the following consequence: Among any group of six people, there is either a subgroup of three people who have never met each other, or there is a subgroup of three people who have all met each other.

This is a consequence of F.P. Ramsey's classic theorem:

Theorem 7 (F.P. Ramsey 1929 [**Ram29**]). *Given any integer c and integers $n_1, \ldots, n_c$, there is a number $R(n_1, \ldots, n_c)$, such that if the edges of a complete graph of order $R(n_1, \ldots, n_c)$ are colored with c different colors, then for some i between 1 and c, it must contain a subgraph of order n_i whose edges are all color i.*

The example above is a consequence of the case $c = 2$ and $n_1 = n_2 = 3$, since it is known that $R(3, 3) = 6$.

In geometry, another classical example of a Ramsey-type theorem is the famous Erdős-Szekeres theorem- another key result related to Tverberg's theorem. It was born out of the "happy ending problem" (so named by P. Erdős because it lead to the marriage of G. Szekeres and E. Klein) which states the following: Any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral. E. Klein's discovery of this fact was one of the original results leading to the development of Ramsey theory. P. Erdős and G. Szekeres generalized this result as follows.

Theorem 8 (P. Erdős, G. Szekeres 1935 [**ES35**]). *For any positive integer N, there exists a number $ES(N)$ such that any set of at least $ES(N)$ points in the plane contains a subset of N points that form the vertices of a convex N-gon.*

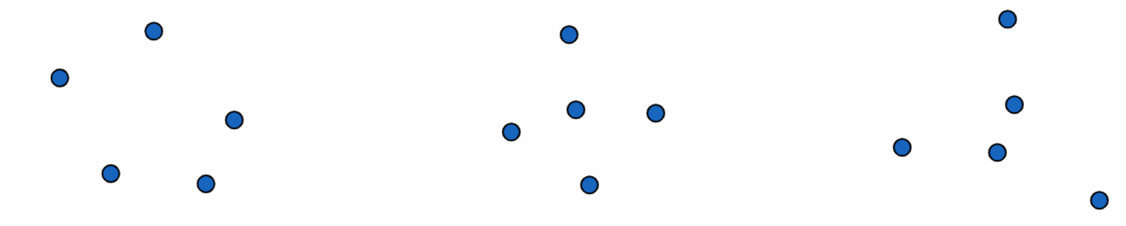


FIGURE 1.9. E. Klein showed that among any five points in general position there is a subset of four that forms a quadrilateral. The three cases (how many points are in the convex hull of the others) are shown here.

We stress that Tverberg's theorem is also of Ramsey-type, although in this case the constant is explicit. Our results proved in Chapter 3 are new Ramsey-Tverberg-type results, where nerve structures are shown to arise once we have sufficiently many points. Our results are also a kind of universality result, in the spirit of A. Por [**Por18**]. A. Por's result is very interesting as it explores how we can control the structure of Tverberg partitions in large data sets. It says that given a big enough data set, we can select a smaller subset whose Tverberg partitions are exactly of a certain type - so called *rainbow partitions*. This is an impressive result because the Tverberg partitions of a data set are not uniquely determined by the order type of the data set, see Figure 1.15. In fact, there is a more sophisticated semi-algebraic predicate that determines whether an m-partition of $(m-1)(d+1)+1$ points is a Tverberg partition. Por's result was conjectured in a related paper about understanding the structure of Tverberg partitions due to B. Bukh, P.-S. Loh, and G. Nivasch [**BLN17**]. They exhibited (using the semi-algebraic predicate machinery mentioned above) an infinite family of configurations of $(m-1)(d+1)+1$ points whose Tverberg partitions are exactly the rainbow partitions, as well as a family of interesting related Ramsey-type results on the structure of Tverberg partitions.

While similar, our results are not constraints on the partition itself, but instead constraints on the intersection pattern of the convex hulls of the subsets in the partition. We will see our results depend on some universal Ramsey-like constants too and we use Ramsey numbers of hypergraphs for our geometric estimates.

To state our results precisely we begin with some terminology and notation typical of geometric topological combinatorics (see [**Mat02**, **Tan13**] for details, especially on simplicial complexes discussed here). Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of convex sets in $\mathbb{R}^d$. The *nerve* $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$

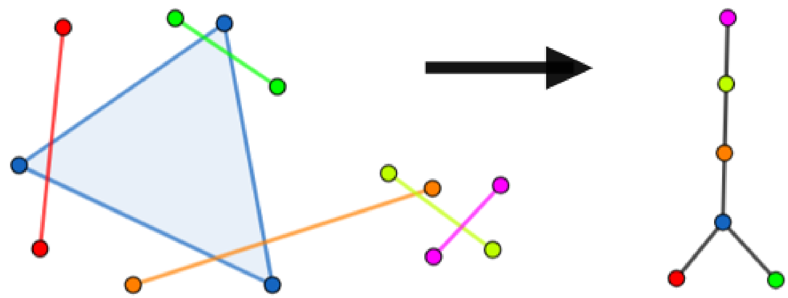


FIGURE 1.10. A partitioned point set on the left and its induced nerve on the right.

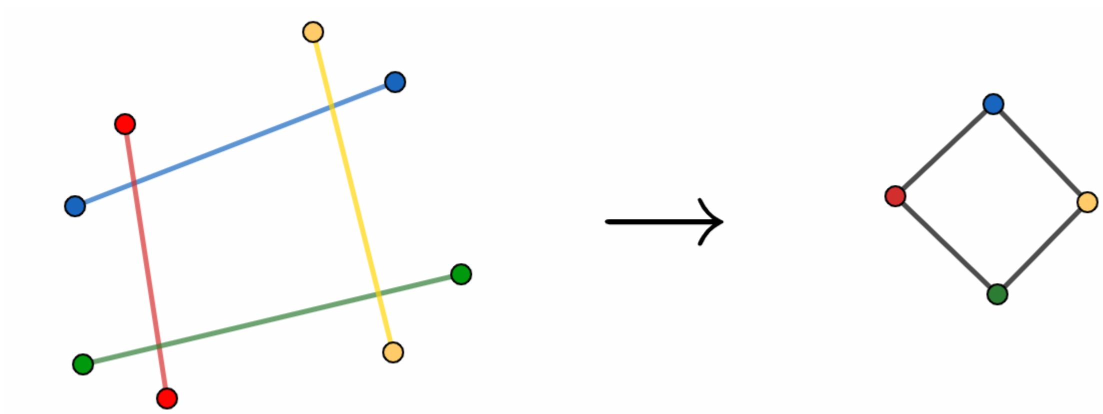


FIGURE 1.11. A partitioned set of eight points in the plane with the four-cycle as its induced nerve.

is the simplicial complex with vertex set $[m] := \{1, 2 \ldots, m\}$ whose faces are $I \subset [m]$ such that $\cap_{i \in I} F_i \neq \emptyset$.

Given a collection of points $S \subset \mathbb{R}^d$ and an n-partition into n color classes $\mathcal{P} = S_1, \ldots, S_n$ of S, we define *the nerve of the partition*, $\mathcal{N}(\mathcal{P})$ to be the nerve complex $\mathcal{N}(\{\text{conv}(S_1), \ldots, \text{conv}(S_n)\})$, where $\text{conv}(S_i)$ is the convex hull of the elements in the color class i. For an example, see Figure 1.10. Similarly, given a partition $\mathcal{P}$, we define the *intersection graph of the partition*, denoted $\mathcal{N}^1(\mathcal{P})$, as the 1-skeleton of the nerve of $\mathcal{P}$.

Given a simplicial complex K, and a finite set of points S in $\mathbb{R}^d$, we say that K is *partition induced on S* if there exists a partition $\mathcal{P}$ of S such that the nerve of the partition is isomorphic to K. We say that K is *d-partition induced* if there exists at least one set of points $S \subset \mathbb{R}^d$ such that K is partition induced on S. For example, as depicted in Figure 1.11, the four-cycle is 2-partition induced. However, it is not 1-partition induced.

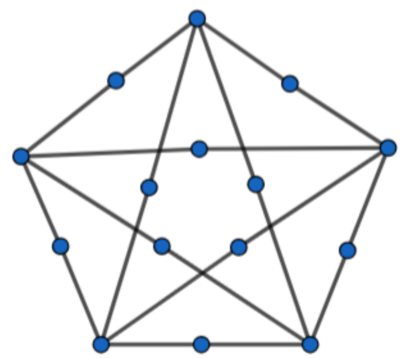


FIGURE 1.12. The barycentric subdivision of the complete graph on five vertices is a 1-dimensional simplicial complex which is not 2-partition induced.

It was shown by G. Y. Perelman [**Per85**] that every d-dimensional simplicial complex is $(2d+1)$-partition induced on some point set. This result is in fact optimal, because the barycentric subdivision of the d-skeleton of a $(2d+2)$-dimensional simplex is not $2d$-partition induced, see [**Weg67**] and [**Tan11**] for details. See Figure 1.12 for an example of a graph which is not 2-partition induced.

Motivated by Tverberg's theorem, we introduce another property of simplicial complexes that is much stronger than being d-partition induced because it has to hold in all point sets once they have sufficiently many points.

DEFINITION 9. *A simplicial complex K is d-*Tverberg *if there exists a constant* $\mathrm{Tv}(K,d)$ *such that K is partition induced on all point sets $S \subset \mathbb{R}^d$ in general position with $|S| > \mathrm{Tv}(K,d)$. The minimal such constant* $\mathrm{Tv}(K,d)$ *is called the* Tverberg number for K in dimension d.

Let us briefly examine the definition of d-Tverberg complexes. First of all, note one can re-state the classical Tverberg's theorem as follows:

THEOREM 10 (Tverberg's theorem rephrased). *The $(m-1)$-simplex is a d-Tverberg complex for all $d \geq 1$, with Tverberg number $(d+1)(m-1)+1$.*

Definition 9 can be compared with earlier work by J.R. Reay and others [**Rea79**], [**Rou09**], [**PS16**], who asked what happens when we only demand that each k of the convex hulls intersect. They looked for the smallest number n of points sufficient so that some partition induces a nerve which contains the $(k-1)$-skeleton of a simplex. In fact, Reay's conjecture says for every $n \leq (d+1)(m-1)$

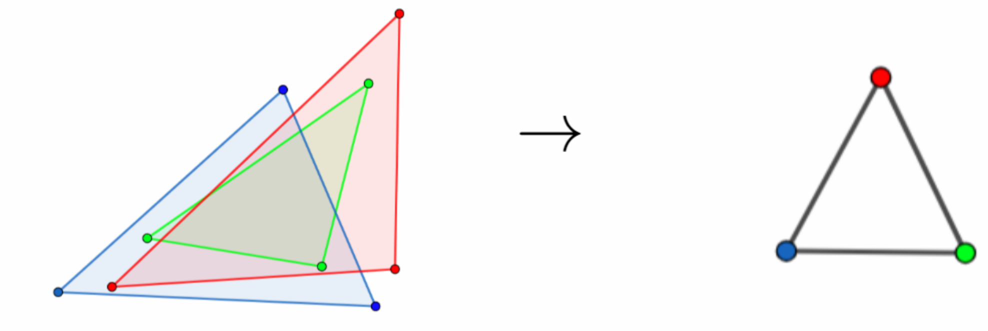


FIGURE 1.13. Tverberg's theorem says that there is a Tverberg numbers $\mathrm{Tv}(K_m, d)$ so that the $m-1$ simplex on m vertices is partition realizable from *any* configuration of at least $\mathrm{Tv}(K_m, d) := (m-1)(d+1)+1$ points in $\mathbb{R}^d$.

there exists an n point set $X \subset \mathbb{R}^d$ such that no partition of X induces the complete graph K_m as its intersection graph. M. A. Perles and M. Sigron have results in this direction [**PS16**], but mention the following special case as evidence that the conjecture is false: Given $1,000,000$ points in $\mathbb{R}^{1000}$, Tverberg's theorem shows you can partition them into $1,000$ parts whose convex hulls all have a point in common. Is there a set of $999,999$ points in $\mathbb{R}^{1000}$ that cannot be partitioned into 1000 parts whose convex hulls intersect pairwise? This seems implausible.

In contrast to Reay's conjecture, we are not studying a relaxed intersection pattern, but instead we are studying sufficient conditions to induce an exact nerve of general kind.

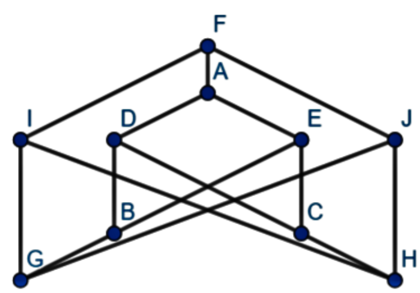


FIGURE 1.14. A 2-partition induced simplicial complex that is not 2-Tverberg.

Definition 9 is most interesting for sets $S \subset \mathbb{R}^d$ in general position. The reason is that for collinear points the only type of nerve complexes possible are those whose graphs are *interval graphs*. Interval graphs have been classified [**LB62**] and in particular are *chordal*. With Definition 9 the 4-cycle graph is not 1-Tverberg, because it is not chordal, but we will show later that it is d-Tverberg for all $d \geq 2$. Similarly, while every d-Tverberg complex K is clearly d-partition induced, the converse is not true. The complex in Figure 1.14 is a graph that is partition induced on some planar point sets, but not for points in convex position, regardless of how many points we use. Thus it is not a 2-Tverberg complex. Details are presented in Section 3.3.

The key contribution of Chapter 3 is to generalize the classical Tverberg's theorem by showing that similar theorems exist where other simplicial complexes -not just simplices- are d-Tverberg complexes too. Before stating our first result, recall that the *k-hypergraph Ramsey number $R_k(m)$* is the least integer N such that every red-blue 2-coloring of all k-subsets of an N-element set contains either a red set of size m or a blue set of size m, where a set is called red (blue) if all k-subsets from this set are red (or respectively blue). See [**CFS10**] and references therein.

THEOREM 11. *All trees and cycles are d-Tverberg complexes for all $d \geq 2$.*

(A) *Every tree T_n on n nodes, is a d-Tverberg complex for $d \geq 2$. The Tverberg number* $\mathrm{Tv}(T_n, d)$ *exists and it is at most $R_{d+1}((d+1)(n-1)+1)$. More strongly,* $\mathrm{Tv}(T_n, 2)$ *is at most* $\binom{4n-4}{2n-2} + 1$.

(B) *Every n-cycle C_n with $n \geq 4$ is a d-Tverberg complex for $d \geq 2$. The Tverberg number exists and* $\mathrm{Tv}(C_n, d)$ *is at most $nd + n + 4d$.*

The proof of Theorem 11 relies on several powerful non-constructive tools such as the Ham-Sandwich theorem (see Section 1.3 [**Mat02**]), a characterization of oriented matroids of cyclic polytopes [**CD00**], and the multi-dimensional version of Erdős-Szekeres theorem (this is due to B. Grünbaum [**Grü67**] and R. Cordovil and P. Duchet [**CD00**], see also Chapter 9 of [**BLVS$^+$93**], and the survey [**MS16**]). These tools are enough to show the existence of a Tverberg number $\mathrm{Tv}(T_n, d)$, but the bounds are far from tight. Details are presented in Section 3.1.

We can also prove the following general lower bound for the Tverberg numbers (see Section 3.3 for the argument).

LEMMA 1. *For any connected simplicial complex K with $n \geq 2$ vertices, if it exists, then* $\mathrm{Tv}(K, d) \geq 2n$.

In addition to this general lower bound, we show that the upper bounds of Theorem 11 can indeed be improved by giving better bounds on the Tverberg numbers of *caterpillar trees*. Caterpillar trees are those in which all the vertices are within distance one of a central path; these include paths and stars. See Section 3.2.

THEOREM 12. *If a tree T_n is a caterpillar tree with n nodes, then T_n is d-Tverberg complex for all d, and its d-Tverberg number* $\mathrm{Tv}(T_n, d)$ *is no more than* $(d+1)(n-1)+1$.

In terms of intersection properties caterpillar graphs have been shown to be precisely the trees that are also interval graphs by J. Eckhoff [**Eck93**]. In other words, the previous theorem implies that a tree T_n is also 1-Tverberg if and only if T_n is a caterpillar tree.

Furthermore, in dimension two we can give some exact Tverberg numbers for trees:

THEOREM 13.

(A) *The 2-Tverberg numbers* $\mathrm{Tv}(S_n, 2)$ *for a star tree with n nodes equals $2n$.*

(B) *The 2-Tverberg numbers of the path and cycle with four nodes are* $\mathrm{Tv}(P_4, 2) = 9$ *and* $11 \leq \mathrm{Tv}(C_4, 2) \leq 13$.

The proof of Theorem 13 (B) requires exhaustive computer enumeration of all possible partitions, over all possible order types of point sets with fewer than ten points. Luckily, these order types were classified in [**AAK02**]. For more details see Appendix A.

Recall that for an ordered set of points $S = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n)$ the *order type* (see 9.3 [**Mat02**]) of S is defined as the mapping assigning to each $(d+1)$-tuple $(i_1, i_2, \ldots, i_{d+1})$ of indices, $1 < i_1 < i_2 < \cdots < i_{d+1} \leq n$, the orientation of the $(d+1)$-tuple $(\boldsymbol{p}_{i_1}, \boldsymbol{p}_{i_2}, \ldots, \boldsymbol{p}_{i_{d+1}})$ (i.e., the sign of the determinant of the corresponding matrix). The order type of S is encoded by the *chirotope* of S which is the sequence of resulting $\binom{n}{d+1}$ signs of possible determinants. This is a vector of $+1$'s and -1's, with $\binom{n}{d+1}$ entries.

The proof of Theorem 13 (B) also uses the following lemma to ensure that it suffices to check one representative configuration of points from each order type, reducing calculations to finitely many cases. See details in the Appendix.

LEMMA 2. *Suppose S_1 and S_2 are two point sets in $\mathbb{R}^d$ with the same order type, and let σ be a bijection from S_1 to S_2 that preserves the orientation of any $(d+1)$-tuple in S_1. Then any partition $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ of S_1 and the corresponding partition of S_2 via σ, denoted $\sigma\mathcal{P} = \{\sigma(P_1), \sigma(P_2), \ldots, \sigma(P_n)\}$, have the same intersection graph $\mathcal{N}^1(\mathcal{P})$.*

Lemma 2 cannot be extended to arbitrary nerve complexes as we see in the example of Figure 1.15. Despite the fact that the chirotope-preserving bijections do not preserve the higher-dimensional skeleton of the nerve of a partition we can still make use of Lemma 2 throughout Chapter 3 because our results are only about *triangle-free* simplicial complexes, thus their nerve complexes equal their 1-skeleton.

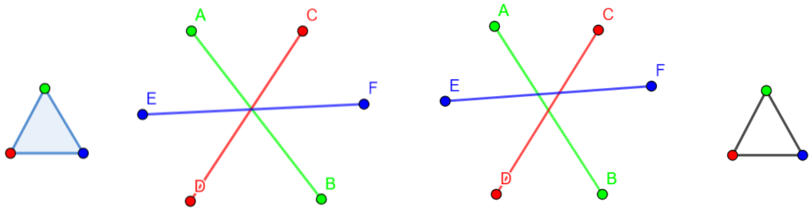


FIGURE 1.15. Only the 1-skeleton of the nerve is preserved by order-preserving bijection.

1.3. Patterns in the classification of random data sets

The focus of Chapter 4 is to develop probabilistic theorems inspired by Tverberg's theorem for use in the foundations of data science. In particular, we give bounds on the probability that m random data classes all contain a point in common in their convex hulls. See Chapter 2 for a discussion of the many applications of these theorems.

We also study probabilistic aspects of another parameter in Tverberg partitions: *Tolerance* is a notion of "robust" intersection in the sense that the convex hulls of the various data classes intersect in such a way that points from any class can be removed without destroying their intersection. See Figure 1.16 for an example of such a partition. Here is the formal definition of the central geometric object we study:

DEFINITION 14. *Given a set $S \subset \mathbb{R}^d$, a* Tverberg m-partition of S with tolerance t *is a partition of S into m subsets $S_1, \ldots S_m$ with the property that all m convex hulls of the S_i intersect after any t-points are removed. In other words, for all $\{x_1, \ldots, x_t\} \in S$, we have*

$$\bigcap_{i \in [m]} \operatorname{conv}(S_i \setminus \{x_1, \ldots, x_t\}) \neq \emptyset.$$

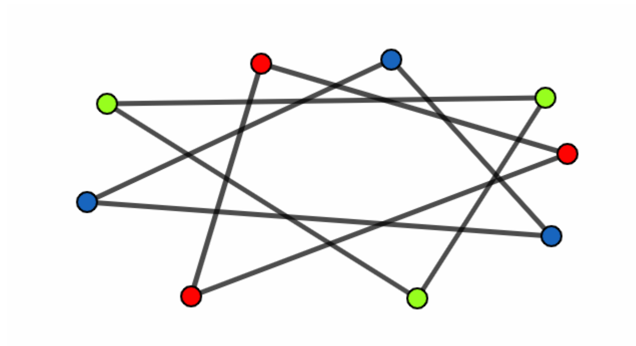


FIGURE 1.16. A Tverberg three-partition with tolerance one. All three convex hulls intersect even if any one point is removed.

The notion of "tolerant Tverberg theorems" was pioneered by D. Larman [**Lar72**] and refined over the years, such as in the following result due to P. Soberón and R. Strausz [**SS12**].

THEOREM 15 (P. Soberón, R. Strausz [**SS12**]). *Every set S with at least $(t+1)(m-1)(d+1)+1$ points in $\mathbb{R}^d$ has at least one Tverberg m-partition with tolerance t.*

More recently, P. Soberón proved the following bound [**Sob18**]. Let N denote the smallest positive integer such that a Tverberg m-partition with tolerance t exists among any N points in dimension d. Then $N = mt + O(\sqrt{t})$ for fixed m and d. The proof of this result relies on the probabilistic method and, as Soberón remarked, can in fact be used to prove a Stochastic Tverberg-type result, which we will revisit later.

Motivating our geometric probability results, we have Cover's theorem, which is a probabilistic version of Radon's lemma.

THEOREM 16 (T.M. Cover 1964 [**Cov65**]). *Consider $n \geq (d+1)$ points in general position in $\mathbb{R}^d$. Suppose the each of the n points are colored red or blue according to independent fair coin flips. Then the probability that the convex hull of the red points intersects the convex hull of the blue points is*

$$1 - 2^{-n+1} \sum_{i=0}^{d} \binom{n-1}{i}.$$

For example, consider a set of six points in the plane in general position. By applying Cover's theorem above, we see that if we color each point red or blue independently with uniform probability, then the convex hulls of the resulting (possibly empty) sets of red and blue points intersect with probability 1/2.

This is actually a generalization of Radon's lemma when the points are in general position. It implies that for $(d+2)$ points in $\mathbb{R}^d$, the probability that a random bi-partition of the points is a Radon partition is strictly positive. Thus there exists at least one such partition! But with sufficiently points, Cover's result shows that in fact "most" bi-partitions of the data set are Radon partitions.

Geometric Probability: Stochastic Tverberg theorems. Before stating our main results, we introduce two models for random partitioned data point sets. In both models will use the term colors instead of subsets, for ease of presentation. Hereafter, when we refer to a continuous distribution on $\mathbb{R}^d$, we mean continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Proofs of the new results stated are in Chapter 4.

Our first model is a so-called *random equi-partition model* i.e., we ensure that every color has the same number of points. More specifically, given integers m and n and a continuous probability distribution D on $\mathbb{R}^d$, we let $\mathcal{E}_{m,n,D}$ denote a random equi-partitioned point set with mn points, consisting of m colors, and n points of each color, distributed independently according to D.

Our second model is a *random allocation model*: Given integers k and m and a continuous probability distribution D on $\mathbb{R}^d$, we let $\mathcal{R}_{m,k,D}$ denote a random point set with k points i.i.d. according to D, which are randomly colored one of m colors with uniform probability ($1/m$ for each color).

For example, using these models we can state T.M. Cover's result as follows:

Theorem 1 (T.M. Cover 1965). *If D is a continuous probability distribution on $\mathbb{R}^d$, then*

$$\mathbb{P}(\mathcal{R}_{2,k,D} \textit{ is Radon}) = 1 - 2^{-m+1} \sum_{k=0}^{d} \binom{m-1}{k}.$$

In particular, we have

$$\mathbb{P}(\mathcal{R}_{2,2(d+1),D} \textit{ is Radon}) = 1/2.$$

Furthermore, for any $\epsilon > 0$ and any sequence of continuous probability distributions $\{D_i\}, i \in \mathbb{Z}_+$ where each D_d is a distribution on $\mathbb{R}^d$, we have

$$\lim_{i\to\infty} \mathbb{P}(\mathcal{R}_{2,(1+\epsilon)2i,D_i} \textit{ is Radon}) = 1$$

and

$$\lim_{i\to\infty} \mathbb{P}(\mathcal{R}_{2,(1-\epsilon)2i,D_i} \text{ is Radon}) = 0.$$

To the best of our knowledge, the first generalization of Cover's 1964 result to more than two colors appeared only recently:

THEOREM 17 (Soberón [**Sob18**]). *Let N, t, d, m be positive integers and let $\epsilon > 0$ be a real number. Given N points in $\mathbb{R}^d$, a random allocation of them into m parts is a Tverberg partition with tolerance t with probablility at least $1-\epsilon$, as long as*

$$t + 1 \le N/m - \sqrt{\frac{1}{2}\left[(d+1)(m-1)N\ln(Nm) + N\ln\left(\frac{1}{\epsilon}\right)\right]}.$$

This result is quite remarkable. For any fixed m, d, and δ, it shows that the probability of a random allocation of of N points in $\mathbb{R}^d$ in m colors having tolerance at least $(1-\delta)N/m$ approaches one as N goes to infinity. On the other hand, by the pigeonhole principle, any allocation of N points into m colors must have one color with at most N/m points. Thus, for a fixed number of colors m, the tolerance of a random partition is asymptotically as high as it could possibly be!

Our new stochastic geometric theorems. Our first result is a geometric probability result similar to P. Soberón's and T.M. Cover's. It yields a stochastic Tverberg theorem for equi-partitions (without tolerance).

THEOREM 18 (stochastic Tverberg theorem for equi-partitions). *Suppose D is a probability distribution on $\mathbb{R}^d$ that is balanced about some point $\boldsymbol{p} \in \mathbb{R}^d$, in the sense that every hyperplane through $\boldsymbol{p}$ partitions D into two sets of equal measure. Then*

$$\left(1 - \left(\frac{1}{2^{n-1}}\sum_{k=0}^{d-1}\binom{n-1}{k}\right)\right)^m \le \mathbb{P}(\mathcal{E}_{m,n,D} \text{ is Tverberg }) \le \left(2(1-2^{-n})^m - (1-2^{-n+1})^m\right)^d.$$

In fact, the previous theorem is asymptotically tight in the number of colors m. This is shown by our next theorem, which establishes an interesting threshold phenomenon for Tverberg partitions. As it is standard in the literature, we say that a sequence of events $X_n, n \ge 1$, occurs with *with high probability* if $\lim_{n\to\infty} P(X_n) = 1$.

THEOREM 19 (Tverberg threshold phenomena for equi-partitions). *Let D be a continous probability distribution in $\mathbb{R}^d$ balanced about some point $\boldsymbol{p} \in \mathbb{R}^d$. Consider the sequence of random equi-partitioned point sets $\mathcal{E}_{m,f(m),D}$, where $m \in \mathbb{N}$, and $n = f(m)$ depends on m. Then $\mathcal{E}_{m,f(m),D}$ is Tverberg with high probability if $f(m) > \log_2(m)$, and $\mathcal{E}_{m,f(m),D}$ is not Tverberg with high probability if $f(m) < \log_2(m)$.*

Remark: It is also interesting to consider the same problem from the "*box convexity*" setting where the convex hull of a set of points is defined to be the smallest box (with sides parallel to the coordinate axes) enclosing those points. Since checking convex hull membership is easier in the box convexity setting, this set up may be more relevant in certain applications. Our method of proof of Theorem 18 also works in box convexity setting, and we obtain the same bounds.

We note that the number of points needed to reach the conclusion in Theorem 19 is independent of the dimension, as in the aforementioned result of P. Soberón [**Sob18**].

The next two theorems adapt both T.M. Cover's result and Theorem 18 to the setting of tolerance.

THEOREM 20 (stochastic Tverberg with tolerance for equi-partition). *Suppose D is a probability distribution on $\mathbb{R}^d$ that is balanced about some point $\boldsymbol{p} \in \mathbb{R}^d$.*

$$\mathbb{P}(\mathcal{E}_{m,n,D} \textit{ is Tverberg with tolerance } t) \geq \left(1 - 2^{-\lfloor n/2d \rfloor} \sum_{i=1}^{t} \binom{\lfloor n/2d \rfloor}{i}\right)^m .$$

For the case of random bi-partitions, we can adapt Cover's result to obtain a stochastic Radon theorem with Tolerance.

THEOREM 21 (stochastic Radon with tolerance for random allocation). *If D is a continuous probability distribution on $\mathbb{R}^d$, then*

$$\mathbb{P}(\mathcal{R}_{2,k,D} \textit{ is Radon with tolerance } t) \geq 1 - \left(2^{-\lfloor k/(2d+2) \rfloor} \sum_{i=0}^{t} \binom{\lfloor k/(2d+2) \rfloor}{i}\right).$$

In particular, we have

$$\mathbb{P}(\mathcal{R}_{2,k,D} \textit{ is Radon with tolerance } \lfloor k/(4d+4) \rfloor) \geq 1/2.$$

Remark: The preceding result yields a weaker expected tolerance than Soberón's result, but the proof is shorter and more elementary.

For random allocations with more than two colors, we will use some developments on random allocation problems, including the following notation. Let $\mathbb{P}(N_n(m)) \leq k)$ denote the probability that, after throwing k balls into m urns uniformly and independently, there are n balls in every urn.

Corollary 1 (stochastic Tverberg for random allocation). *Suppose D is a probability distribution on $\mathbb{R}^d$ that is balanced about some point $\boldsymbol{p} \in \mathbb{R}^d$. Then*

(1)

$$\mathbb{P}(\mathcal{R}_{m,k,D} \textit{ is Tverberg with tolerance } t) \geq \mathbb{P}(N_n(m) \leq k) \left(1 - 2^{-\lfloor n/2d \rfloor} \sum_{i=1}^{t} \binom{\lfloor n/2d \rfloor}{i}\right)^m .$$

(2) *For the case of Tverberg without tolerance, we also have*

$$\mathbb{P}(\mathcal{R}_{m,k,D} \textit{ is Tverberg}) \geq \mathbb{P}(N_n(m) \leq k) \left(1 - \left(2^{-n+1} \sum_{k=0}^{d-1} \binom{n-1}{k}\right)\right)^m .$$

(3) *Lastly, we also have the following asymptotic result:*

Consider the sequence of random partitioned point sets $\mathcal{R}_{m,f(m),D}$, $m \in \mathbb{N}$, where $n = f(m)$ depends on m. Then $\mathcal{R}_{m,f(m),D}$ is Tverberg with high probability if $f(m) > m \log_2(m) \ln(\ln(m))$.

These results are improvements on Soberón's bound when the number of colors is large relative to the desired tolerance. In particular, given a suitably distributed data set of n points, this result shows that a random allocation of the data into less than $\frac{n}{\log_2(n)\ln(\ln(n))}$ subsets is likely to be a Tverberg partition.

1.4. Patterns in the classification of discrete and categorical data

In Chapter 5 we focus on new versions of Tverberg-type theorems where some of the coordinates of the points are restricted to discrete subsets of a Euclidean space. The associated discrete Tverberg numbers are much harder to compute than their classical real-version counterparts (see for instance the complexity discussion of [**Onn91**]). These results are important for work in statistics and optimization, where researchers wish to compute centerpoints. We will see this in detail in Chapter 2. For now we introduce the geometric tools needed in those applications.

We begin our work remembering the following unpublished Tverberg-type result of J.P. Doignon. Consider n points with coordinates in $\mathbb{Z}^2$ and a positive integer $m \geq 3$. If $n \geq 4m-3$, then the points can be partitioned into m subsets whose convex hulls contain a common point in $\mathbb{Z}^2$. According to J. Eckhoff [**Eck00**] this result was stated by J.P. Doignon in a conference.

A partition of points where the intersection of the convex hulls contains at least one lattice point is called an *integer m-Tverberg partition* and such a common point is an *integer Tverberg point* for that partition. Regarding the case $m = 2$, the integer 2-Tverberg partitions are called *integer Radon partitions*. Any configuration of at least six points in $\mathbb{Z}^2$ admits an integer Radon partition. This was proved by J.P. Doignon in his PhD thesis [**Doi75**] and later discovered independently by S. Onn [**Onn91**]. All these values for $\mathbb{Z}^2$ are optimal as shown by following examples. The 5-point configuration $\{(0,0),(0,1),(2,0)(1,2),(3,2)\}$, exhibited by Onn in the cited paper, has no Radon partition. See Figure 1.17. To address the optimality when $m \geq 3$, consider the set $\{(i,i),(i,-i+1)\colon i = -m+2,-m+3,\dots,m-2,m-1\}$. The case where $m = 4$ is depicted in Figure 1.18. (According to J. Eckhoff [**Eck00**], this set was proposed by J.P. Doignon during the aforementioned conference.) This set has $4m-4$ points but cannot have an integer m-partition, as any lattice point is contained in a half-space with less than m points from the set. So regardless of the partition, by the pigeonhole principle each point can always be separated by a line from at least one of the subsets.

More generally, one can define the *Tverberg number* $\mathrm{Tv}(S,m)$ for any subset S of $\mathbb{R}^d$ and an integer $m \geq 2$ as the smallest positive integer n with the following property: Any multiset of n points in S admits a partition into m subsets $A_1, A_2, \dots, A_m$ with

$$\left(\bigcap_{i=1}^{m} \mathrm{conv}(A_i)\right) \cap S \neq \emptyset.$$

(Here, by "partition of a multiset", we mean that each element of a multiset A is contained in a number of submultisets $A_1,\dots,A_m$ so that the sum of its multiplicities in the A_i is equal to its multiplicity in A.) If no such number exists, we say that $\mathrm{Tv}(S,m) = \infty$. Note that Doignon's

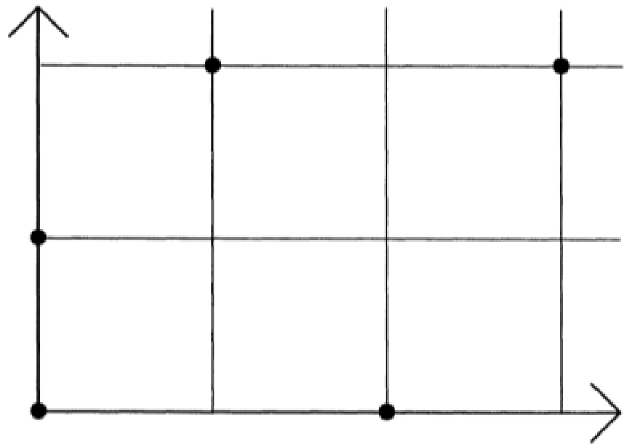


FIGURE 1.17. Five lattice points with no integer Radon partition.

theorem, together with the discussion that follows, allows us to say

$$\mathrm{Tv}(\mathbb{Z}^2, m) = \begin{cases} 6 & \text{if } m = 2, \\ 4m - 3 & \text{otherwise.} \end{cases}$$

The first theorem we prove in Chapter 5 generalizes Doignon's theorem. We determine the exact m-Tverberg number (when m is at least three) for any discrete subset S of $\mathbb{R}^2$, as considered in [**DLLHRS17**]. Before stating this result we recall the *Helly number* $\mathrm{H}(S)$ of a discrete subset S of $\mathbb{R}^d$ as the smallest positive integer with the following property: Suppose $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$, and that $\cap\mathcal{G}$ intersects S in at least one point for every subfamily $\mathcal{G}$ of $\mathcal{F}$ having at most $\mathrm{H}(S)$ members. Then $\cap\mathcal{F}$ intersects S in at least one point. If no such integer exists, we say that $\mathrm{H}(S) = \infty$. Then we have the following theorem. (The theorem is stated for S with finite Helly number, as any $S \subset \mathbb{R}^d$ with $\mathrm{H}(S) = \infty$ has $\mathrm{Tv}(S, m) = \infty$ for all $m \geq 2$ [**Lev51**].)

THEOREM 22. *Suppose S is a discrete subset of $\mathbb{R}^2$ with $\mathrm{H}(S) < \infty$. If $m \geq 3$, then*

$$\mathrm{Tv}(S, m) = \mathrm{H}(S)(m - 1) + 1.$$

Regarding the case $m = 2$, if $\mathrm{H}(S) \leq 3$, then

$$\mathrm{Tv}(S, 2) = \mathrm{H}(S) + 1,$$

and if $\mathrm{H}(S) \geq 4$, then

$$\mathrm{H}(S) + 1 \leq \mathrm{Tv}(S, 2) \leq \mathrm{H}(S) + 2,$$

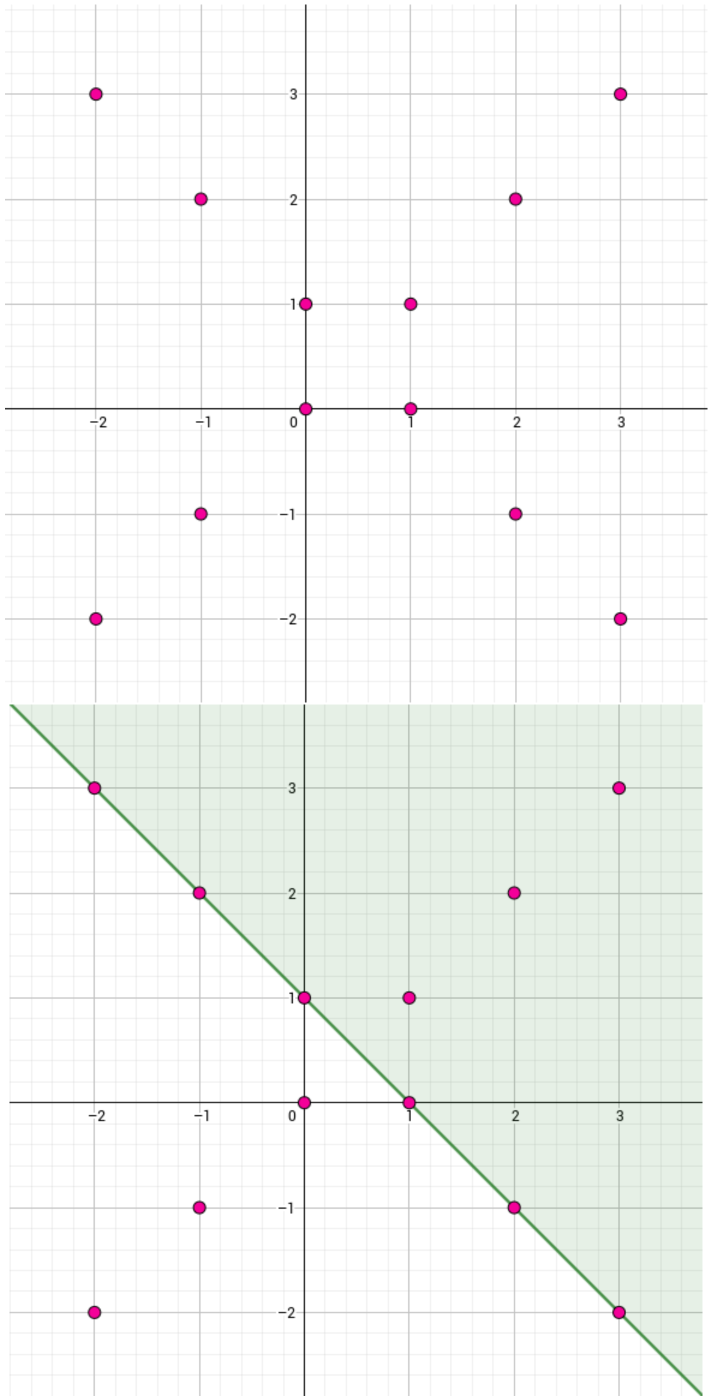


FIGURE 1.18. An example of twelve points without an integer Tverberg four-partition. The bottom picture shows that for any lattice point, we can find some half plane containing that point and only three points of the set.

and both values are possible.

In particular we present a proof of Doignon's theorem, the special case of Theorem 22 where $S = \mathbb{Z}^2$.

Remark: Theorem 22 shows that S-Tverberg numbers of planar sets are very closely related to S-Helly numbers (see [**Ave13**, **AGS+17**] and the references there). However, for the case $\mathrm{H}(S) = 4$, the bounds on $\mathrm{Tv}(S, 2)$ given above cannot be improved. For example, $S' = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\mathbb{Z}^2$ both have Helly number four, but $\mathrm{Tv}(\mathbb{Z}^2, 2) = 6$, while the pigeonhole principle implies that $\mathrm{Tv}(S', 2) = 5$. Our second main result in Chapter 5 improves the upper bound on the integer Tverberg numbers for the three-dimensional case $S = \mathbb{Z}^3$.

THEOREM 23. *The following inequality holds for all $m \geq 2$:*

$$\mathrm{Tv}(\mathbb{Z}^3, m) \leq 24m - 31.$$

Our third main result in Chapter 5 is an inequality that will be used to derive improved bounds on S-Tverberg numbers when S is a product of a Euclidean space with some subset S' of a Euclidean space.

THEOREM 24. *Let $S' \subset \mathbb{R}^j$ be a subset of a Euclidean space. Then for all positive integers k and all $m \geq 2$, we have*

$$\mathrm{Tv}(S' \times \mathbb{R}^k, m) \leq \mathrm{Tv}(S', \mathrm{Tv}(\mathbb{R}^k, m)).$$

For example, choosing S of the form $\mathbb{Z}^j \times \mathbb{R}^k$ leads to the "mixed integer" case. Then Theorem 24 implies that, for all positive integers j, k and all $m \geq 2$, we have

$$\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq \mathrm{Tv}(\mathbb{Z}^j, \mathrm{Tv}(\mathbb{R}^k, m)).$$

Moreover, we will use Theorem 24 to obtain the following bound:

$$2^j(m-1)(k+1) + 1 \leq \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq j2^j(m-1)(k+1) + 1. \tag{1.1}$$

Our fourth main result is a generalization of *J. Pach's positive-fraction selection lemma* [**Pac98**] (see [**KKP+15**] for related bounds). Here is J. Pach's result:

THEOREM 25 (J.Pach [**Pac98**]). *Given an integer d, there exists a constant c_d such that for any set P of n points in $\mathbb{R}^d$, there exists a point $\boldsymbol{q} \in \mathbb{R}^d$, and $(d+1)$ disjoint subsets of P, say $P_1, \ldots, P_{d+1}$,* *such that $|P_i| \geq c_d \cdot n$ for all i and the simplex defined by every transversal of $P_1, \ldots, P_{d+1}$ contains $\boldsymbol{q}$. (By "transversal", we mean a set containing exactly one element from each P_i.)*

For example, in the case of the plane, it is known that $c_2 = 12$; Given n red points, n white points, and n blue points, we can select $\frac{n}{12}$ red, $\frac{n}{12}$ white, and $\frac{n}{12}$ blue points in such a way that *all* red-white-blue triangles for the resulting sets have a point $\boldsymbol{q}$ in common.

Unfortunately the point $\boldsymbol{q}$ need not be an integer point; furthermore, the proof uses the so-called "second selection lemma" that currently does not exist for integer points (see Pach [**Pac98**] and Matoušek [**Mat02**, Chapter 9]). In Section 5.4, we strengthen the above theorem, such that, as a consequence, the theorem now extends to the integer case—indeed, to any scenario where one has points of high *half-space depth* in the following sense:

Given a finite set P of points in $\mathbb{R}^d$ and a point $\boldsymbol{q} \in \mathbb{R}^d$, we say that $\boldsymbol{q}$ is *of half-space depth t with respect to P* if any half-space containing $\boldsymbol{q}$ contains at least t points of P (when the context is clear, we will simply say that $\boldsymbol{q}$ is of *depth t*). Then here is our theorem, proved in Section 5.4.

Theorem 26. *For any integer $d \geq 1$ and real number $\alpha \in (0,1]$, there exists a constant $c_{d,\alpha}$ such that the following holds. For any set P of n points in $\mathbb{R}^d$ and any point $\boldsymbol{q} \in \mathbb{R}^d$ of half-space depth at least $\alpha \cdot n$, there exist $(d+1)$ disjoint subsets of P, say $P_1, \ldots, P_{d+1}$, such that*

- *$|P_i| \geq c_{d,\alpha} \cdot n$ for $i = 1, \ldots, (d+1)$, and*
- *every simplex defined by a transversal of $P_1, \ldots, P_{d+1}$ contains $\boldsymbol{q}$.*

Remark: Our proof yields a constant $c_{d,\alpha}$ whose value is exponential in the dimension d.

Note that the existence of integer points of high half-space depth (Lemma 9) together with Theorem 26 implies the following integer version of the positive-fraction selection lemma.

Corollary 2. *Let P be a set of $n \geq (d+1)$ points in $\mathbb{Z}^d$. Then there exists a point $\boldsymbol{q} \in \mathbb{Z}^d$, and $(d+1)$ disjoint subsets of P, say $P_1, \ldots, P_{d+1}$, such that $|P_i| \geq c_{d,2^{-d}} \cdot n$ for all $i = 1, \ldots, (d+1)$, and the simplex defined by every transversal of $P_1, \ldots, P_{d+1}$ contains $\boldsymbol{q}$.*

Remark: In particular, this implies that $\boldsymbol{q}$ belongs to at least $\left(\lceil c_{d,2^{-d}} \cdot n \rceil !\right)^d$ *distinct* Tverberg partitions, with each such Tverberg partition containing $\lceil c_{d,2^{-d}} \cdot n \rceil$ sets.

CHAPTER 2

New applications of Tverberg-type theorems

In this chapter we discuss several applications and implications of Tverberg-type theorems. Helly's, Radon's, and Carathéodory's theorems are three important but basic theorems in the theory of convexity. Though each of these theorems has found applications in numerous topics in mathematics [**DLGMM19**], the applications of Helly's and Carathéodory's theorem have historically been more prominent than those of Radon's lemma and other Tverberg-type results. The goal of this chapter is to demonstrate some of the important consequences of Radon and Tverberg-type theorems. Much of machine learning aims to partition data in a way to gain insight in the form of prediction or inference, and Tverberg-type theorems can be used to understand those partitions.

Our first application is in logistic regression. Linear classifiers form an important class of predictive models. A consequence of the separating hyperplane theorem for convex sets is that two sets of points are linearly separable if and only if their convex hulls do not intersect. This natural relationship between convexity properties of partitions and linear classifiers is especially fruitful in the theory of maximum likelihood estimation.

2.1. Performance guarantees for maximum likelihood estimation

Logistic regression is perhaps the most widely used non-linear model in multivariate statistics and supervised learning [**MN89**]. Statistical inference for this model relies on the theory of maximum likelihood estimation. In the binary classification case, given n independent observations $(\boldsymbol{x}_i, y_i), i = 1, \ldots, n$, logistic regression links the response $y_i \in \{-1, 1\}, i = 1, \ldots, n$ to the covariates $\boldsymbol{x}_i \in \mathbb{R}^d$ via the logistic model

$$\mathbb{P}(y_i = 1 | \boldsymbol{x}_i) = \sigma(\boldsymbol{x}_i^T \boldsymbol{b}), \qquad \sigma(t) := \frac{e^t}{1 + e^t};$$

here $\boldsymbol{b} \in \mathbb{R}^p$ is the unknown vector of regression coefficients. In this model, the log-likelihood is given by

$$l(\boldsymbol{b}) = \sum_{i=1}^{n} - \log(1 + \exp(-y_i \boldsymbol{x}_i^T \boldsymbol{b}))$$

and, by definition, the *maximum likelihood estimate* (MLE) is any maximizer of this functional.

One difficulty arising in machine learning is that the MLE does not exist in all situations. In fact, given two data sets, say one of red points (where $y_i = 1$), and one of blue points (where $y_i = -1$), it is well-known that an MLE exists if and only if the convex hulls of the blue points intersects the convex hull of the red points [**AA84**, **Sil81**]. Although an appealing criterion for existence, this geometric characterization leads to another question: *How much training data do we need (as a function of the dimension of the covariates of the data) before we expect an MLE to exist with high probability?*

The seminal work of T.M. Cover [**Cov65**] (adapting a technique originally due to L. Schläfli [**Sch50**]) provides an answer in a special case. When applied to logistic regression, Cover's main result states the following: assume that the $\boldsymbol{x}_i$'s are drawn i.i.d. from a continuous probability distribution F and that the class labels are independent from $\boldsymbol{x}_i$, and have equal marginal probabilities; i.e., $\mathbb{P}(y_i = 1|\boldsymbol{x}_i) = 1/2$. Then Cover shows that as d and n grow large in such a way that $d/n \to k$, the convex hulls of the data points asymptotically overlap - with probability tending to one - if $k < 1/2$, whereas they are separated - also with probability tending to one - if $k > 1/2$. (When the class labels are not independent from the $\boldsymbol{x}_i$, the problem is more difficult and was recently addressed by E. J. Candès and P. Sur [**SC18**].)

One can view Cover's result as a stochastic analogue of Radon's lemma in combinatorial geometry [**Rad21**]. Radon's lemma says that any $(d+2)$-point set A in $\mathbb{R}^d$ can be partitioned into two subsets A_1, A_2 with $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) \neq \emptyset$. Cover's result shows that with more points, the conclusion of Radon's lemma can be reached with high probability from a random partition: For large d and $\epsilon > 0$, given a set A of $(1+\epsilon)2d$ points, one can obtain subsets A_1 and A_2 of A with $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) \neq \emptyset$ by simply assigning each $\boldsymbol{x} \in A$ to A_1 or A_2 by flipping a fair coin.

In Chapter 4, we develop the connection between geometric probability (Cover's result), discrete geometry (Tverberg-type results), and the conditions for the existence of MLEs. Tverberg-type results are natural generalizations of "Stochastic Separation Theorems", i.e., the case of two colors such as treated by Cover [**Cov65**]. Cover's result has since become the first of a whole family of "stochastic separability" results arising in many topics such as maximum likelihood estimation [**FGM18**, **SC18**, **AA84**, **Sil81**], error correction in machine learning [**XLJ18**], and computational geometry [**GBRT19**, **SMdBM17**], to name a few. Chapter 4 develops the generalization of this stochastic separation problem to more than two colors, and adds an additional parameter called tolerance.

Given a binary classification algorithm, there are two common approaches to extend it multi-class classification: *"one-vs-rest"* and *"one-vs-one"*. In "one-vs-rest", we train C separate binary classification models. Each classifier f_c for $c \in \{1, \ldots, C\}$ is trained to determine whether or not an example is part of class c or not. To predict the class for a new sample $\boldsymbol{x}$, we run all C classifiers on $\boldsymbol{x}$ and choose the class with the highest score: $\hat{y} = \arg\max_{c \in \{1,\ldots,C\}} f_c(\boldsymbol{x})$. In "one-vs-one" regression, we train $\binom{C}{2}$ separate binary classification models, one for each possible pair of classes. To predict the class for a new sample $\boldsymbol{x}$, we run all $\binom{C}{2}$ classifiers on $\boldsymbol{x}$ and choose the class with the most "votes."

To apply "one-vs-one" multinomial logistic regression, we would like to ensure that the MLE exists between the data corresponding to every pair of labels. Our stochastic Tverberg theorem implies this.

THEOREM 27 (stochastic Tverberg theorem applied to multinomial regression). *Fix $\epsilon > 0$. Assume that the $\boldsymbol{x}_i$'s are drawn i.i.d. from a continuous probability distribution F which is balanced about some point $\boldsymbol{p} \in \mathbb{R}^d$ and that the class labels are independent from $\boldsymbol{x}_i$, and have equal marginal probabilities; i.e., $\mathbb{P}(y_i = k|\boldsymbol{x}_i) = 1/m$ for all $k \in \{1, \ldots, m\}$.*
Letting the number of data points $f(m)$ grow as a function of the number of labels m, the MLE exists between the data corresponding to every pair of labels with high probability as long as

$$f(m) > (1 + \epsilon)m \log_2(m) \ln(\ln(m)).$$

The same bound applies to "one-vs-rest" logistic regression, since MLE existence in that case is a weaker condition. The parameter of tolerance is also significant in studying MLE existence. A natural observation is that Tverberg partitions with tolerance t correspond to robust MLE existence, in the sense that any t points (possibly corrupted or outlier data) can be removed and still the convex hulls of the data with each label intersect. The various special cases of Stochastic Tverberg theorems are thus useful in different kinds of classification problems, and these observations are summarized in Table 2.1.

In fact, the parameter of tolerance is also similar to an important parameter used to guarantee to the convergence speed of first order methods for finding MLEs. Recently, when studying binomial logistic regression, Freund, Grigas and Mazumunder [**FGM18**] introduced the following notion to quantify the extent that a dataset is non-separable (where $\boldsymbol{a}^-$ denotes the negative part of $\boldsymbol{a}$):

$$\begin{aligned} \text{DegNSEP}^* := \min_{\boldsymbol{b}\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n}[y_i\boldsymbol{b}^T\boldsymbol{x}_i]^- \\ \text{s.t. } \|\boldsymbol{b}\| = 1 \end{aligned}$$

DegNSEP* is thus the smallest (over all normalized models $\boldsymbol{b}$) average misclassification error of the model $\boldsymbol{b}$ over the n observations. They showed that the condition number DegNSEP* informs the computational properties and guarantees of the standard deterministic first-order steepest descent solution method for logistic regression. Let us now briefly discuss how the parameter of tolerance for Radon partitions (Tverberg 2-partitions) can be viewed as a discrete analogue of DegNSEP*. Define PertSEP* as the smallest (or more precisely, the infimum thereof) perturbation $\Delta\boldsymbol{X}$ of the feature data $\boldsymbol{X}$ which will render the perturbed problem intstance $(\boldsymbol{X}+\Delta\boldsymbol{X},\boldsymbol{y})$ separable. Namely,

$$\begin{aligned} \text{PertSEP}^* := \inf_{\Delta\boldsymbol{X}} \frac{1}{n}\|\Delta\|_{\cdot,1} \\ \text{s.t. } (\boldsymbol{X}+\Delta\boldsymbol{X},\boldsymbol{y}) \text{ is separable.} \end{aligned}$$

In Proposition 2.4 of [**FGM18**] it is shown that DegNSEP* = PertSEP*.

We introduce another parameter PertSEP^*_0 simply defined as the L_0 norm of the smallest perturbation of the feature data $\boldsymbol{X}$ which will render the perturbed problem instance $(\boldsymbol{X}+\Delta\boldsymbol{X},\boldsymbol{y})$ separable. Namely,

$$\begin{aligned} \text{PertSEP}^*_0 := \inf_{\Delta\boldsymbol{X}} \frac{1}{n}\|\Delta\|_{\cdot,0} \\ \text{s.t. } (\boldsymbol{X}+\Delta\boldsymbol{X},\boldsymbol{y}) \text{ is separable.} \end{aligned}$$

The following theorem shows that the tolerance of a Radon partition is given by PertSEP^*_0:

THEOREM 28. *Suppose that $\boldsymbol{X}=\boldsymbol{X}_1\cup\boldsymbol{X}_2$, $|\boldsymbol{X}|=n$ is a Radon partition with tolerance precisely equal to t. Then viewing $\boldsymbol{X}$ as a labeled dataset (with $(\boldsymbol{X},\boldsymbol{y})=\{(\boldsymbol{x},-1):\boldsymbol{x}\in\boldsymbol{X}_1\}\cup\{(\boldsymbol{x},1,):\boldsymbol{x}\in\boldsymbol{X}_2\}$), we have that*

$$\mathit{PertSEP}^*_0 = t/n.$$

Proof. Let M denote the minimal number of points perturbed among any perturbation that makes $(\boldsymbol{X}, \boldsymbol{y})$ separable, and N denote the minimal number of points needing to be removed from $(\boldsymbol{X}, \boldsymbol{y})$ to make $(\boldsymbol{X}, \boldsymbol{y})$ separable. Then PertSEP$^*{}_0(\boldsymbol{X}, \boldsymbol{y})$ is equal to M/n, and the tolerance t of $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ is equal to N. It suffices to show that $M = N$. To see that $M \geq N$, note if $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$ in $\boldsymbol{X}$ are moved so that the resulting set $(\boldsymbol{X'}, \boldsymbol{y'})$ is separable, then $(\boldsymbol{X} \setminus \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}, \boldsymbol{y} \setminus \{y_1, \ldots, y_N\})$ is also separable. To see that $M \leq N$, suppose that $(\boldsymbol{X} \setminus \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}, \boldsymbol{y} \setminus \{y_1, \ldots, y_M\})$ is separable by a hyperplane. Then moving $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$ to the appropriate sides of the hyperplane determined by h, we can construct a separable dataset $(\boldsymbol{X'}, \boldsymbol{y'})$, obtained from moving M points from $(\boldsymbol{X}, \boldsymbol{y})$. $\square$

Theorem 28, combined with a result of Soberón, implies Corollary 3, which roughly says that PertSEP$^*{}_0$ of a randomly bi-partitioned point set asymptotically approaches $1/2$.

Corollary 3. *For any sequence* $\{\mathcal{R}_{(2,k,d)}\}, k \in \mathbb{N}$ *of partitioned point sets, and any* $\epsilon > 0$*, we have* $|PertSEP^*{}_0(\mathcal{R}_{(2,k,d)}) - 1/2| < \epsilon$ *with high probability.*

In fact, for fixed d and m, Corollary 3 can be extended to the multi-class setting. In other words, for a large randomly m-partitioned data set, we expect PertSEP$^*{}_0$ of every pair of data points to be close to $1/2$:

Theorem 29. *Fix* $\epsilon > 0$*. For any sequence* $\{\mathcal{R}_{(m,k,d)}\}, k \in \mathbb{N}$ *of* m*-partitioned point sets* $\mathcal{R}_{(m,k,d)} = \{X_1, \ldots, X_m\}$,

we have

$$\lim_{k \to \infty} \left(\min_{X_i, X_j \in \mathcal{R}_{(m,k,d)}} PertSEP^*{}_0(X_i \cup X_j) = 1/2 \right)$$

with high probability.

These results show how Tverberg's theorem is related to prediction problems in machine learning, but Tverberg's theorem is also helpful in inference problems in machine learning.

Most inference problems are *unsupervised* learning problems: For every observation $i = 1, \ldots, n$ we observe a vector of measurements $\boldsymbol{x}_i$ but no associated response y_i. So the situation is called unsupervised because we lack a respond variable that can supervise our analysis. What sort of

Deterministic	Stochastic	Likely MLE Existence
Radon	[**Cov65**]	pair of datasets
Tverberg	Thms 18,19	all pairs of datasets
Radon with tolerance	Thm 21	pair of datasets with outliers removed
Tverberg with tolerance	Thms 1,20, [**Sob18**]	all pairs of datasets with outliers removed

TABLE 2.1. Stochastic analogues of Tverberg's theorem and their implications for existence of MLEs. By "Likely MLE Existence", we mean that one can bound below the probability of MLE existence as a function of the amount of input data according to the corresponding theorems in the "Stochastic" column.

statistical analysis is possible? We can seek to understand the relationships between the variables or between the observations using cluster analysis or topological data analysis. Nerves of partitions are a useful object of study in both of these fields.

2.2. Topological data analysis and clustering

A motivation for studying which nerves are induced by sufficiently large data sets comes from clustering and data classification [**CM13, DMW17**]. Clustering algorithms aim to "color" or "label" data points by groups that share common characteristics (see [**GMW07, Jai10**] and references there). Classification is then a partitioning of the data set. Two sets of points will intersect if they share members with both characteristics. When doing a classification researchers face the question, is the proposed partition of data showing intrinsic data properties, unique to the particular input data points, or is this a mere artifact appearing in *all* data sets after having a sufficiently large data set? Tverberg-type theorems with altered nerves are relevant to data science when analyzing the statistical significance of a proposed classification in large-scale data sets. For example, the Mapper algorithm proposed by G. Carlsson, F. Mémoli, and G. Singh, [**SMC07**] is a method for topological summarization which studies a special nerve complex constructed from a point cloud.

The idea behind Mapper can be summarized as follows. Suppose we have point cloud data $X \subset \mathbb{R}^n$ representing a shape, for example a circle. We fix some *filter* function, which is a function $f : X \to \mathbb{R}$. Then we choose an open cover $C = U_1, U_2, \ldots, U_k$ of the image $f(X)$. For each U_i with $i \in [k]$, we consider the pullbacks $X_i = f^{-1}(U_i)$. Since C is a cover of $f(X)$, we have $\bigcup_{i\in[n]} X_i = X$ but the X_i are not necessarily disjoint. Now for each X_i, we apply a clustering algorithm to split it into subsets X_{ik}. The Mapper algorithm then outputs the nerve complex of the convex hulls of

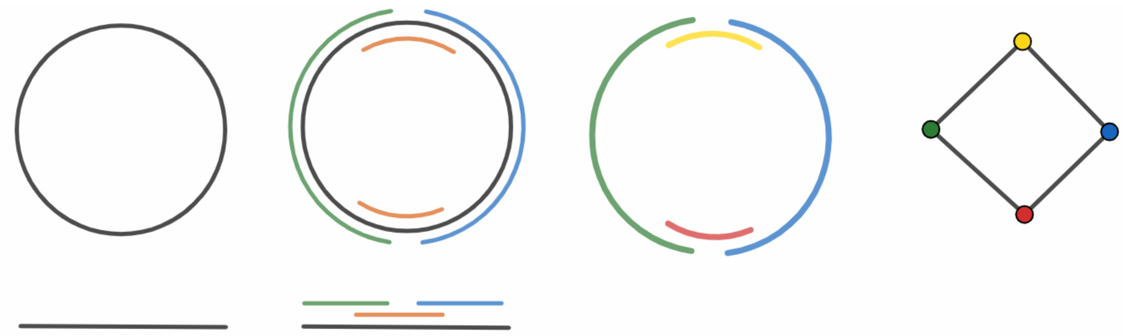


FIGURE 2.1. Example of the mapper algorithm applied to a circular point cloud.
From left to right:
1. We choose the projection onto the first coordinate as the filter function.
2. Take the pullback of the green, orange, and blue cover of the image.
3. Cluster the orange points into yellow and red clusters.
4. Construct the resulting nerve complex.

the X_{ik}. For an example of this process, see Figure 2.1. We refer the reader to [**Sin08**, **Car09**] for more details.

The result is a combinatorial object encoding some of the structure of the data, and can be useful for dimensionality reduction of the data. Remarkably, this algorithm can be applied to a point cloud representing a human hand, and recover a tree with one root, and five "branches" corresponding to each finger. But the results of Chapter 3 show that every tree is a Tverberg complex, so every sufficiently large data set can be partitioned in such a way that the nerve is a tree with one root and five "branches". On the other hand, the main results of Chapter 4 show that such a nerve arising randomly is unlikely if the data set is large. Combined, these two notions can be used to give a better idea of the statistical significance of the result from Mapper.

In cluster analysis, the *k-means* algorithm is ubiquitous in industry due to its $O(n)$ runtime, simple interpretability, and easy implementation. The k-means clustering algorithm aims to partition n points into k clusters so that each point belongs to the cluster with the nearest mean. A common approach to reach this objective is to start by randomly partitioning the points into k clusters, and then iteratively move the points to the cluster whose mean they are closest to (updating the cluster means after each iterate).

Let us consider what happens to the nerve of the partitions generated by this algorithm. If this algorithm runs until no more improvement is possible, the resulting partition of points corresponds to a partition of the data space into *Voronoi cells*. Voronoi cells are a decomposition of the data space into regions corresponding to the set of points closest to a given point. See Figure 2.2 for an example. If we are using the Euclidean metric, Voronoi cells are convex, and thus the nerve of the resulting partition consists of just k vertices. On the other hand, applying the stochastic Tverberg theorems of Chapter 4, we see that for large data sets the nerve of the original random partition is likely the $k-1$ simplex. All together the iterative process of moving points from one subset to another in the k-means algorithm induces a sequence of simplicial complexes on k vertices using the nerve map, and this perspective may be useful for understanding the convergence of the k-means algorithm.

A major motivation for further study of the k-means algorithm is that the resulting clusters may be sensitive to the original random cluster at the start of the algorithm. An improvement would be to aim to find a k-clustering that minimizes the average distance from each point to its cluster mean. Unfortunately, it is not well understood when the iterative algorithm reaches this globally optimal partition. Studying nerves of partitions is another way to understand the state space of possible partitions. This may be of interest to study this convergence of the k-means algorithm or other iterative clustering algorithms.

2.3. Computing centerpoints in optimization

The problem of finding Tverberg partitions is of interest applied toward finding *centerpoints*. A *centerpoint* of an n-point data set $S \subset \mathbb{R}^d$ is a point $\boldsymbol{p}$ such that every half space containing $\boldsymbol{p}$ has at least $\frac{n}{d+1}$ points in S. Centerpoints are useful in a variety of contexts, such as in "divide and conquer" schemes for optimization. See [**DLGMM19**] for further discussion. Unfortunately, obtaining a centerpoint is difficult, and the current best randomized algorithm constructs a centerpoint in time $O(n^{d+1}+n\log n)$ [**Cha04**]. Thus finding an approximate centerpoint of a set is useful. We define a point of *half-space depth k in S* as a point $\boldsymbol{p}$ such that every half-space containing $\boldsymbol{p}$ contains at least k points in S.

Tverberg's theorem implies that every data set has a centerpoint, as the Tverberg point of a Tverberg partition must be a point of half-space depth one in each of the $m = \lceil \frac{n}{d+1} \rceil$ color classes. Hence an effective version is desirable as a method to obtain centerpoints. The proof of Radon's lemma is constructive and, in fact, one of the most notable randomized algorithms for computing approximate centerpoints works by repeatedly replacing subsets of $d+2$ points by their Radon

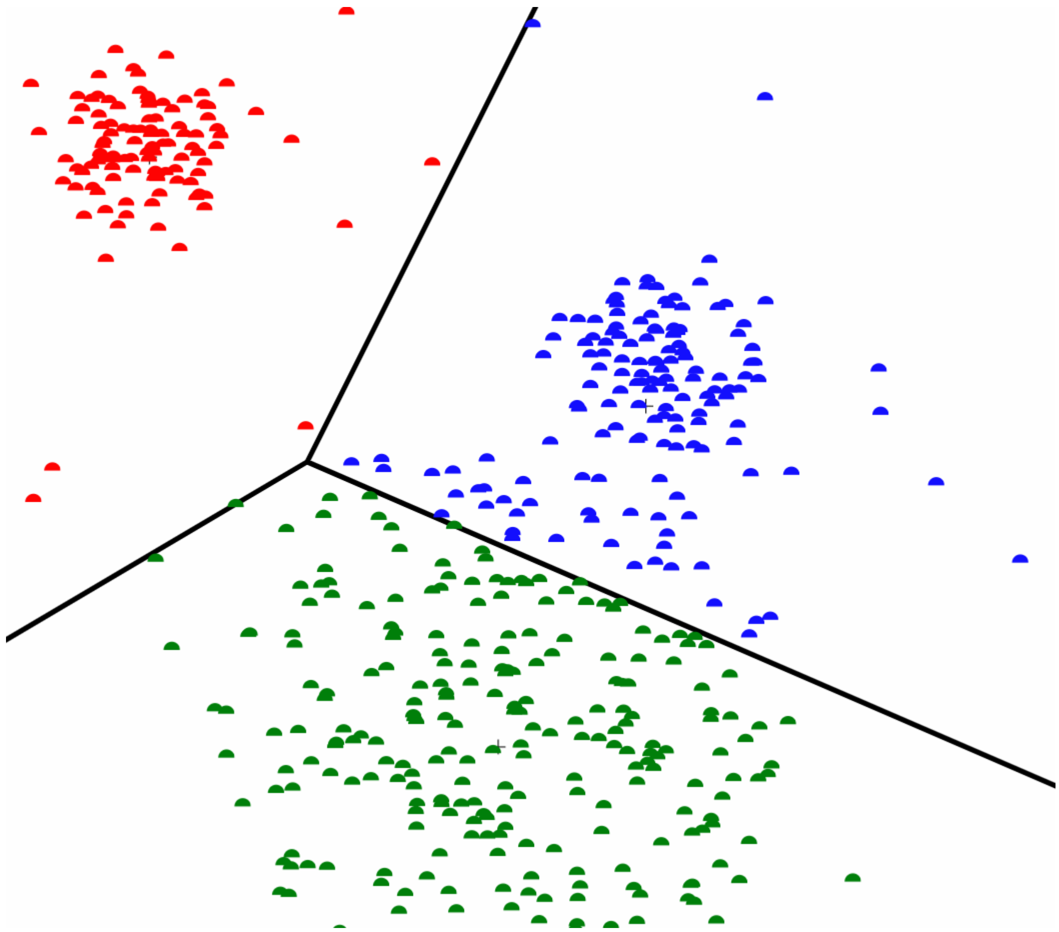


FIGURE 2.2. Output of K-means algorithm and corresponding Voronoi decomposition. Image credit: Wikipedia

point [**CEM$^+$96**]. In contrast, no known polynomial time algorithm exists for computing exact Tverberg points. Thus, fast algorithms for approximate Tverberg points have been introduced in [**MW13**, **RS16**].

If one is interested in non-deterministic algorithms for finding Tverberg partitions, the main results of our paper can be used to give expected performance of algorithms where we obtain Tverberg partitions by random choice.

In particular, Theorem 19 suggests a trivial algorithm for finding a Tverberg partition among a set of suitably distributed points. According to Theorem 19, a random equipartition of m points into less than $m/log_2(m)$ sets should produce a Tverberg partition with high probability. This trivial non-deterministic algorithm was also suggested by Soberón, except using a random allocation rather than equi-partition. Our asymptotic results improve the bounds on expected performance of Soberón's proposed algorithm (random allocation) as well, though our performance bounds for Random equi-partitions are still better. We summarize the performance and time complexity of various algorithms for obtaining Tverberg partitions below.

TABLE 2.2. Approximate Tverberg Partitions.

Method	**Number of Colors**	**Time complexity**
Tverberg	$\lfloor (m+1)/(d+1) \rfloor$	PPAD (Unknown if polynomial)
Mulzer and Werner [**MW13**]	$m/(4d+1)^3$	$d^{O(\log d)}m$
Random equi-partition	$O(\frac{m}{\log_2(m)})$	$O(m)$
Random Allocation	$O(\frac{m}{\log_2(m)(\ln(\ln(m)))})$	$O(m)$

Obtaining centerpoints in discrete sets is even more desirable, since discrete optimization problems are almost always more difficult than their continuous counterparts. The same motivations in optimization appear, especially since integer valued and binary variables appear in many classic optimization problems. Our Tverberg type results over discrete sets in Chapter 5 can be applied to obtain centerpoint theorems in these contexts. It also shows that rather than looking for centerpoints directly, one can instead seek Tverberg partitions instead, since they are guaranteed to exist over these sets. For example, applying Theorem 22, we see that any discrete subset of the plane with Helly number three and cardinality n has a center point of depth $\lceil n/3 \rceil$ and such a centerpoint is in fact witnessed by at least one Tverberg partition into $\lceil n/3 \rceil$ sets.

CHAPTER 3

Tverberg-type theorems with altered nerves

The main results of this chapter demonstrate that, Tverberg's theorem is but a special case of a more general situation where other intersection patterns can be induced as the nerve of the convex hulls of a partition of any sufficiently large set of points. In particular, we prove that, given sufficiently many points, all trees and cycles, can also be induced by at least one partition of a point set. We also discuss how some complexes cannot be achieved this way, even for arbitrarily large sets of point sets.

3.1. A Tverberg theorem for trees and cycles

3.1.1. Proof of Theorem 11 (A) in the plane. Because the case of dimension two exemplifies the key ideas very well and because we can provide a better bound, we first give the proof of Theorem 11 (A) in the plane. To summarize the proof, first, we show in Theorem 30 that the result holds if the points are arranged as the vertices of a convex polygon. Second, given any set $\bar{S}$ with at least $\binom{4n-4}{2n-2}+1$ points in the plane, we apply the Erdős-Szekeres theorem to deduce that $\bar{S}$ has a sub-configuration S of $2n$ points in convex position. Then we apply Theorem 30 to obtain a partition of S whose nerve is the tree T_n, and finally, in Lemma 3, we prove we can extend the partition of S to the rest of $\bar{S}$ while preserving the nerve. Later in Subsection 3.1.2 we present the general case in $\mathbb{R}^d$ following a similar strategy, but some of the key steps are different.

THEOREM 30. *Let T_n be a tree with n nodes, and let $S \subset \mathbb{R}^2$ be any $2n$ point set in convex position. Then S admits a partition $\mathcal{P}$ such that its nerve $\mathcal{N}(\mathcal{P})$ is isomorphic to T_n.*

PROOF. The proof is by induction on n, the number of vertices in T_n. For an example of the construction see Figure 3.1.

For $n = 1$, the tree consists of a single node and S is a set of two points in $\mathbb{R}^2$. Coloring both points with color 1 will trivially satisfy the theorem. When $n = 2$, the only tree with two vertices is K_2. By Radon's theorem any set of four points in S, say $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$ in counterclockwise order, can be partitioned with intersection graph K_2. Note that in this case, coloring the points in $S = S_1 \cup S_2$ with two alternating colors $\boldsymbol{s}_1 = 1, \boldsymbol{s}_2 = 2, \boldsymbol{s}_3 = 1, \boldsymbol{s}_4 = 2$ will yield the required partition.

For performing the induction step, we can assume T_n was obtained from a tree T_{n-1} by adding the leaf node v_n to a node $v_r \in T_{n-1}$ such that $\{v_n, v_r\}$ is an edge of T_n. Note that in our labeling of the n nodes, r may not be $n-1$, but all trees are constructed by a sequence of leaf additions.

By the induction hypothesis, for any set S' with exactly $2n-2$ points in convex position in $\mathbb{R}^2$, there exists a partition $\mathcal{P}'$ of S' into $n-1$ color classes, where each color $i \in \{1, 2, \ldots n-1\}$ is used twice, such that $T_{n-1} = \mathcal{N}(\mathcal{P}')$. Thus, we may assume that there exists a two-to-one "coloring function" $\mathcal{C} : S' \to [n-1]$ that associates two points in S' with a color i, (the color of node v_i).

Let S be a set of $2n$ points in convex position in $\mathbb{R}^2$, ordered in a clockwise manner, say $S = \{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{2n}\}$, and assume without loss of generality that $\boldsymbol{s}_1$ is at twelve o' clock. Next, consider the set $S' := S \setminus \{\boldsymbol{s}_2, \boldsymbol{s}_{2n}\}$. To this set S' we can apply the induction hypothesis, it is properly colored and gives T_{n-1}. Now we show how to add color n to the remaining points in S to give T_n. There are two cases.

Case 1 If $\mathcal{C}(\boldsymbol{s}_1) = r$, then extend $\mathcal{P}'$ to a partition $\mathcal{P}$ of S by assigning color n to the points $\boldsymbol{s}_2$ and $\boldsymbol{s}_{2n}$. Thus $\mathcal{P} = \mathcal{P}' \cup \{\boldsymbol{s}_2, \boldsymbol{s}_{2n}\}$. Let L_n be the line through $\boldsymbol{s}_2$ and $\boldsymbol{s}_{2n}$. Observe that on one side of L_n, say L_n^+, there is only $\boldsymbol{s}_1$. Then the other points in S' are contained in the other open half plane L_n^-. In particular, one point, say $\boldsymbol{s}_j$, is such that $\mathcal{C}(\boldsymbol{s}_j) = r$. Thus $\boldsymbol{s}_1$ and $\boldsymbol{s}_j$ have color r. Then $\operatorname{conv}(\boldsymbol{s}_2, \boldsymbol{s}_{2n})$ and $\operatorname{conv}(\boldsymbol{s}_1, \boldsymbol{s}_j)$ intersect so $\mathcal{N}(\mathcal{P})$ contains the edge (r, n). Furthermore, for any $i \neq n, r$, we have that $\mathcal{N}(\mathcal{P})$ does not contain the edge (i, n), since the points with color i are contained in L_n^- and so their convex hull cannot intersect $\operatorname{conv}(\boldsymbol{s}_2, \boldsymbol{s}_{2n})$. Thus the nerve of $\mathcal{P}$ is T_n.

Before starting Case 2 consider the relabeling of $S' := S \setminus \{\boldsymbol{s}_2, \boldsymbol{s}_{2n}\} = \{\boldsymbol{x}_1 = \boldsymbol{s}_1, \boldsymbol{x}_2 = \boldsymbol{s}_3, \ldots, \boldsymbol{x}_{2n-2} = \boldsymbol{s}_{2n-1}\}$.

Case 2 If $\mathcal{C}(\boldsymbol{s}_1) \neq r$, then we know that on one side of the line L_n (through $\boldsymbol{s}_2$ and $\boldsymbol{s}_{2n}$) there are two points in S', say $\boldsymbol{x}_i, \boldsymbol{x}_{i+k}$ (as above) such that $\mathcal{C}(\boldsymbol{x}_i) = \mathcal{C}(\boldsymbol{x}_{i+k}) = r$ for $i \geq 3$ and $1 < k \leq (2n-2) - i$. Apply to S' the following new coloring $\bar{\mathcal{C}} : S' \to [n-1]$ defined as

$\bar{\mathcal{C}}(\boldsymbol{x}_j) = \mathcal{C}(\boldsymbol{x}_{(j+2n-i-1)}) \bmod(2n-2)$. That is, the rotation that sends the corresponding color in $\boldsymbol{x}_i$ to $\boldsymbol{x}_1$. Observe that this rotation preserves all the intersection patterns that existed before (by Lemma 2), and thus $\mathcal{N}(\mathcal{P}')$ is T_{n-1}. Lastly, we are now in the position to apply Case 1 again, so the theorem follows.

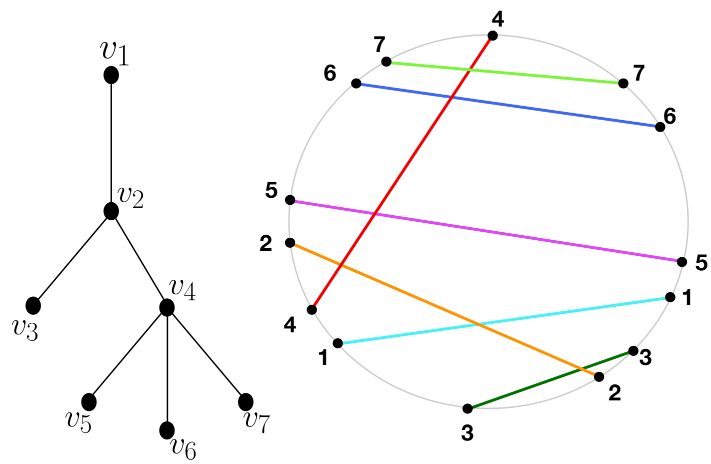


FIGURE 3.1. Example of a tree with seven nodes and shown as partition induced on a set S of 14 points in convex position.

This completes the proof that any set S of $2n$ points in convex position in the plane have a partition whose nerve is isomorphic to any given tree T_n. $\square$

To extend our result to the case that the points are in general position, we will use a famous theorem in combinatorial geometry, the Erdős-Szekeres theorem. This theorem says that every sufficiently large set of points in general position contains a subset of k points in convex position. The fact that this number $N = N(k, 2)$ exists for every k was first established in a seminal paper of Erdős and Szekeres, [**ES35**] who proved the following bounds on $N(k, 2)$.

$$2^{k-2} + 1 \leq N(k, 2) \leq \binom{2k-4}{k-2} + 1.$$

A handful of recent papers have improved the upper bound (see for instance [**MS16**] for an excellent survey and a very recent paper by A. Suk [**Suk17**] showing that $N(k, 2) = 2^{k+o(k)}$).

By the Erdős-Szekeres Theorem we know that $\binom{4n-4}{2n-2}+1$ points always contain a $2n$-gon. Then, we can use Theorem 30. Finally we explain how to extend the partition (or coloring) given by Theorem 30 to the rest of the points in $\bar{S}$.

DEFINITION 31. *Let S be a set of points in $\mathbb{R}^d$ and let $\mathcal{P} = S_1, \ldots, S_n$ be an n-partition of S into n color classes that yields a specific nerve $\mathcal{N}(\mathcal{P})$. We say that a $\mathcal{P}$ is* extendable *if for all $\bar{S}$ containing S, there is a partition $\bar{\mathcal{P}} = \bar{S}_1 \ldots \bar{S}_n$ of $\bar{S}$ extending $\mathcal{P}$ (meaning $S_i \subset \bar{S}_i$ for all i) such that $\mathcal{N}(\bar{\mathcal{P}})$ isomorphic to $\mathcal{N}(\mathcal{P})$.*

Observe that in general, such an extension is not necessarily possible, for example, Figure 3.2 shows a set of six vertices, and a partition in three color classes (see left side of the figure), that is not extendable. Note that any extension that includes the midpoint will change the intersection pattern (see right side of the figure). Surprisingly, in the case of the nerves of the partitions obtained in Theorem 30 and (Theorem 32 in the next subsection), this extension is possible.

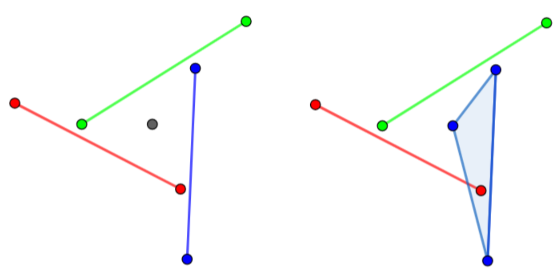


FIGURE 3.2. An example of a non-extendable partition.

LEMMA 3. *Let T_n be a given tree on n nodes and let S be a set of $2n$ points in convex position in the plane. Then the partition $\mathcal{P}$ obtained in proof of Theorem 30 is extendable.*

PROOF. Let $\bar{S}$ be an arbitrary finite set of points in $\mathbb{R}^d$, such that $S \subset \bar{S}$. We begin by assuming that the "color partition function" $\mathcal{C} : S \to [n]$ is the one given in Theorem 30. It yields a partition $\mathcal{P}$ of S with nerve $\mathcal{N}(\mathcal{P})$ isomorphic to T_n and n is the last color added. Recall that we denoted by v_r the node in T_{n-1} such that $\{v_n, v_r\}$ is the leaf of T_n in which we added v_n.

The extension of $\mathcal{P}$ of S will be given through induction on n, by a "color partition function" $\bar{\mathcal{C}_n} : \bar{S} \to [n]$ as follows:

a) For $n = 1$, let $\bar{\mathcal{C}}_1(\boldsymbol{x}) = 1$ for every point in $\bar{S}$.

b) For the induction step, the extension $\bar{\mathcal{C}}_{n-1} : \bar{S} \to [n-1]$ exists by induction hypothesis. Here is how we obtain the extension $\bar{\mathcal{C}}_n$:

Let S_j denote the set of points in S of color v_j, or j for simplicity. Consider the line L_n through S_n (it is given by points $\boldsymbol{s}_2$ and $\boldsymbol{s}_{2n}$ in Theorem 30), and recall that this line leaves only one element of S_r in one side, say L_n^+, and the rest of the points of S in the other side L_n^-. We define $\bar{\mathcal{C}} : \bar{S} \to [n]$ as follows: $\bar{\mathcal{C}}_n(\boldsymbol{x}) = \bar{\mathcal{C}}_{n-1}(\boldsymbol{x})$ when $\boldsymbol{x} \in L_n^-$, $\bar{\mathcal{C}}_n(\boldsymbol{x}) = r$ when $\boldsymbol{x} \in \operatorname{conv}(S_r)$, and, finally, $\bar{\mathcal{C}}_n(\boldsymbol{x}) = n$ when $\boldsymbol{x} \in \operatorname{cl}(L_n)^+$ but $\boldsymbol{x} \notin \operatorname{conv}(S_r)$. Here $\operatorname{cl}(L_n)^+$ denotes the closed half-plane at the right of L_n.

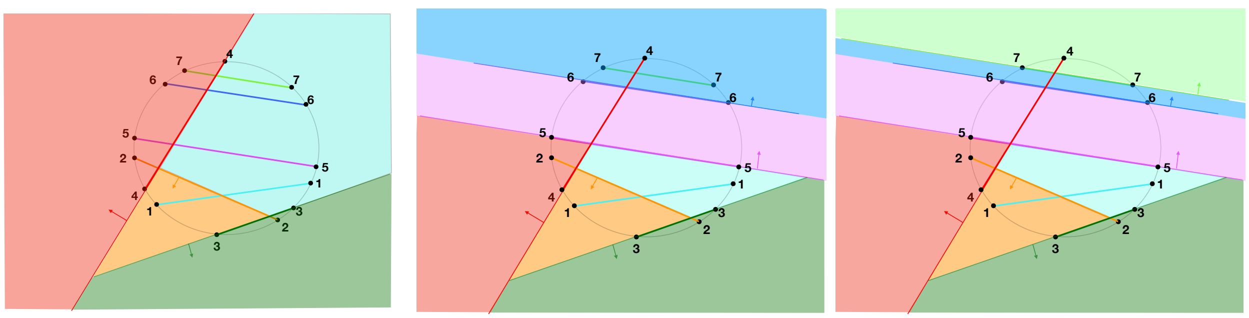


FIGURE 3.3. The extension of the partition obtained in Figure 3.1. The left figure, is the extension up to $n = 4$, the central figure is the extension up to $n = 6$, and the right figure is up to $n = 7$.

Observe that, by the induction process, the intersection pattern of $\bar{S}_1, \dots \bar{S}_{n-1}$ are the same in $L_n{}^-$ by construction. Furthermore, $\operatorname{cl}(L_n)^+$ does not intersect any other element in the partition, so no new intersections occur. $\square$

3.1.2. Proof of Theorem 11 (A) in $\mathbb{R}^d$. Next, we will show a general dimension version of Theorem 30. The pattern of the proof is very similar to the planar case, but we will need to use properties of cyclic polytopes and their oriented matroids. A parametrized curve $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^d$ is a *d-order curve* (sometimes called *alternating*) when no affine hyperplane H in $\mathbb{R}^d$ meets the curve in more than d points. An example is the famous *moment curve*. See [**Stu87**], [**CD00**], [**BLVS$^+$93**].

In what follows we will use *ordered cyclic d-polytopes* $C_m(d)$ which are obtained as the convex hull of m vertices $S := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots \boldsymbol{x}_m\}$ along a d-order curve in $\mathbb{R}^d$ and thus, we may order the vertices of this polytope in an increasing sequential manner, say $\alpha(t_1) = \boldsymbol{x}_1 < \alpha(t_2) = \boldsymbol{x}_2 < \cdots < \alpha(t_m) = \boldsymbol{x}_m$. Ordered cyclic polytopes are very special because every subpolytope is also cyclic with respect to the same vertex order, i.e., the corresponding oriented matroid is *alternating*. Alternating means the chirotope has all positive signs. See Section 9.4 in the book [**BLVS$^+$93**].

THEOREM 32. *Let T_n be any tree with n nodes, and let S be the vertices of an ordered cyclic d-polytope $C_m(d)$ with $m = (n-1)(d+1)+1$ vertices in $\mathbb{R}^d$. Then, there exists a partition $\mathcal{P}$ of S such that the nerve $\mathcal{N}(\mathcal{P})$ is isomorphic to T_n.*

PROOF. Let $C_m(d)$ be an ordered cyclic d-polytope, with m vertices and assume as before $S := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_m\}$ along the curve. As in the case of the plane, the proof will be given by induction on n the number of nodes of the tree T_n.

If $n = 1$, again there is nothing to prove. If $n = 2$, the only tree with two vertices is K_2. Then by Radon's theorem, any set of $d+2$ points in S can be partitioned in $S = S_1 \cup S_2$ with $2 \leq |S_i| \leq d$ for $i \in \{1, 2\}$ and intersection graph K_2, see Figure 3.4 on the left.

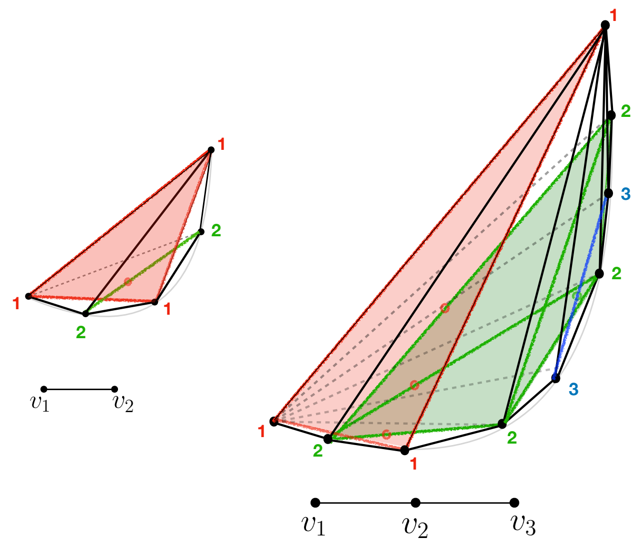


FIGURE 3.4. Two examples, on the left, we show a tree on two nodes shown as a partition in a set S of five vertices of the cyclic polytope $C_5(3)$. On the right side we represent a tree on three nodes as a partition of the nine vertices of another cyclic polytope in $\mathbb{R}^3$, this time $C_9(3)$.

For the induction step, suppose T_n was obtained from T_{n-1} by adding the node v_n to a node $v_r \in T_{n-1}$ such that $\{v_n, v_r\}$ is a leaf of T_n, and assume that T_{n-1} is the nerve of some set $\mathcal{N}(\mathcal{P}')$ where the set S' are the vertices of the ordered cyclic polytope with exactly $(n-1)(d+1)-d$ vertices in $\mathbb{R}^d$ and $\mathcal{P}' = \{S'_1, S'_2, \ldots, S'_{n-1}\}$ are the color classes with color $1, 2, \ldots n-1$ respectively, via a "coloring function" $\mathcal{C}' : S' \to [n-1]$.

Let k the maximum number in $[n]$ such that $\boldsymbol{x}_k$ is in S'_r. Next in $C_m(d)$ consider the subpolytope Q of $(n-1)(d+1)-d$ vertices, obtained as the convex hull $\text{conv}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k, \boldsymbol{x}_{k+d+2}, \ldots \boldsymbol{x}_m)$, and R is the polytope consisting of the convex hull of the complement of Q and $\boldsymbol{x}_k$, thus $R = \text{conv}(\{\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{k+d+1}\})$. Note both Q and R are ordered cyclic polytopes and $Q \cap R = \{\boldsymbol{x}_k\}$. Thus by the induction hypothesis there exists a partition of the vertices of Q into $n-1$ color classes whose nerve is isomorphic to T_{n-1} as before. Next, by Radon's Lemma there exists a partition into two color classes A and B of the $d+2$ vertices of R.

Say $\boldsymbol{x}_k \in A$, then define a "coloring function" $\mathcal{C} : S \to [n]$ in the following way: $\mathcal{C}(\boldsymbol{x}) = \mathcal{C}'(\boldsymbol{x})$ if $\boldsymbol{x}$ is a vertex of Q, $\mathcal{C}(\boldsymbol{x}) = r$ if $\boldsymbol{x} \in A$, and, finally, $\mathcal{C}(\boldsymbol{x}) = n$ if $\boldsymbol{x} \in B$. That is $S_n \cap S_r \neq \emptyset$ and no further intersections occur. By the construction the parts of $\mathcal{P}$ consist of the n color classes determined by the coloring $\mathcal{C}$. The nerve $\mathcal{N}(\mathcal{P})$ is isomorphic to T_n. $\square$

In dimension two, we relied on Erdős-Szekeres to build a convex polygon. For the general case in $\mathbb{R}^d$, we need a multi-dimensional version of Erdős-Szekeres theorem that follows from an application of the hypergraph Ramsey theorem [**CFS10**]. The theorem we need was first given by Grünbaum (Exercise 7.3.6 in [**Grü67**]) and Cordovil and Duchet [**CD00**] using oriented matroid methods. See Proposition 9.4.7 of [**BLVS$^+$93**] for a short proof. The theorem shows the existence of a number $N = N(k, d)$ such that every set of N points in general position in $\mathbb{R}^d$ contains the vertices of an ordered cyclic d-polytope. N is bounded from above by the hypergraph Ramsey number $R_{d+1}(m)$ (see the Introduction) ensuring the existence of an alternating oriented matroid (hence an ordered cyclic polytope with m vertices).

According to [**Stu87**] when the oriented matroid is alternating, then its cyclic d-polytope is on a d-order curve in $\mathbb{R}^d$ and every subpolytope of it is also cyclic. This is quite a useful fortuity, since it is well known, that in odd dimensions there exist combinatorial cyclic polytopes with that some subpolytopes which are not cyclic (see page 104 of the same paper). By these facts, we know that if $\bar{S}$ is a set of points in general position in $\mathbb{R}^d$ with at least $R_{d+1}((n-1)(d+1)+1)$ points, then $\bar{S}$ contains a set S consisting of the $m = (n-1)(d+1)+1$ vertices of an ordered cyclic d-polytope $C_m(d)$.

To finish the proof we just need to "extend", as we did in the case of the plane, the partition given in Theorem 32 (for the vertices of $C_m(d)$) to a partition $\bar{P}$ of $\bar{S}$ in such a way that the nerve $\mathcal{N}(\bar{\mathcal{P}})$

is preserved. Lemma 4 below guarantees that this is always possible, finishing the proof of Theorem 11 (A).

Lemma 4. *Let T_n be a given tree and let S be the vertices of an ordered cyclic polytope with $m = (n-1)(d+1)+1$ vertices in $\mathbb{R}^d$. Then the specific partition $\mathcal{P}$ obtained in Theorem 32 is extendable.*

Proof. Let $\bar{S}$ be an arbitrary finite set of points in $\mathbb{R}^d$, such that $S \subset \bar{S}$. Let S_j denote the set of points in S of color v_j, or j for simplicity. Let us begin by assuming that the "color partition function" $\mathcal{C} : S \to [n]$, given in Theorem 32, yields a partition $\mathcal{P}$ of S with nerve $\mathcal{N}(\mathcal{P})$ isomorphic to T_n. The extension of $\mathcal{P}$ of S will be given by induction on the number of nodes n.

a) In the case $n = 1$ assign $\bar{\mathcal{C}_1}(\boldsymbol{x}) = 1$ for every point in $\bar{S}$.
b) For the induction step note that the induction hypothesis guarantees the extension $\bar{\mathcal{C}}_{n-1} : \bar{S} \to [n-1]$ exists.

To begin observe that polytopes Q and R defined in Theorem 32 satisfy that $Q \cap R = \{\boldsymbol{x}_k\}$ so $(Q \setminus \{\boldsymbol{x}_k\}) \cap (R \setminus \{\boldsymbol{x}_k\}) = \emptyset$. Therefore, there exists a $(d-1)$-hyperplane H that separates these two sets and leaves points of color r in both sides of the hyperplane. Furthermore, R is completely contained in the closure of one of the sides of this hyperplane, say H_n^-. The "color partition function" $\bar{\mathcal{C}_n} : \bar{S} \to [n]$ is given as follows:

$\bar{\mathcal{C}_n}(\boldsymbol{x}) = \bar{\mathcal{C}}_{n-1}(\boldsymbol{x})$ if $\boldsymbol{x} \in H_n^-$, $\bar{\mathcal{C}_n}(\boldsymbol{x}) = r$ if $\boldsymbol{x} \in S_r$, and $\bar{\mathcal{C}_n}(\boldsymbol{x}) = n$ if $\boldsymbol{x} \in \mathrm{cl}(H_n^+)$ and $\boldsymbol{x} \notin S_n$.

As before $\mathrm{cl}(H_n^+)$ is the closed half-hyperplane containing only points in R of color n and color r.

Observe that by the induction process the intersection pattern of $\bar{S}_1, \dots \bar{S}_{n-1}$ is the same in ${H_n}^-$ by construction, and $\mathrm{cl}(H_n^+) \cap S_r \neq \emptyset$ yields the leaf $\{v_r, v_n\}$. Furthermore $S_n \subset \mathrm{cl}(H_n^+)$ does not intersect any other elements in the partition since they are contained in H_n^-, so no further intersections occur.

$\square$

3.1.3. Proof of Theorem 11 (B). Suppose that $\bar{S}$ is a set of at least $nd + n + 4d$ points in general position in $\mathbb{R}^d$. We start by projecting the points onto a generic 2-plane H where we can assume, without loss of generality, that the points of $\bar{S}$ have distinct projections onto it. Let S be the projection of $\bar{S}$, now planar points.

LEMMA 5. *There exists a circle C containing all these projected planar points in S, and a subdivision C' of C into n sectors such that:*

(i) Each sector contains at least $d+1$ points.

(ii) No two adjacent sectors form a combined angle of more than π radians.

PROOF. We start by picking a line L_1 with at least $\lfloor \frac{nd+n+4d}{2} \rfloor$ points on both sides of L_1. Denote by L_1^- and L_1^+ respectively, the open half-spaces defined by L_1 and M_1^+, M_1^- the points of S on the two half-spaces of L_1. Applying the Ham Sandwich Theorem (see Section 1.3 in [**Mat02**]) to the sets M_1^- and M_1^+, we can find a line L_2 so that L_1 and L_2 together separate the plane into four regions, say R_1, R_2, R_3 and R_4 with at least$\lfloor \frac{nd+n+4d}{4} \rfloor$ projected points in each region. Note that $\lfloor \frac{nd+n+4d}{4} \rfloor \geq d+1$ points.

Denote by $\boldsymbol{p}$ the point in the plane where L_1 and L_2 intersect, and let C be a circle centered at $\boldsymbol{p}$ that contains all the projected points. Now we choose arcs emanating from $\boldsymbol{p}$ to subdivide each of the four regions R_1,R_2,R_3, and R_4 into as many subregions, containing at least $d+1$ points (note that each of the R_i have at least $d+1$ points in them by construction). This can be done as follows:

If R_i' has at least $2d+2$ points, then take a line emanating from $\boldsymbol{p}$ and rotate it until it divides R_i into two regions, one with $d+1$ points denoted R_{i1}, and the other with the remaining (at least $d+1$) points in R_i denoted R_i'. Otherwise do nothing. Repeating this process as many times as possible, we will obtain a subdivision of each R_i into subregions, all but one of which have exactly $d+1$ points, and none of which have more than $2d+1$ points. We call the final regions of this recursive process sectors.

Since the original four regions R_1, R_2, R_3, R_4 satisfy property claim (ii) of the lemma, and process of subdivision is made to show claim (i) holds after subdividing the four regions, all we have left to do is to check there are n sectors. For this, let k_1, k_2, k_3, and k_4 denote the respective number of sectors formed from each of the four regions, and j_1, j_2, j_3 and j_4 denote the number of points in each region. It suffices to show that $k_1 + k_2 + k_3 + k_4 \geq n$ because we can always merge adjacent sectors within the same region R_i, while preserving claims (i) and (ii).

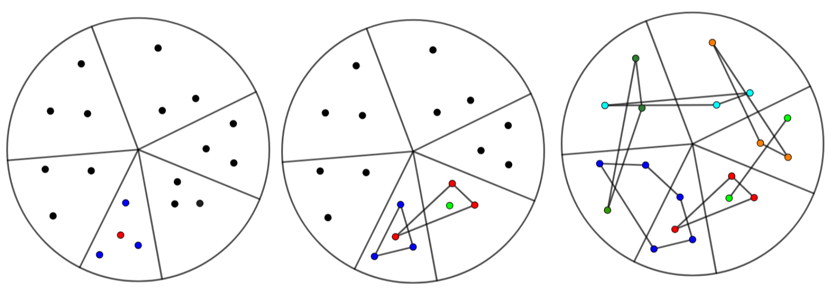
Our procedure for generating subdivisions guarantees that $j_i \leq k_i(d+1)+d$ for all $i = 1, 2, 3, 4$. Summing these inequalities we get $j_1 + j_2 + j_3 + j_4 \leq (k_1 + k_2 + k_3 + k_4)(d+1) + 4d$, so

$$nd + n + 4d \leq (k_1 + k_2 + k_3 + k_4)(d+1) + 4d,$$

which implies that $(k_1 + k_2 + k_3 + k_4) \geq n$. This completes the proof of the lemma. □

Now we will use the the subdivision C', whose existence is guaranteed by Lemma 5, to find our desired partition of the data points whose partition nerve is an n-cycle.

We construct a partition one sector at a time. In the first step, we notice that one of the n sectors, say Q_1, has at least $d+2$ points by the pigeonhole principle.
Use Radon's Lemma to partition the points in Q_1 into two sets S_1 and S_2 so that the convex hulls of S_1 and S_2 intersect.



From left to right: Illustration of the construction at steps 1 and 2, and the resulting partition.

In the second step, we denote the slice counterclockwise to Q_1 as Q_2.
By Radon's Lemma, any point $\boldsymbol{x}_2$ in S_2 from step 1, combined with the (at least) $d+1$ points in Q_2 can be partitioned into two sets S_2' and S_3 so that the convex hulls of S_2 and S_3 intersect. Without loss of generality we can assume that $\boldsymbol{x}_2 \in S_2'$, and then set $S_2 = S_2 \cup S_2'$. In step k, where $3 \leq k \leq n-1$, we continue in the same way. We denote the slice counterclockwise to Q_{k-1} as Q_k. By Radon's Lemma, any point $\boldsymbol{x}_k$ in S_k from step $(k-1)$, combined with the (at least) $d+1$ points in Q_{k+1} can be partitioned into two sets S_k' and S_{k+1} so that the convex hulls of S_k' and S_{k+1} intersect. Without loss of generality we can assume that $\boldsymbol{x}_{k-1} \in S_{k-1}'$, and then set $S_k = S_k \cup S_k'$. Finally, in step $n-1$ we set $S_1 = S_1 \cup S_n$.

We claim that the nerve of the resulting partition $\mathcal{P} = \{S_1, S_2, \ldots, S_n\}$ is the n-cycle.
This is a consequence of two facts: We used Radon's Lemma to guarantee that any two subsets

appearing in the same sector have intersecting convex hulls. Each subset appears in at most two sectors, and since two adjacent sectors have a combined angle of at most π radians there is a line separating any two subsets that do not appear in the same sector. Thus we have that $\mathrm{conv}(S_i) \cap \mathrm{conv}(S_j) \neq \emptyset$ if and only if there is some sector containing points from both S_i and S_j. If we let $\mathbf{v}_i$ denote the vertex of the nerve corresponding to subset S_i, we see that the edges of $\mathcal{N}(\mathcal{P})$ consist precisely of $(\mathbf{v}_n, \mathbf{v_1})$ and $(\mathbf{v_i}, \mathbf{v_{i+1}})$ where $i \in [n-1]$. This completes the proof.

3.2. Improved Tverberg numbers of special trees and in low dimensions

3.2.1. Proof of Theorem 12: Better bounds for Tverberg numbers of caterpillar trees. To make the notation easier, we adopt the following convention throughout the proof of Theorem 12: All point sets $S \subset \mathbb{R}^d$ are indexed in increasing order with respect to their first coordinate. That is, if $S = \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, with $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{id})$, then we assume that $x_{11} \leq x_{21} \leq \cdots \leq x_{n1}$. Furthermore, by rotating the axes, we can assume that no two points have the same first coordinate and that the previous inequalities are strict.

We first prove the special case of stars in Theorem 12 as a lemma. A caterpillar is a sequence of stars, thus we can later use induction again.

Lemma 6. *For any $(d+1)(n-1)+1$ points in $\mathbb{R}^d$, we can find a partition of those points with nerve St_n, the star tree on n vertices (i.e., with $(n-1)$ spokes).*

Proof of Lemma 6. We prove this by induction on n. For $n = 1$, the partition of one point to get St_1 is obvious. Now assume the result is true for some n. We need to show that any $(d+1)n+1$ points can be partitioned with partition nerve St_{n+1}. Let $M = (n-1)(d+1)+1$. By induction hypothesis, the subset $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M\} \subset S$ admits a partition $\mathcal{P} = \{A_1, \ldots, A_n\}$ with $\mathcal{N}(\mathcal{P}) \simeq St_n$. Without loss of generality, assume that A_1 is the central vertex of the star graph. Let $\boldsymbol{x} \in S$ be some point in A_1. By Radon's lemma, there is a way to partition the $d+2$ points $\boldsymbol{x}, \boldsymbol{x}_M, \boldsymbol{x}_{M+1}, \ldots \boldsymbol{x}_{M+d+1}$ into two sets X, Y with $\mathrm{conv}(X) \cap \mathrm{conv}(Y) \neq \emptyset$, and we can assume that $\boldsymbol{x} \in X$. The set Y intersects $A_1 \cup X$ but does not intersect any of A_i, $2 \leq i \leq n$, because every point in Y has larger first coordinate than any point in A_i. Then we see $\{A_1 \cup X, A_2, \ldots, A_n, Y\}$ is a partition which will induce the star graph St_n. $\square$

Proof of Theorem 12. Now we prove that for every caterpillar tree T_n with at most n nodes, every set S with at least $(d+1)(n-1)+1$ points in $\mathbb{R}^d$ admits a partition $\mathcal{P}$ with $\mathcal{N}(\mathcal{P}) \simeq T_n$. An illustration of the partition constructed in the proof is given in the Figure 3.6. The proof is by induction on the length of the central path in T_n, which we will denote by m. The induction hypothesis says that for every $m \in \mathbb{N}$ and any caterpillar tree T_n with n vertices and a central path of length m the following two statements hold:

(1) Every set S of $(d+1)(n-1)+1$ points in $\mathbb{R}^d$ admits a partition $\mathcal{P}$ with $\mathcal{N}(\mathcal{P}) \simeq T_n$

(2) Denote by v the last vertex of the central path, and denote by St_{k+1} the star subgraph induced by v and its k neighbors. Then the subsets in $\mathcal{P}$ corresponding to vertices in St_{k+1} are comprised of the $(d+1)k+d+1$ points in S with largest first coordinate.

If the length of the central path is one, both parts of the induction hypothesis follow by applying Lemma 6. Assume the result holds if the central path is of length m. We consider caterpillar graphs which have central paths of length $m+1$. Let G be such a graph with n vertices. We consider the endpoint of the path v_{m+1} and the vertex prior v_m. If we consider the subgraph of G consisting of the path $v_1, \ldots, v_m$ and all vertices adjacent to it except v_{m+1}, this is a caterpillar graph with a path of length m. Let p denote the number of vertices of this graph. By induction hypothesis, we can represent this graph using the $(d+1)(p-1)+1$ points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{(d+1)(p-1)+1}$. We will have the partition $\{A_1, \ldots, A_p\}$ where we take A_1 to be the set corresponding to v_m. Then take a point $\boldsymbol{x} \in A_1$ and the next $d+1$ points $\boldsymbol{x}_{(d+1)(p_1)+2}, \ldots, \boldsymbol{x}_{(d+1)(p_1)+d+2}$ to have a Radon partition X, Y with $\boldsymbol{x} \in X$. Our new partition will be $\{A_1 \cup X, A_2, \ldots, A_p, Y\}$. Y will correspond to the vertex v_{m+1} and will not intersect any of the other sets due to having larger first coordinate. In addition, $A_1 \cup X$ will not intersect any new sets by how we have arranged the points due to the induction hypothesis. Now as in the proof of the lemma, we can add new sets by considering $(d+1)$ points in iteration for each of the other vertices adjacent to v_{m+1}. Since there were $n-p$ vertices and we used $(d+1)$ points for each, in total we used $(d+1)(n-1)+1+(d+1)(n-p) = (d+1)(n-1)+1$ points. This is the desired number.

Thus we have proved the induction hypothesis. To complete the proof of the theorem, we note that if we have more than $(d+1)(n-1)+1$ points, we can apply the induction hypothesis to find the desired partition of the $(d+1)(n-1)+1$ points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_{(d+1)(n-1)+1}$, then add any remaining points to the subset corresponding to the endpoint of the central path in the caterpillar graph. $\square$

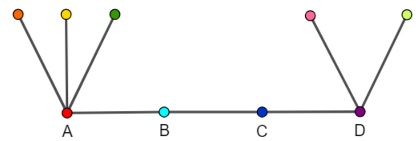


FIGURE 3.5. An example caterpillar graph G with 9 vertices.

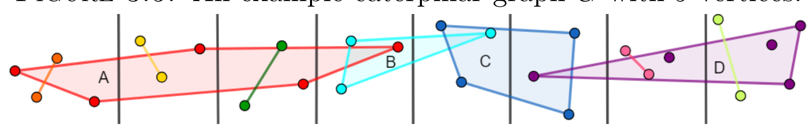


FIGURE 3.6. An example of how a set of points can be partitioned with nerve G. The vertical lines indicate how we start with a Radon partition of the leftmost $d+2$ points, then partition the points from left to right, considering $d+1$ more points at each step. Notice there are extra points on the right, which are added to the subset corresponding to the last vertex on the central path.

3.2.2. Proof of Theorem 13: Tverberg numbers of trees in dimension two. Now we focus on the situation in two dimensions.

LEMMA 7. *Let $S \in \mathbb{R}^2$ be a set of $2n$ points in the plane. Let $L_{\boldsymbol{p}_1\boldsymbol{p}_2}$ denote the line segment between points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Suppose that there exists $\boldsymbol{p}_1, \boldsymbol{p}_2 \in X$ such that $L_{\boldsymbol{p}_1\boldsymbol{p}_2}$ divides the remaining points into two sets A, B each of size $n-1$ and such that for any $\boldsymbol{a} \in A, \boldsymbol{b} \in B$, we have that $L_{\boldsymbol{ab}}$ intersects $L_{\boldsymbol{p}_1\boldsymbol{p}_2}$. Then it is possible to pair off elements $\boldsymbol{a}_i \in A$, $\boldsymbol{b}_i \in B$, such that for $i,j = 1,\ldots,n-1$, $i \neq j$, $L_{\boldsymbol{a}_i\boldsymbol{b}_i}$ does not intersect $L_{\boldsymbol{a}_j\boldsymbol{b}_j}$.*

PROOF. Suppose we have points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as hypothesized and partition the remaining points into A and B. Let L be the line between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. To pair off the points, we consider the vertices of $conv(A \cup B)$. Since L separates the points of A and B, we must have that there are a pair of adjacent vertices of $conv(A \cup B)$ such that one, $\boldsymbol{a}_1$, is a member of A and the other $\boldsymbol{b}_1$, a member of B. The segment between this pair cannot intersect the segment between any other pair of points as this segment forms the boundary of the convex hull. We pair off these two points and then consider $conv(A \setminus \{\boldsymbol{a}_1\} \cup B \setminus \{\boldsymbol{b}_1\})$. We see that L separates $A \setminus \{\boldsymbol{a}_1\}$ and $B \setminus \{\boldsymbol{b}_1\}$, so we can repeat this argument to pair off $\boldsymbol{a}_2$ and $\boldsymbol{b}_2$. Continuing in this fashion until we have paired off all the elements, we will have a pairing $(\boldsymbol{a}_1, \boldsymbol{b}_1), (\boldsymbol{a}_2, \boldsymbol{b}_2), \ldots, (\boldsymbol{a}_n, \boldsymbol{b}_n)$ where $L_{\boldsymbol{a}_i\boldsymbol{b}_i}$ does not intersect $L_{\boldsymbol{a}_j\boldsymbol{b}_j}$ for $i \neq j$. $\square$

Proof of Theorem 13 part (A). Let $A \in \mathbb{R}^2$ be a collection of $2n$ points in general position in the plane. Our goal will be to find a pair of points which can separate the remaining points into two sets of equal size so we can apply the above lemma. This will not always be possible, so we will try to make the size of the two sets as close as possible.

To do this, we will consider the vertices of the convex hull of A. We pick arbitrarily a vertex $\boldsymbol{p}_1$ of $conv(A)$ and order the remaining vertices $\boldsymbol{p}_2, \ldots, \boldsymbol{p}_k$ in counter-clockwise order where k is the number of vertices. For $i = 2, \ldots, k$, we divide the remaining vertices of A into two sets B_i, C_i where B_i is the set of vertices in A to the left of $L_{\boldsymbol{p}_1\boldsymbol{p}_i}$ and C_i is the set of vertices to the right of $L_{\boldsymbol{p}_1\boldsymbol{p}_i}$. We note that the size of B_i decreases from $2n-2$ to 0 as i increases and the size of C_i increases from 0 to $2n-2$.

We consider two cases. The first case is that there exists i such that $|B_i| = |C_i| = n-1$ and then we can apply the above lemma as the line segment between every pair of points in $B_i \times C_i$ intersects $L_{\boldsymbol{p}_1\boldsymbol{p}_i}$ since $L_{\boldsymbol{p}_1\boldsymbol{p}_i}$ separates B_i and C_i and $\boldsymbol{p}_1, \boldsymbol{p}_i$ are vertices of conv(A). Then we have a pairing $(\boldsymbol{b}_1, \boldsymbol{c}_1), \ldots, (\boldsymbol{b}_{n-1}, \boldsymbol{c}_{n-1})$ where for any two pairs the segments do not intersect, but each intersects $L_{\boldsymbol{p}_1\boldsymbol{p}_i}$. Then the partition $\{\{\boldsymbol{b}_1, \boldsymbol{c}_1\}, \ldots, \{\boldsymbol{b}_{n-1}, \boldsymbol{c}_{n-1}\}, \{\boldsymbol{p}_1, \boldsymbol{p}_i\}\}$ is a partition which induces the star graph S_n. For an example of this case and how to partition the points, see Figure 3.7.

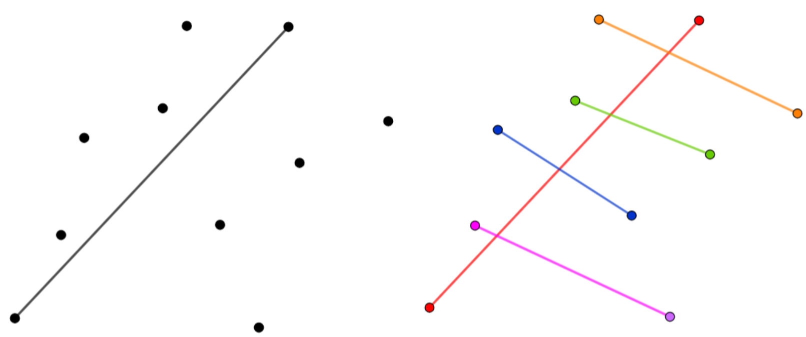


Figure 3.7. In the first case, there is a partition which divides the remaining points into two sets of equal size. Then we can pair off points such that the segment connecting them intersects the dividing line, but no other segment.

The second case is that there does not exist such an i. In this case, we find i such that $|B_i| > |C_i|$ and $|B_{i+1}| < |C_{i+1}|$. Set $D = \{\boldsymbol{p}_1, \boldsymbol{p}_i, \boldsymbol{p}_{i+1}\}$ and notice that conv(D) must contain at least one point of S in it's interior. D will form the center vertex of our star graph. See Figure 3.6 for a depiction of this central triangle.

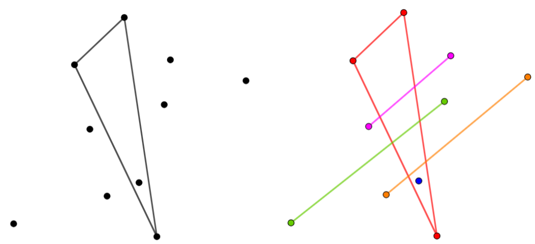


FIGURE 3.8. In the second case, we find a central dividing triangle of a given point configuration. Then we pair off as many points on opposite sides of the triangle as possible using Lemma 7, and make points in the interior of the triangle singletons until we have n subsets. Any extra points are added to the subset containing the central dividing triangle.

Now, using the above lemma, pair off points from B_{i+1} and C_i to form disjoint segments which will intersect $conv(D)$, and let every point in the interior of D be a singleton (which will not intersect any of the segments since the points are in general position). Denote this partition as $\mathcal{P}$.

$\mathcal{N}(\mathcal{P})$ is clearly a star graph, so it suffices to show $\mathcal{N}(\mathcal{P})$ has at least n nodes (we can merge the subsets corresponding to any extra nodes with D, as $conv(D)$ already intersects every subset). To see this, first note that average number of points in each subset of $\mathcal{P}$ is at most two, since $\mathcal{P}$ has one subset of size three, at least one singleton, and the rest of the subsets are either singletons or pairs. On the other hand, the average number of points in each subset is equal to $2n$ divided by the number of subsets, so there must be at least n subsets in $\mathcal{P}$. Thus $\mathcal{N}(\mathcal{P})$ has at least n nodes, as was to be shown.

$\square$

Now we move to the proof of Theorem 13 part (B): As a consequence of Lemma 2, when enumerating partition induced graphs it is enough to consider the partitions of combinatorial types of point sets. We can check whether a given simplex complex is 2-partition induced on a representative for each order type.

To complete part (B) we relied on an explicit computer enumeration of all order types on small points set provided by [**AAK02**]. There exists exactly one point configuration for which it is impossible to generate P_4. This point configuration is displayed in Figure 3.9. Its specific coordinatization is $A(222, 243)$, $B(238, 13)$, $C(131, 50)$, $D(154, 105)$, $E(166, 145)$,$F(134, 106)$ $G(174, 188)$, $H(18, 51)$. For every other point configuration, we found a partition which induced the path graph P_4. From this we assert that $\mathrm{Tv}(P_4, 2) \geq 9$. Since we also found a partition inducing P_4 for every single order type on nine points we are sure that $\mathrm{Tv}(P_4, 2) = 9$ because in the case 10 or more points, we can use the weaker bound given in the proof of Theorem 12 part (B).

Similarly for the cycle C_4 we have the configuration with coordinates $A(0,0)$, $B(8,5)$, $C(18,3)$, $D(7,4)$, $E(14,5)$, $F(10,8)$, $G(11,7)$, $H(14,17)$, $I(11,6)$, $J(12,12)$, which gives the desired lower bound. The upper bound is given by following the proof of Theorem 11 part (B), except starting with any set of 13 points (the bound given in the theorem is higher since it accounts for divisibility issues that can occur in certain cases).

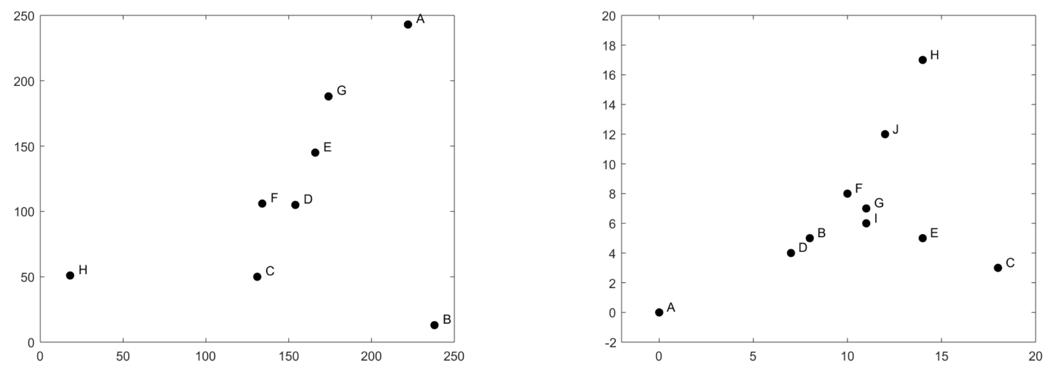


FIGURE 3.9. Two point configurations which cannot be partitioned to induce, respectively, P_4 (top on eight points) nor C_4 (bottom on ten points).

3.3. An important example and some auxiliary results

PROOF OF LEMMA 1. Suppose by contradiction $\mathrm{Tv}(K, d) < 2n$. Let $S \subset \mathbb{R}^d$ be a set of points in convex position with $|S| = \mathrm{Tv}(K, d)$. By the pigeonhole principle, if we partition S into n disjoint subsets, there must be at least one subset that is a singleton $\{\boldsymbol{x}\}$. Since K is connected, the node corresponding to the singleton $\{\boldsymbol{x}\}$ is connected, by an edge, to at least one other node, implying that $\{\boldsymbol{x}\}$ is in the convex hull of another subset. However, this is a contradiction as the points are in convex position. $\square$

PROOF OF LEMMA 2. To show that $\mathcal{N}^1(\mathcal{P}) = \mathcal{N}^1(\sigma(\mathcal{P}))$ it suffices to show that $\mathrm{conv}(P_{i_1}) \cap \mathrm{conv}(P_{i_2}) \neq \emptyset$ if and only if $\mathrm{conv}(\sigma(P_{i_1})) \cap \mathrm{conv}(\sigma(P_{i_2})) \neq \emptyset$ for all $i_1, i_2 \in [n]$. Suppose $\mathrm{conv}(P_{i_1}) \cap \mathrm{conv}(P_{i_2}) \neq \emptyset$. Then they contain respectively P'_{i_1} and P'_{i_2} which are an inclusion minimal Radon partition of S_1. Since σ is an order-preserving bijection, σ is an isomorphism between oriented

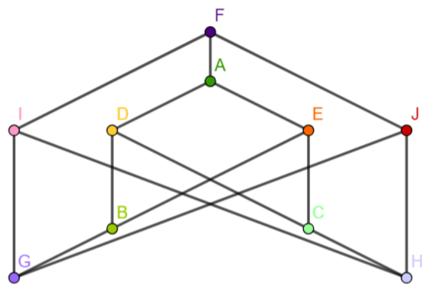


FIGURE 3.10. Example of a graph K which is not 2-Tverberg.

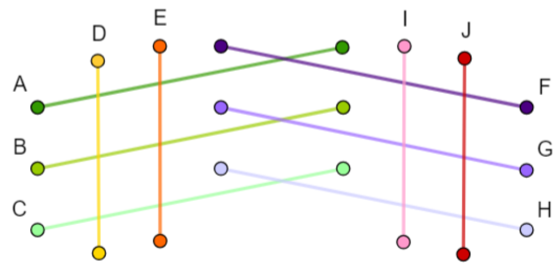


FIGURE 3.11. Partitioned point set with nerve K.

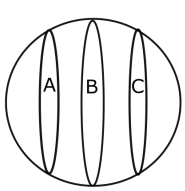
matroids (see, for instance [**RGZ97**]) determined by S_1 and S_2. The minimal Radon partitions in S_1 correspond to the circuits of the oriented matroids and therefore are preserved under σ. Thus $\mathrm{conv}(\sigma(P'_{i_1})) \cap \mathrm{conv}(\sigma(P'_{i_2})) \neq \emptyset$. The reverse implication is shown by the reasoning applied to σ^{-1}. $\square$

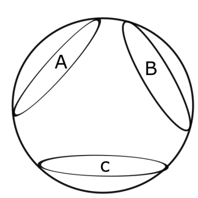
As we mentioned in the Introduction, the graph K in Figure 3.10 is 2-partition induced (in particular by the partitioned point set in Figure 3.11), but not 2-Tverberg, as implied by the following lemma:

LEMMA 8. *Suppose S is any set of points in convex position in $\mathbb{R}^2$. Then the graph K in Figure 3.10 is not partition induced on S.*

PROOF. We note that since K is a triangle free graph, it suffices to show that it is not the intersection graph of any partition of points in convex position. We argue by contradiction. Suppose that there is a set of points in convex position partitioned so that they have the graph above as their intersection graph. By Lemma 2 we may assume the points are arranged on the boundary of a disc $\mathcal{D}$. Denote the convex hull of the points corresponding to each node i by region i. In the rest of the proof of Lemma 8, we will rely on the following.

CLAIM 33. *Consider the independent set of nodes $\{A, B, C\}$ in Figure 3.10. Up to exchanging their labels (note that the graph is symmetric about A, B, C), there are two possible arrangements of the regions A,B, and C, pictured in Figures 33 and 33.*





OF THE CLAIM. The region $\mathcal{M} - B$ has two connected components. If regions A and C lie in different connected components of $\mathcal{M} - B$, then regions A, B, and C must be arranged as in Figure 33. Otherwise, A and C lie in the same connected component, say $\mathcal{N}$, of $\mathcal{M} - B$. If we walk clockwise around the boundary of $\mathcal{N}$, we can only alternate twice between being in regions A and C, reducing to the two possibilities shown. $\square$

By the claim, we see that A, B, and C must be arranged (up to symmetry) as in one of the two cases pictured above. If they are arranged as in Figure 33, note that regions E and F both intersect regions A, B, and C. In that case it is easy to see that regions E and F must intersect, which is a contradiction.

If the regions are arranged as in Figure 33, consider that regions D, F, G, and H. Note that region D intersects regions A, B, C. Also, region F is disjoint from all the regions B through H, while intersecting region A. Similarly, region G is disjoint from all the regions A through H except B. Also region H is disjoint from all the regions A through H except C. Considering the two cases: F, G, H lie in the same connected component of $\mathcal{M} - D$, or F, G, H lie in different connected components of $\mathcal{M} - D$, it is easy to see that, in both cases, F, G, and H must be arranged as A, B, and C are in Figure 33. Then I, J are disjoint but both intersect F, G and H, which is a contradiction by the argument above. Thus K cannot be the nerve of a set of points in convex position. $\square$

CHAPTER 4

Stochastic Tverberg theorems

The main results proved in this chapter provide bounds on the probability that a randomly partitioned point set has the property that the convex hulls of each subset have at least one point in common. In particular, we show that in the case of a distribution that is balanced about a point, there is a threshold phenomena for such an event. In that case, roughly speaking, a partition of mn random points into m sets of n points each is likely to be a Tverberg partition if $n > \log_2(m)$, and unlikely to be a Tverberg partition if $n < \log_2(m)$.

4.1. Stochastic Tverberg theorems for equi-partitioned points

Proof of the lower bound in Theorem 18. After a possible translation, can assume without loss of generality that D is balanced about the origin. We will prove that

$$\left(1 - \left(2^{-n+1}\sum_{k=0}^{d-1}\binom{n-1}{k}\right)\right)^m \leq \mathbb{P}(\mathcal{P}_{m,n,D} \text{ is Tverberg})$$

by bounding from below the probability that the origin is a Tverberg point. We may assume without loss of generality that none of the randomly selected points are at the origin. The origin is then a Tverberg point as long as the points from each color contain the origin in their convex hull. This is equivalent to showing no color has all of its points contained in one hemisphere. For a fixed color, the probability of the n points of that color being contained in one hemisphere was computed by Wagner and Welzl [**WW01**]. This result of Wagner and Welzl generalized the celebrated result of Wendel [**Wen62**] who addressed the case when D is centrally symmetric. (Moreover, in the same paper Wagner and Welzl showed that this probability is at least the probability given in Equation 4.1 below, with equality if and only if D is balanced about the origin.)

$$\left(2^{-n+1}\sum_{k=0}^{d-1}\binom{n-1}{k}\right). \tag{4.1}$$

Using this to compute the probability that none of the m color classes is contained in one hemisphere we obtain the desired bound above.

□

Proof of the upper bound in Theorem 18. Again, we assume without loss of generality that D is balanced about the origin. We will first treat the case $d = 1$, and then explain how to obtain the bound for arbitrary d. To bound the probability of a Tverberg partition from above, we bound the probability of the complement below. We let E denote the event that the convex hulls have empty intersection. In dimension one, E is contained in the event that there is at least one color class with all points less than zero, and at least one color class with all points greater than zero. Since we assume that the origin equipartitions D, we can rephrase this as the probability that among m people each flipping n fair coins, there is at least one person with all heads and at least one person with all tails. In other words, denoting by H and T the events that at least one person gets all heads or tails respectively, we have $\mathbb{P}(E) \geq \mathbb{P}(H \cap T)$. We have

$$\mathbb{P}(H \cap T) = \mathbb{P}(H) + \mathbb{P}(T) - \mathbb{P}(H \cup T) = \mathbb{P}(H) + \mathbb{P}(T) - (1 - \mathbb{P}((H \cup T)^c).$$

Since $\mathbb{P}(H) = \mathbb{P}(T) = (1 - 2^{-n})^m$ and $\mathbb{P}((H \cup T)^c) = 1 - (1 - 2^{-n+1})^m$, this yields

$$\mathbb{P}(H \cap T) = 1 + (1 - 2^{-n+1})^m - 2(1 - 2^{-n})^m.$$

The probability of a Tverberg partition is thus bounded as follows

$$\mathbb{P}(\mathcal{P}_{m,n,D} \text{ is Tverberg}) \leq 1 - \mathbb{P}(E) \leq 1 - \mathbb{P}(H \cap T) = 2(1 - 2^{-n})^m - (1 - 2^{-n+1})^m.$$

This proves the desired bound for dimension 1. For higher dimensions, we note that if we let p_i, denote the projection onto the i-th axis for $i \leq d$, we have that the signs of $p_1(x), \ldots, p_d(x)$ are independent Bernoulli random variables with probability $1/2$ (as the hyperplane orthogonal to the i-th axis equipartitions D by the assumption that D is balanced about the origin). Thus to have a Tverberg partition, we must have that no pair of the color classes are separated by the origin after projecting onto the d coordinate axes. Since these d events are independent, the probability of this happening is bounded as follows.

$$\mathbb{P}(\mathcal{P}_{m,n,D} \text{ is Tverberg}) \leq \left(2(1-2^{-n})^m - (1-2^{-n+1})^m\right)^d.$$

$\square$

4.2. Threshold phenomena for Tverberg partitions

Proof of Theorem 19. We will show that $\mathcal{P}_{m,f(m),D}$ is Tverberg with high probability if $f(m) > \ln(m)/\ln(2)$. Fix an $\epsilon > 0$. We set $n = (1+\epsilon)\log_2(m)$ and apply the lower bound in Theorem 18 to deduce that

$$\mathbb{P}(\mathcal{P}_{m,n,D} \text{ is Tverberg }) \geq \left(1 - \left(2^{-(1+\epsilon)*\log_2(m)+1} \sum_{k=0}^{d-1} \binom{n-1}{k}\right)\right)^m$$

$$= \left(1 - \left(2m^{-(1+\epsilon)} \sum_{k=0}^{d-1} \binom{n-1}{k}\right)\right)^m.$$

Choosing C so that $Cn^d \geq 2\sum_{k=0}^{d-1}\binom{n-1}{k}$, we have

$(1 - Cn^d m^{-(1+\epsilon)})^m \leq \mathbb{P}(\mathcal{P}_{m,n,D} \text{ is Tverberg })$.

We will show that the limit as m approaches infinity of the left hand side is bigger than $e^{-\delta}$ for any $\delta > 0$. Fix $\delta > 0$. As $n^d \sim O(\ln(m)^d)$, there exists an M such that $Cn^d m^{-\epsilon} < \delta$ for all $m \geq M$. Consequently $(1 - Cn^d m^{-(1+\epsilon)})^m > (1-\delta m^{-1})^m$ for all $m \geq M$. Thus

$$\lim_{m\to\infty}(1 - Cn^d m^{-(1+\epsilon)})^m \geq \lim_{m\to\infty}(1-\delta m^{-1})^m = e^{-\delta}.$$

Since δ was arbitrary, we see that the probability of a Tverberg partition tends to 1.

Now we show that $\mathcal{P}_{m,f(m),D}$ is not Tverberg with high probability if $f(m) < \log_2(m)$. As before, we fix an ϵ greater than zero apply the upper bound in Theorem 18 with $n = (1-\epsilon)log_2(m)$ to obtain

$$\mathbb{P}(\mathcal{P}_{m,n,D} \text{ is Tverberg }) \leq \left(2(1-m^{-1+\epsilon})^m - (1-2m^{-1+\epsilon})^m\right)^d.$$

For any $\gamma > 0$, when m is large, both terms inside the parentheses are smaller than $(1-\gamma m^{-1})^m$. Since $\lim_{m\to\infty}(1-\gamma m^{-1})^m = e^{-\gamma}$, the probability of a Tverberg partition converges to zero as m approaches infinity.

□

Proof of Theorem 20. Again, we assume without loss of generality that D is balanced about 0. Let S denote the set of points of some fixed color. Then we assume that $|S| = n$, and we can partition S into $\lfloor n/2d \rfloor$ subsets $S_1, \dots, S_{\lfloor n/2d \rfloor}$ with $S_i \geq 2d$ for each i. By Wagner and Welzl's result (Equation 4.1 above), for each i, $\mathrm{conv}(S_i)$ contains the origin with probability at least 1/2. By independence, the probability that less than $t+1$ of the S_i contain the origin is less than $2^{-\lfloor n/2d \rfloor} \sum_{i=1}^{t} \binom{\lfloor n/2d \rfloor}{i}$. On the other hand, if at least $t+1$ of the $\mathrm{conv}(S_i)$ contain the origin, then by pigeonhole principle $\mathrm{conv}(S \setminus \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_t\})$ contains the origin for any $\boldsymbol{x}_1, \dots \boldsymbol{x}_t \in S$. Thus, with probability at least $1 - 2^{-\lfloor n/2d \rfloor} \sum_{i=1}^{t} \binom{\lfloor n/2d \rfloor}{i}$, we have that $\mathrm{conv}(S) \setminus \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_t\})$ contains the origin. Since this probability is independent for each of the m colors, the result follows. □

Using a similar strategy combined with Cover's result, we give the proof of Theorem 21 below.

Proof of Theorem 21. Given k points in $\mathbb{R}^d$ colored red and blue by random allocation, we arbitrarily partition them into $\lfloor k/(2d+2) \rfloor$ groups of size at least $2d+2$. By Cover's result, for each fixed group, the convex hulls (of the red and blue points) in that group intersect with probability at least 1/2. For each of the $\lfloor k/(2d+2) \rfloor$ groups, we think of the event that the convex hulls in that group intersect as a "success". Then the probability that at least $t+1$ groups have intersecting convex hulls is bounded below by the probability that a binomial process with $\lfloor k/(2d+2) \rfloor$ trials and success probability 1/2 has at least $t+1$ total successes. Computing this binomial probability yields the theorem. (If at least $t+1$ groups have intersecting convex hulls, then removing at most t points leaves at least one group with intersecting convex hulls.) □

4.3. Stochastic Tverberg theorems for randomly allocated points

Proof of Corollary 1. We split the proof according to the three respective parts of the statement.

(1) The probability that a random allocation of k points into m colors is an m-Tverberg partition with tolerance t is bounded below by the probability that a random allocation of k points into m colors has at least n points per color, times the probability that an equipartition of nm points into m colors is Tverberg with tolerance t. The result for Tverberg with tolerance then follows from Theorem 20.

(2) The result for the special case of Tverberg without tolerance then follows the same reasoning as part (1), except using Theorem 18 in place of Theorem 20.

(3) To show the asymptotic result, we use a result on urn models due to Erdös and Renyi [**ER61**] saying that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{N_m(n)}{n} < \log(n) + (m-1)\log(\log(n)) + x\right) = exp\left(-\frac{e^{-x}}{(m-1)!}\right).$$

This implies that for any $\epsilon > 0$ and sequence of $\log(\log(m))\log_2(m)(1+\epsilon)$ points allocated into m urns, we have at least $\log_2(m)(1+\epsilon/2)$ points in each urn with high probability. Then we apply Theorem 19, which says that any equi-partition of a point set into m colors and $\log_2(m)(1+\epsilon/2)$ points per color is Tverberg with high probability.

□

CHAPTER 5

Tverberg-type theorems in discrete sets

In this chapter we prove Tverberg-type results for points in discrete sets. These results build on prior work in discrete geometry, where much work has been done to extend the classic "pillars of convexity" - Helly's, Radon's, and Carathéodory's theorems to a wider class of convexity spaces, see [**DLGMM19**, **QT17**]. Among these, Radon's lemma seems the most difficult to obtain tight bounds for in other convexity spaces. The more general problem of determining Tverberg numbers in other convexity spaces is difficult, even if the Radon numbers are known (see [**Eck00**] and [**Buk10**]). Thus we preface the main proofs with a discussion of some of the prior work on the generalization of Radon's lemma and Tverberg's theorem to discrete spaces, as well as introducing some important related results.

5.0.1. Preliminaries. The problem of computing the Tverberg number for $\mathbb{Z}^d$ with $d \geq 3$ seems to be challenging. It has been identified as an interesting problem since the 1970's [**GS79**] and yet the following inequalities are almost all that is known about this problem: for the general case, J. A. De Loera et al. [**DLLHRS17**] proved

$$2^d(m-1)+1 \leq \mathrm{Tv}(\mathbb{Z}^d, m) \leq d2^d(m-1)+1, \quad \text{for } d \geq 1 \text{ and } m \geq 2. \tag{5.1}$$

Two special cases get better bounds:

$$\mathrm{Tv}(\mathbb{Z}^3, 2) \leq 17 \qquad \text{and} \qquad 5 \cdot 2^{d-2}+1 \leq \mathrm{Tv}(\mathbb{Z}^d, 2) \quad \text{for } d \geq 1. \tag{5.2}$$

The left-hand side inequality is due to K. Bezdek and A. Blokhuis [**BB03**] and the right-hand side was proved by J. P. Doignon in his PhD thesis (and rediscovered by Onn).

Previously established bounds for the "mixed integer" case include the bounds for the Radon number (2-Tverberg number) found by G. Averkov and R. Weismantel [**AW12**].

$$2^j(k+1)+1 \leq \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, 2) \leq (j+k)2^j(k+1)-j-k+2.$$

Later, J. A. De Loera et al. [**DLLHRS17**] gave the following general bound for all mixed integer Tverberg numbers:

$$\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq (j+k)2^j(m-1)(k+1)+1.$$

Note that (1.1) above is a simultaneous improvement of both of these.

Previous bounds and related work on more general S-Tverberg numbers can also be found in [**DLLHRS17**], including the following bound for any discrete $S \subset \mathbb{R}^d$:

$$\mathrm{Tv}(S, m) \leq \mathrm{H}(S)(m-1)d+1.$$

The following lemma about integer points of high half-space depth is used throughout the paper. See [**BO16**] for a proof and related results.

Lemma 9. *Consider a multiset A of points in $\mathbb{Z}^d$. If $|A| \geq 2^d(m-1)+1$ (counting multiplicities), then there is a point $\boldsymbol{q} \in \mathbb{Z}^d$ of half-space depth m in A.*

The proof of this result follows the standard proof of Rado's centerpoint theorem, except instead of Helly's theorem, we use the following version of Helly's theorem for the integer lattice.

Theorem 34 (J.P. Doignon 1973 [**Doi73**], D.E. Bell 1976 [**Bel76**], H.E. Scarf 1977 [**Sca77**]). *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If $(\bigcap \mathcal{K}) \cap \mathbb{Z}^d \neq \emptyset$ for all $\mathcal{K} \subset \mathcal{F}$ of cardinality at most 2^d, then $(\bigcap \mathcal{F}) \cap \mathbb{Z}^d \neq \emptyset$.*

For example, in the two dimensional lattice, the integer Helly theorem says that if any four convex sets in a larger collection contain a lattice point in common, then there must be a lattice point shared by every convex set in the collection. This beautiful result was discovered independently by J.P. Doignon, D.E. Bell, and H.E. Scarf. Key applications include fast non-deterministic algorithms for infeasibility certificates in integer and mixed integer optimization, and a short proof of a centerpoint theorem for the integer lattice. In our case, it serves as a foundational tool for the extension of Tverberg's theorem to discrete sets. For more generalizations, and further details see [**ADLS17**, **AGS⁺17**, **DLLHORP17**]

The paper is organized as follows. In Section 5.1, we prove Theorem 22 using a somewhat similar strategy to B. J. Birch's proof of the planar case of the original Tverberg theorem [**Bir59**]. In Section 5.2, we prove Theorem 23 using techniques reminiscent of those in [**DLLHRS17**]. In

Section 5.3, we prove Theorem 24 (adapting an approach by W. Mulzer and D. Werner [**MW13**, Lemma 2.3]) and collect some consequences of the main theorems presented above, including (1.1). Finally, in Section 5.4, we prove Theorem 26 by proving a new lemma and adapting the methods of J. Pach in [**Pac98**].

5.1. Tverberg numbers over discrete subsets of the plane

We start with the proof of the special case $S = \mathbb{Z}^2$ because it nicely illustrates the techniques of the more general proof of Theorem 22.

5.1.1. Proof of the special case $S = \mathbb{Z}^2$. The theorem will follow easily from the following two lemmas, the first covering the case $m \geq 3$ and the second the case $m = 2$.

LEMMA 10. *Consider a multiset A of points in $\mathbb{Z}^2$ with $|A| \geq 4m - 3$ and $m \geq 3$. If $\boldsymbol{p} \notin A$ is a point of depth m, then there is an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point.*

LEMMA 11. *Consider a multiset A of points in $\mathbb{Z}^2$ with $|A| \geq 6$. If $\boldsymbol{p} \notin A$ is a point of depth two, then there is a Radon partition of A with $\boldsymbol{p}$ as Tverberg point.*

PROOF OF SPECIAL CASE $S = \mathbb{Z}^2$. Consider a multiset A of at least $4m - 3$ points in $\mathbb{Z}^2$. By Lemma 9, A has an integer point $\boldsymbol{p}$ of depth m. If $\boldsymbol{p}$ is an element of A with multiplicity μ, then take the singletons $\{\boldsymbol{p}\}$ as μ of the sets in the Tverberg partition. Then $\boldsymbol{p}$ is a point of depth $m - \mu$ of the remaining $4m - \mu - 3$ points. If $\mu \geq m$, we are done, and if $\mu = m - 1$, the point $\boldsymbol{p}$ is in the convex hull of the remaining points and we take them to be the last set in the desired partition. If $\mu \leq m - 3$, according to Lemma 10, there is an $(m - \mu)$-Tverberg partition of the remaining points with $\boldsymbol{p}$ as Tverberg point. There is thus an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point. The case $\mu = m - 2$ is treated similarly with the help of Lemma 11 in place of Lemma 10. □

PROOF OF LEMMA 10. Since $\boldsymbol{p}$ is not in A, up to a radial projection, we can assume that the points of A are arranged in a circle around $\boldsymbol{p}$. Define q and r to be respectively the quotient and the remainder of the Euclidean division of $|A|$ by m. Define moreover e to be $\lceil \frac{r}{q} \rceil$.

Suppose first that $\boldsymbol{p}$ is a point of depth $m + e$. Since $qe \geq r$, we can choose k_i with $i \in [q]$, and $0 \leq k_i \leq e$, such that $k_1 + k_2 + \cdots + k_q = r$. Then we arbitrarily select a first point in A, and label

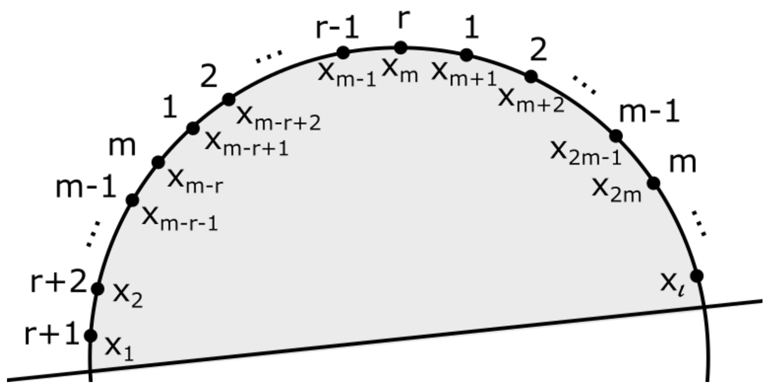


FIGURE 5.1. Labeling of the points in the half-plane H_-.

clockwise the points with elements in $[m]$ according to the following pattern:

$$1, 2, \ldots, m, 1, 2, \ldots, k_1, 1, 2, \ldots, m, 1, 2, \ldots, k_2, \ldots, 1, 2, \ldots, m, 1, 2, \ldots, k_q.$$

Each half-plane delimited by a line passing through $\boldsymbol{p}$ contains at least $m+e$ consecutive points in this pattern and thus has at least one point with each of the m different labels. Partitioning the points so that each subset consists of all points with a fixed label, we therefore obtain an m-Tverberg partition with $\boldsymbol{p}$ as Tverberg point.

Suppose now that $\boldsymbol{p}$ is not a point of depth $m+e$. There is thus a closed half-plane H_+, delimited by a line passing through $\boldsymbol{p}$, with $|H_+ \cap A| < m+e$. The complementary closed half-plane to H_+, which we denote by H_-, is such that $|H_- \cap A| > 4m - 3 - (m+e)$. Define ℓ to be $|H_- \cap A|$. Since $e \leq \frac{m}{3}$, we have $\ell \geq 2m$. Denote the points in $H_- \cap A$ by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell$, where the indices are increasing when we move clockwise. We label $\boldsymbol{x}_i$ with $r+i$ from $\boldsymbol{x}_1$ to $\boldsymbol{x}_{m-r}$, and then label $\boldsymbol{x}_{m-r+j}$ with j from $\boldsymbol{x}_{m-r+1}$ to $\boldsymbol{x}_m$. We then continue labeling the points of A, still moving clockwise, using labels $1, 2, \ldots, m, \ldots, 1, 2, \ldots m, 1, 2, \ldots r$. See Figure 5.1 for an illustration of the labeling scheme.

The labeling pattern is such that any sequence of m consecutive points either has all m labels, or contains the two consecutive points $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$. Let us prove that any closed half-plane H delimited by a line passing through $\boldsymbol{p}$ contains at least one point with each label. Once this is proved, the conclusion will be immediate by taking as subsets of points those with same labels, as above.

If such an H does not simultaneously contain $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$, then H contains at least one point with each label. Consider thus a closed half-plane H delimited by a line passing through $\boldsymbol{p}$ and containing $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$. Note that according to Farkas' lemma ([**Sch03**] Theorem 5.3) , $\boldsymbol{x}_{m+1}$ cannot be separated from $\boldsymbol{x}_1$ and $\boldsymbol{x}_\ell$ by a line passing through $\boldsymbol{p}$, since they are all in H_-. This means that either H contains $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m+1}$, or H contains $\boldsymbol{x}_{m+1}, \boldsymbol{x}_{m+2}, \ldots, \boldsymbol{x}_\ell$. In any case, H contains a point with each label. $\square$

Proof of Lemma 11. As before, we assume that the points in A are arranged on a circle centered at $\boldsymbol{p}$. If $|A|$ is even, it clearly suffices to label the points in order, alternating between 1 and 2. We may therefore assume that $|A|$ is odd, and thus $|A| \geq 7$. If $\boldsymbol{p}$ is a point of depth three, it suffices to label the points alternating labels between 1 and 2, except with two consecutive points labeled 1. If $|A|$ is odd but $\boldsymbol{p}$ is not a point of depth three, then $|A| \geq 7$ and there is a half-plane H_+ containing $\boldsymbol{p}$ with $|H_+ \cap A| = 2$. The complementary half-plane H_- has $|H_- \cap A| \geq 5$ and we follow a similar strategy as in the second half of Lemma 10. Namely, we denote the points in $H_- \cap A$ by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell$, where the indices are increasing when we move clockwise. Then we label $\boldsymbol{x}_1$ with 2, $\boldsymbol{x}_2$ with 1, $\boldsymbol{x}_3$ with 1, and $\boldsymbol{x}_4$ with 2. We continue this pattern for $\alpha \geq 5$, labeling $\boldsymbol{x}_\alpha$ with 1 if α is odd, and $\boldsymbol{x}_\alpha$ with 2 if α is even. For the remaining points in A we continue labeling clockwise, alternating between the labels 1 and 2.

The labeling pattern is such that any sequence of 2 consecutive points either has both labels, or contains the two consecutive points $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. As in Lemma 10 it suffices to show that any closed half-plane H delimited by a line passing through $\boldsymbol{p}$ contains at least one point with each label.

If such an H does not simultaneously contain $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, then H contains at least one point with each label. Consider thus a closed half-plane H delimited by a line passing through $\boldsymbol{p}$ and containing $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. Note that according to Farkas' lemma, $\boldsymbol{x}_3$ cannot be separated from $\boldsymbol{x}_1$ and $\boldsymbol{x}_4$ by a line passing through $\boldsymbol{p}$, since they are all in H_-. This means that either H contains $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, or H contains $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$. In any case, H contains a point with each label. $\square$

5.1.2. Proof of the general case.

The proof of the general case is split into three lemmas addressing the lower bound, the upper bound for $\mathrm{H}(S) \geq 4$, and the upper bound for $\mathrm{H}(S) \leq 3$, respectively.

Lemma 12. *For any discrete set $S \subset \mathbb{R}^2$ with finite Helly number $\mathrm{H}(S)$, we have $\mathrm{Tv}(S, m) > \mathrm{H}(S)(m-1)$.*

Proof. It suffices to exhibit a subset $R \subseteq S$, of cardinality $|R| = \mathrm{H}(S)(m-1)$, with the property that no point in S is of half-space depth m with respect to R. By Lemma 2.6 in [**ADLS17**], there exists a set R' of $\mathrm{H}(S)$ points in S in convex position with the property that $\mathrm{conv}(R') \cap S = R'$. Let R be the multiset given by taking each point in R' with multiplicity $(m-1)$, so $|R| = \mathrm{H}(S)(m-1)$. No points of $S - R$ are in $\mathrm{conv}(R)$. Since R' was taken to be in convex position, for any point in R, there exists a line such that one side of that line has at most $m-1$ points in R. Thus S cannot contain a point of half-space depth m with respect to R. $\square$

Lemma 13. *For any discrete set $S \subset \mathbb{R}^2$ with finite Helly number $\mathrm{H}(S) \geq 4$, we have $\mathrm{Tv}(S, m) \leq \mathrm{H}(S)(m-1)+1$ whenever $m \geq 3$. For the case $m = 2$, we have $\mathrm{Tv}(S, 2) \leq \mathrm{H}(S) + 2$.*

Proof. The proof of Lemma 13 is the same as the proof of Theorem 1. In particular, we can use Lemmas 2 and 3 as they are stated, except that we use the following result (Theorem 2 in [**BO16**] with μ being the uniform probability measure on A) in place of Lemma 9. For any discrete discrete subset S of a Euclidean space with finite Helly number $\mathrm{H}(S)$, and any set $A \subseteq S$ with $|A| \geq \mathrm{H}(S)(m-1)+1$, there exists a point $\boldsymbol{p} \in S$ that is of half-space depth m with respect to A. $\square$

Lemma 14. *For a discrete set $S \subset \mathbb{R}^2$ with finite Helly number $\mathrm{H}(S) \leq 3$, we have $\mathrm{Tv}(S, m) \leq \mathrm{H}(S)(m-1)+1$.*

Proof. The case $\mathrm{H}(S) = 1$ implies that S consists of a single point, so the result trivially follows. If $\mathrm{H}(S) = 2$, it must be that all points in S are collinear (as any set containing a non-degenerate triangle has Helly number at least 3), and thus we can take median of any set with at least $2(m-1)+1$ points in S as the desired m-Tverberg point. Thus for the remainder of the proof we assume that $\mathrm{H}(S) = 3$.

Given any set A of $\mathrm{H}(S)(m-1) = 3m-2$ points in S, there exists an m-Tverberg partition, say $\mathcal{P}$ by the classical Tverberg theorem. We denote by $K_1, \ldots, K_m$ the m convex hulls of the subsets in $\mathcal{P}$. As $\bigcap_{1 \leq i \leq m} K_i$ is a nonempty polygon, say Q, (possibly just a point or line segment) we pick an arbitrary vertex $\boldsymbol{q}$ of Q.

It suffices to show that $\boldsymbol{q} \in S$. We can assume that $\boldsymbol{q}$ is not a vertex of any K_i, since otherwise $\boldsymbol{q} \in A \subseteq S$.

Since $\boldsymbol{q}$ is a vertex of Q, it must be contained in a one dimensional face F_1 of at least one K_i. Since $\boldsymbol{q}$ is not a vertex of any K_i, in fact $\boldsymbol{q}$ is in the relative interior of F_1. For $\boldsymbol{q}$ to be a vertex of Q, it must also be in another one dimensional face, say F_2, of some other K_i, such that F_1 is not parallel to F_2. Moreover, $\boldsymbol{q}$ must be in the relative interior of F_2, and we also have $F_1 \cap F_2 = \{\boldsymbol{q}\}$.

Denote by $\{\boldsymbol{a}, \boldsymbol{b}\}$ and $\{\boldsymbol{c}, \boldsymbol{d}\}$ the vertices of F_1 and F_2 respectively. We have that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in S$ are the vertices of a convex quadrilateral with diagonals intersecting at $\boldsymbol{q}$, by the assumption that F_1 and F_2 are non parallel. Out of the four triangles $\mathrm{conv}(\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}), \mathrm{conv}(\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}), \mathrm{conv}(\{\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d}\}), \mathrm{conv}(\{\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\})$, any three have at least one vertex in common, and therefore intersect in S. Since $\mathrm{H}(S) = 3$, the four triangles therefore all intersect in S. This intersection point is $\boldsymbol{q}$, the point where the diagonals of the quadrilateral intersect. □

5.2. Tverberg numbers over the three-dimensional lattice

We state the following lemma without proof; it is a consequence, upon close inspection of the argument, of the proof of the main theorem in the already mentioned paper by K. Bezdek and A. Blokhuis [**BB03**].

LEMMA 15. *Consider a multiset A of at least 17 points in $\mathbb{R}^3$ and a point $\boldsymbol{p}$ of depth 3 in A. There is a bipartition of A into two subsets whose convex hulls contain $\boldsymbol{p}$.*

Next, we prove the following.

LEMMA 16. *Consider a multiset A of points in $\mathbb{R}^3$ with $|A| \geq 24m - 31$ and $m \geq 2$. If $\boldsymbol{p} \notin A$ is a point of depth $3m - 3$, then there is an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point.*

PROOF. Since $\boldsymbol{p}$ is not an element of A, we assume without loss of generality that the points of A are located on a sphere centered at $\boldsymbol{p}$, as in the proof of Theorem 22.

We claim that there exist pairwise disjoint subsets $X_1, X_2, \ldots, X_{m-2}$ of A, each having $\boldsymbol{p}$ in its convex hull and each being of cardinality at most 4. (Here "pairwise disjoint" means that each element of A is present in a number of X_i's that does not exceed its multiplicity in A.) We proceed by contradiction. Suppose that we can find at most $s < m - 2$ such subsets X_i's. Then, by Carathéodory's theorem, $\boldsymbol{p}$ is not in the convex hull of the remaining points in A. Therefore there is a half-space H_+ delimited by a plane containing $\boldsymbol{p}$ such that $H_+ \cap A \subseteq \bigcup_{i=1}^{s} X_i$. On the other hand, since each X_i contains $\boldsymbol{p}$ in its convex hull (and we can assume the X_i are minimal with respect to containing $\boldsymbol{p}$), we have $|H_+ \cap X_i| \leq 3$ for all $i \in [s]$. Therefore $|H_+ \cap A| \leq |H_+ \cap (\bigcup_{i=1}^{s} X_i)| \leq 3s < 3(m-2)$, which is a contradiction since $\boldsymbol{p}$ is a point of depth $3m-3$ in A. There are thus $m-2$ disjoint subsets $X_1, X_2, \ldots, X_{m-2}$ as claimed.

Let X denote $\bigcup_{i=1}^{m-2} X_i$. Consider an arbitrary half-space H_+ delimited by a plane containing $\boldsymbol{p}$. Since $|H_+ \cap X_i| \leq 3$ for all i, we have $|H_+ \cap X| \leq 3(m-2)$. Furthermore $|H_+ \cap A| \geq 3m-3$, so $|H_+ \cap (A \setminus X)| \geq 3$. Since H_+ is arbitrary, $\boldsymbol{p}$ is a point of depth 3 of $A \setminus X$. Also, $|A \setminus X| \geq |A| - 4(m-2) \geq 20m - 23 \geq 17$, so Lemma 15 implies that $A \setminus X$ can be partitioned into two sets whose convex hulls contain $\boldsymbol{p}$. With the subsets X_i, we have therefore an m-Tverberg partition of A, with $\boldsymbol{p}$ as Tverberg point. $\square$

From these two lemmas we can now finish the proof of Theorem 23.

Proof of Theorem 23. Consider a multiset A of $24m-31$ points in $\mathbb{Z}^3$. The case $m=2$ is the already mentioned result by K. Bezdek and A. Blokhuis. Assume that $m \geq 3$. Applying Lemma 9, A has an integer point $\boldsymbol{p}$ of depth $3m-3$. If $\boldsymbol{p}$ is an element of A with multiplicity μ, then take the singletons $\{\boldsymbol{p}\}$ as μ of the sets in the Tverberg partition.

If $\mu \geq m$, we are done. If $\mu = m-1$, the point $\boldsymbol{p}$ is still in the convex hull of points in A, and thus we are done. And if $\mu \leq m-2$, the point $\boldsymbol{p}$ is still a point of depth $3m - \mu - 3 \geq 3(m-\mu) - 3$ of the remaining $24m - \mu - 31 \geq 24(m-\mu) - 31$ points. Thus, we may apply Lemma 16 to get an $(m-\mu)$-Tverberg partition of the remaining points, with $\boldsymbol{p}$ as Tverberg point, and conclude the result. $\square$

5.3. Tverberg numbers over mixed spaces

In this section, we prove Theorem 24. We adapt an approach by W. Mulzer and D. Werner [**MW13**, Lemma 2.3] and show how the results of our paper can be combined to improve known bounds and to determine new exact values for the Tverberg number in the mixed integer case, as well as better bounds for certain S-Tverberg numbers.

Proof of Theorem 24. Let $t = \mathrm{Tv}(\mathbb{R}^k, m) = (m-1)(k+1)+1$. Choose a multiset A in $S' \times \mathbb{R}^k$ with $|A| \geq \mathrm{Tv}(S', t)$. It suffices to prove that A can be partitioned into m subsets whose convex hulls contain a common point in $S' \times \mathbb{R}^k$.

Let A' be the projection of A onto S'. Since $|A'| \geq \mathrm{Tv}(S', t)$, there is a partition of A' into t submultisets $Q'_1, \ldots, Q'_t$ whose convex hulls contain a common point $\boldsymbol{q}$ in S'. The Q'_i are the projections onto S' of t disjoint subsets Q_i forming a partition of A. For each $i \in [t]$, we can find a point $\boldsymbol{q}_i \in \mathrm{conv}(Q_i)$ projecting onto $\boldsymbol{q}$.

The t points $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_t$ belong to $\{\boldsymbol{q}\} \times \mathbb{R}^k$. As $t = \mathrm{Tv}(\mathbb{R}^k, m)$, there exists a partition of $[t]$ into $I_1, \ldots, I_m$ and a point $\boldsymbol{p} \in \{\boldsymbol{q}\} \times \mathbb{R}^k$ such that $\boldsymbol{p} \in \mathrm{conv}\left(\bigcup_{i \in I_\ell} \boldsymbol{q}_i\right)$ for all $\ell \in [m]$. For each $\ell \in [m]$, define A_ℓ to be $\bigcup_{i \in I_\ell} Q_i$. We have, for each $\ell \in [m]$

$$\boldsymbol{p} \in \mathrm{conv}\left(\bigcup_{i \in I_\ell} \boldsymbol{q}_i\right) \subseteq \mathrm{conv}\left(\bigcup_{i \in I_\ell} \mathrm{conv}(Q_i)\right) = \mathrm{conv}(A_\ell)$$

and the A_ℓ form the desired partition. □

Here are the new bounds and exact values we get:

(1) $\mathrm{Tv}(\mathbb{Z} \times \mathbb{R}^k, m) = 2(m-1)(k+1)+1$.
(2) $\mathrm{Tv}(\mathbb{Z}^2 \times \mathbb{R}^k, m) = 4(m-1)(k+1)+1$.
(3) $\mathrm{Tv}(\mathbb{Z}^3 \times \mathbb{R}^k, m) \leq 24(m-1)(k+1)-7$.
(4) $2^j(m-1)(k+1)+1 \leq \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq j2^j(m-1)(k+1)+1$.
(5) If $S' \subset \mathbb{R}^2$ with finite Helly number $\mathrm{H}(S')$, then

$$\mathrm{Tv}(S' \times \mathbb{R}^k, m) \leq \mathrm{H}(S')(m-1)(k+1)+1.$$

The lower bound in (4) is obtained by repeated applications of Proposition 1 below. The upper bounds follow from Theorem 24, combined with the fact that $\mathrm{Tv}(\mathbb{Z}, m) = 2m-1$, Theorem 22 for $S = \mathbb{Z}^2$, Theorem 23, the upper bound in Equation (5.1), and Theorem 22 respectively.

Proposition 1. *Let j and k be two non-negative integers. Then we have*

$$\mathrm{Tv}(\mathbb{Z}^{j+1} \times \mathbb{R}^k, m) > 2\,\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) - 2.$$

We prove Proposition 1 by following the idea of the proof of Proposition 2.1 in [**Onn91**].

Proof of Proposition 1. Assume toward a contradiction that

$$\mathrm{Tv}(\mathbb{Z}^{j+1} \times \mathbb{R}^k, m) \leq 2\,\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) - 2.$$

Choose A to be a set of $\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) - 1$ points in $\mathbb{Z}^j \times \mathbb{R}^k$ with no m-Tverberg partition. Let $A_i = \{(\boldsymbol{a}, i) \colon \boldsymbol{a} \in A, i = \{0,1\}\}$. Since $A_0 \cup A_1 \subset \mathbb{Z}^{j+1} \times \mathbb{R}^k$ has cardinality $2\,\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) - 2$, there exists an m-Tverberg partition $Y_1, Y_2, \ldots, Y_m$ of $A_0 \cup A_1$ with $\boldsymbol{p} \in \bigcap_{i\in[m]} \mathrm{conv}(Y_i)$. Furthermore $\boldsymbol{p}$ is in $\mathbb{Z}^{j+1} \times \mathbb{R}^k$. That implies either $\boldsymbol{p} \in \mathrm{conv}(A_0)$ or $\boldsymbol{p} \in \mathrm{conv}(A_1)$. In either case A_0 or A_1 has an m-Tverberg partition, a contradiction with our choice of A. $\square$

5.4. A generalized positive-fraction selection lemma

Our proof relies on the simplicial partition theorem of J. Matoušek, used in a similar manner as in [**MR17**], which states the following.

Theorem 35 ([**Mat92**]; see also [**Cha00**]). *Given an integer $d \geq 1$ and a parameter r, there exists a constant $c_d \geq 1$ such that for any set P of n points in $\mathbb{R}^d$, there exists an integer s and a partition $\{P_1, \ldots, P_s\}$ of P such that*

- *for each $i = 1, \ldots, s$, $\frac{n}{r} \leq |P_i| \leq \frac{2n}{r}$, and*
- *any hyperplane intersects the convex hull of less than $c_d \cdot r^{1-\frac{1}{d}}$ sets of the partition.*

The constant c_d is independent of P and depends only on d.

We now prove the following key lemma.

Lemma 17. *For any integer $d \geq 1$, there exists a constant c_d such that the following holds. For any set P of n points in $\mathbb{R}^d$ and a real number $\alpha \in (0,1]$, there exists a partition $\mathcal{P} = \{P_1, \ldots, P_r\}$, $r = \left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil$, of P such that*

- *$\frac{n}{2r} \leq |P_i| \leq \frac{2n}{r}$ for each $i = 1, \ldots, r$, and*
- *the convex hull of any transversal Q of $\mathcal{P}$ contains all points in $\mathbb{R}^d$ of half-space depth at least $\alpha \cdot n$.*

Proof. Apply the simplicial partition theorem (Theorem 35) to P with $r = \left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil$, and let the resulting partition be $\{P'_1, \ldots, P'_s\}$. Note that as $\frac{n}{r} \leq |P'_i| \leq \frac{2n}{r}$ for each $i = 1, \ldots, s$, we have

$\frac{r}{2} \le s \le r$. Now partition arbitrarily each of $r-s$ biggest sets in $\{P'_1, \ldots, P'_s\}$ into two equal parts, and let the resulting partition be $\{P_1, \ldots, P_r\}$. Clearly each set of this partition has size in the interval $\left[\frac{n}{2r}, \frac{2n}{r}\right]$. This proves the first part. Note also that each hyperplane intersects the convex hull of at most twice as many sets, i.e., less than $2c_d \cdot r^{1-\frac{1}{d}}$ sets of the partition $\{P_1, \ldots, P_r\}$.

To see the second part, let $\boldsymbol{c}$ be any point of half-space depth at least $\alpha \cdot n$, and Q any transversal of $\mathcal{P}$. For contradiction, assume that $\boldsymbol{c} \notin \operatorname{conv}(Q)$. Then there exists a hyperplane H containing $\boldsymbol{c}$ in one of its two open half-spaces, say H^-, and containing $\operatorname{conv}(Q)$ in the half-space H^+. We will show that then there exists an index $i \in \{1, \ldots, r\}$ such that $P_i \subseteq H^-$. But then $P_i \cap Q = \emptyset$, a contradiction to the fact that Q is a transversal of $\mathcal{P}$.

It remains to show the existence of a set $P_i \in \mathcal{P}$ such that $P_i \subseteq H^-$. Towards this, we bound $|P \cap H^-|$. Each point of P lying in H^- belongs to a set $P' \in \mathcal{P}$ such that either

- $P' \subseteq H^-$, in which case we are done, or
- P' is not contained in H^-. As H^- contains at least one point of P', we must have $\operatorname{conv}(P') \cap H \neq \emptyset$. As argued earlier, there are less than $2c_d \cdot r^{1-\frac{1}{d}}$ such sets.

Thus we have

$$\left|P \cap H^-\right| < 2c_d \cdot r^{1-\frac{1}{d}} \cdot \frac{2n}{r} = \frac{4c_d \cdot n}{\left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil^{\frac{1}{d}}} \le \alpha \cdot n. \tag{5.3}$$

On the other hand, as $\boldsymbol{c}$ has half-space depth at least $\alpha \cdot n$ and $\boldsymbol{c} \in H^-$, we have $|P \cap H^-| \ge \alpha n$, a contradiction to inequality (5.3). □

Remark: In particular, for $r = \left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil$, there exist at least $\left(\frac{n}{2r}\right)^r$ r-sized subsets, each of whose convex hull contains all integer points of depth at least $\alpha \cdot n$.

Proof of Theorem 26. Given the point set P in $\mathbb{R}^d$ and a point $\boldsymbol{q} \in \mathbb{R}^d$ of half-space depth $\alpha \cdot n$, apply Lemma 17 with P and α to get a partition consisting of $r \ge d+1$ sets, where $r = \left\lceil \left(\frac{4c_d}{\alpha}\right)^d \right\rceil$. By discarding at most $\frac{n}{2}$ points of P, we can derive a partition on the remaining points of P, say $\mathcal{P} = \{P_1, \ldots, P_r\}$, such that the P_i's are equal-sized disjoint subsets of P, i.e., $|P_i| = \frac{n}{2r}$ for all $i = 1, \ldots, r$. Furthermore, every transversal of $\mathcal{P}$ contains all points in $\mathbb{R}^d$ of half-space depth at least αn, and thus $\boldsymbol{q}$.

For each transversal Q of $\mathcal{P}$, the point $\boldsymbol{q}$ lies in the convex hull of Q, and by Carathéodory's theorem, there exists a $(d{+}1)$-sized subset of Q whose convex hull also contains $\boldsymbol{q}$. By the pigeonhole principle, there must exist $(d+1)$ sets of $\mathcal{P}$, say the sets $P_1, \ldots, P_{d+1}$, such that at least

$$\frac{\left(\frac{n}{2r}\right)^r}{\binom{r}{d+1}\left(\frac{n}{2r}\right)^{r-(d+1)}} \geq \frac{\left(\frac{n}{2r}\right)^{(d+1)}}{\left(\frac{er}{d+1}\right)^{d+1}} = \frac{1}{\left(\frac{er}{d+1}\right)^{d+1}} \cdot \prod_{i=1}^{d+1} |P_i|. \tag{5.4}$$

distinct transversals of $\{P_1, \ldots, P_{d+1}\}$ contain $\boldsymbol{q}$.

The rest of the proof follows the one of J. Pach [**Pac98**]. In brief, we view the P_i's as parts of a $(d{+}1)$-partite hypergraph with vertices corresponding to points in P and a hyperedge corresponding to each transversal of $\mathcal{P}$ containing $\boldsymbol{q}$. As there are $\Omega\left(n^{d+1}\right)$ such transversals by inequality (5.4), we apply a weak form of the hypergraph version of Szemerédi's regularity lemma (see [**Mat92**] Theorem 9.4.1) to derive the existence of constant-fraction sized subsets $P'_1 \subseteq P_1, \ldots, P'_{d+1} \subseteq P_{d+1}$ such that the following is true, for some constant c'_d:

For *any* $P''_1 \subseteq P'_1, \ldots, P''_{d+1} \subseteq P'_{d+1}$, with $|P''_i| \geq c'_d \cdot |P'_i|$ for $i = 1, \ldots, d+1$, we have the property that there exists at least one transversal of $\left\{P''_1, \ldots, P''_{d+1}\right\}$ whose convex hull contains $\boldsymbol{q}$.

Then the same-type lemma ([**BV98**] Theorem 2) applied to $\left\{P'_1, \ldots, P'_{d+1}, \{\boldsymbol{q}\}\right\}$ gives constant-fraction sized subsets $X_1 \subseteq P'_1, \ldots, X_{d+1} \subseteq P'_{d+1}$ such that each transversal of $\{X_1, \ldots, X_{d+1}\}$ has the same order type with respect to $\boldsymbol{q}$.

We can set up the parameters for the same-type lemma and the weak regularity lemma such that $|X_i| \geq c'_d \cdot |P'_i|$, for all $i = 1, \ldots, d+1$. Then the weak regularity lemma implies that there exists at least one transversal of $\{X_1, \ldots, X_{d+1}\}$ that contains $\boldsymbol{q}$. However, as each transversal of $\{X_1, \ldots, X_{d+1}\}$ has the same order type, it must be that *each* transversal of $\{X_1, \ldots, X_{d+1}\}$ contains $\boldsymbol{q}$. These are the required subsets.

The size of each X_i is a constant-fraction of n, say $|X_i| \geq c_{d,\alpha} \cdot n$, where the constant $c_{d,\alpha}$ depends on the constants in inequality (5.4), in the weak regularity lemma and in the same-type lemma. All of these depend only on α and d. □

CHAPTER 6

Further remarks and open problems

In this thesis we generalized Tverberg's theorem in a variety of ways: We showed that many simplicial complexes, called Tverberg complexes, are always induced as the nerve of some partition of any sufficiently large set of points in a fixed dimension. We also showed that with enough points, most partitions into a fixed number of sets are in fact Tverberg partitions. Lastly, we proved new bounds on the extension of Tverberg's theorem to spaces with special coordinate constraints. But there are many more open questions arising from these topics. We conclude with a list of open questions for each of these three new directions. Here are a few questions arising in the study of Tverberg complexes in Chapter 3:

(1) What is the exact value of $\mathrm{Tv}(T_n, d)$ where T_n is a tree with n nodes? Is $(d+1)(n-1)+1$ the correct value? What about the case of $d = 2$?

(2) What is the computational complexity of determining if a point configuration can partition induce a given graph?

(3) What is the computational complexity of computing the Tverberg numbers of a given Tverberg complex, such as a tree?

(4) Are there topological versions of Tverberg theorems for other simplicial complexes?

(5) Is there a graph G which is not 3-Tverberg?

(6) Is there a complex K which is not d-Tverberg for any d?

(7) Is there a complex K and $i, j \in \mathbb{N}$, $i < j$ so that K is i-Tverberg but not j-Tverberg?

Here are some open problems regarding the Stochastic aspects of Tverberg's theorem discussed in Chapter 4:

(1) Is there a stochastic Tverberg theorem which contains our results, as well as the results of P. Soberón, as a special case?

(2) Is there a threshold phenomena for when the nerve of a random partition is a connected simplical complex?

(3) Can we prove stochastic Tverberg theorems for the case that the points in each subset are sampled from different distributions? This would be desirable for further applications in maximum likelihood estimation.

(4) Can one obtain convergence results for general steepest descent algorithms based on the PertSEP$*_0$ parameter we introduced?

(5) Does the same threshold phenomena exhibited in Theorem 19 occur with arbitrary distributions (not just those that are balanced about a point $\boldsymbol{p}$)?

Lastly, here are some open problems regarding Tverberg's theorem over discrete sets discussed in Chapter 5:

(1) What is the Radon number of the three dimensional lattice?

(2) What are the asymptotics of the Tverberg numbers over $\mathbb{Z}^d$.

(3) For each d, do the m-Tverberg numbers for $\mathbb{Z}^d$ have a closed form for large enough m? For example, for $\mathbb{Z}^2$, there is one exceptional case when $m = 2$, then for higher m we have that the Tverberg number is equal to $4m - 3$. Does this hold in every dimension?

(4) J. A. De Loera conjectured that all integer linear program feasibility problems can be reduced to the problem of determining whether some set of points admits an integer Tverberg partition. Is this true?

(5) Can the probabilistic methods in Chapter 4 be used to improve the bounds on integer Tverberg numbers?

(6) One can also study more general Tverberg complexes over $\mathbb{Z}^d$. In that case, it would be interesting to know if there is a complex that is Tverberg over $\mathbb{Z}^d$ for some d, but not over $\mathbb{R}^d$ (this may be possible, since working over $\mathbb{Z}^d$ could allow one to "avoid" certain undesired intersections to obtain a specific complex).

APPENDIX A

Computing Tverberg numbers of trees

This appendix contains the code we used to compute the Tverberg number of small trees by enumerating order types. We also include some example computations for simple trees. The code was written by Dominic Yang.

We are able to find lower bounds for the Tverberg number of small graphs via an enumeration of all point sets, and determining for each point set which partitions can be induced. Obviously, as there are infinitely many point configurations in the plane, an explicit enumeration is impossible; however, there is a way to classify the combinatorial properties of small point sets (whether line segments through points cross, possible triangulations, etc.) into a finite number of point configurations. One way of classifying point configurations is through the order type of a set. For example, on four points, there are two order types, one with one point contained in the interior of the convex hull of the other three and the other with all four points in convex position. Its order type can be computed as assigning the triples $(1,2,3),(1,3,4),(1,4,2),(2,3,1),(2,1,4),(2,3,4),(3,4,2),(3,1,2),$ $(3,4,1),(4,2,1),(4,2,3),$ and $(4,1,3)$ to -1 for counter-clockwise orientation and the remaining ordered triples to 1 for clockwise orientation.

The order type encodes various combinatorial properties of point sets. If two point sets have the same order type, then if two segments cross in one set, their corresponding segments in the other order set should cross. If we have a valid triangulation in one set, it remains valid when mapped onto the other set. More importantly for our purposes, if two point sets have the same order type, then the intersection graphs induced by the two sets remain the same.

O. Aichholzer, et. al., have provided a catalog of representative point configurations for all order types up to $n = 10$ [**AAK02**]. Their method of generating each order type is done by generating a list of candidates of "pseudo order types" and then group these candidates into equivalence classes based on their order types. Then, they realize an actual point set for each possible class of order

Number of Points	Number of Sets
3	1
4	2
5	3
6	16
7	135
8	3315
9	158817
10	14309547
11	2334512907

TABLE A.1. Number of Sets with a Given Order Type

type until they reach the required number of order types, which is known from the literature. This catalog of point configurations can be found in [**Aic06**].

They have used these point configurations to test small cases for determining questions of isomorphisms between triangulations and the number of triangulations as well as the crossing number of complete graphs and problems related to finding Hamiltonian cycles on complete graphs. As can be seen in Table A.1, the number of order types grows exponentially, so enumeration of order types for $n > 10$ is out of the question (in part because O. Aichholzer has not provided so large a quantity in his database).

A.1. Partitions

Given a point set of n points we are interested in all the ways which we are able to partition this point set into m unlabeled subsets. The amount of ways in which we can partition this set in such a manner is known as the **Stirling number of the second kind** and we denote this value by $\left\{{n \atop m}\right\}$. For example all the ways to partition the set of 4 elements $\{1, 2, 3\}$ into two parts are $\{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}$ and so $\left\{{3 \atop 2}\right\} = 3$.

One simple property that can be derived regarding this quantity is that

$$\left\{{n+1 \atop m}\right\} = m\left\{{n \atop m}\right\} + \left\{{n \atop m-1}\right\}.$$

To see this, we note that to count the number of ways to partition $n + 1$ objects, we have two choices for where to put the $n + 1$-th object: we can place it in its own singleton set, in which case we have to partition the remaining n objects into $m - 1$ sets, or we partition the n objects into k sets and then we have m choices for where to include the $n + 1$-th object. In the first case we have

$\left\{ {n \atop m-1} \right\}$ partitions and in the second we have $m\left\{ {n \atop m} \right\}$ partitions. With the base cases $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$ for $n > 0$, we can define Stirling Numbers for all $n, m \geq 0$. We include a table of all the values up to $n = k = 10$ in Table A.2.

TABLE A.2. Stirling Numbers up to $n = m = 10$. Each row represents a value for n and each column a value for m.

	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0
3	0	1	3	1	0	0	0	0	0	0	0
4	0	1	7	6	1	0	0	0	0	0	0
5	0	1	15	25	10	1	0	0	0	0	0
6	0	1	31	90	65	15	1	0	0	0	0
7	0	1	63	301	350	140	21	1	0	0	0
8	0	1	127	966	1701	1050	266	28	1	0	0
9	0	1	255	3025	7770	6951	2646	462	36	1	0
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

As n grows large, it was shown in [**MW58**] that we have a rough estimate for fixed m that $\left\{ {n \atop m} \right\} \sim \frac{m^n}{m!}$. This shows that the growth is exponential for fixed n, so we should expect that the explicit enumeration of partitions quickly becomes computationally intractable.

A.2. Enumerating partitions over each of the possible order types

Using the classification from [**AAK02**] of all order types, it is relatively straight forward to determine all possible different graphs which can be induced by partitioning a given point set by brute force enumeration. It is just a matter of listing all possible partitions into k subsets and then determining the intersection graph from checking the intersection of convex hulls. An explicit algorithm for producing the intersection graph of a point set X and a partition σ into k sets is given in Figure A.1.

```
function IntersectionGraph(X,σ, k)
    Let A ∈ {0,1}^{k×k}
    for i = 1,...,k do
        for j = i,...,k do
            if i = j then
                A_{i,i} = 0
            else
                if Intersects(σ^{-1}(i), σ^{-1}(j)) then
                    A_{i,j} = A_{j,i} = 1
                else
                    A_{i,j} = A_{j,i} = 0
                end if
            end if
        end for
    end for
end function
```

FIGURE A.1. Algorithm for computing the intersection graph of a given partition σ of a point set X onto k subsets.

```
function Intersects(X, Y)
    C_X ← conv(X), C_Y ← conv(Y)
    if |C_X| < |C_Y| then
        Swap C_X and C_Y
    end if
    if |C_X| = 2 and |C_Y| = 2 then
        L_X ← Line(C_X), L_Y ← Line(C_Y)
        return SegmentIntersect(L_X, L_Y)
    else if |C_X| > 2 and |C_Y| = 1 then
        return InPolygon(C_Y, C_X)
    else
        return EdgeIntersect(C_Y, C_X) or InPolygon(C_Y, C_X) or InPolygon(C_X, C_Y)
    end if
end function
```

FIGURE A.2. Algorithm for determining if the convex hulls of two given sets X and Y intersect.

We also present the method used for determining if two given convex sets intersect. The method used to compute the convex hull and for determining if a point is contained in a polygon or if polygons intersect are MATLAB routines.

An alternative implementation based on checking the feasibility of the linear program defined by adjoining the representations of the convex polygons by inequalities was also considered and implemented. This was deemed to run slower than the MATLAB functions for determining the intersection of polygon edges however.

Checking if a graph has a Tverberg number of n now simply is a matter of enumerating all order types and checking for every single one if there is a single partition which induces that graph.

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