Notes on
Partial Differential Equations

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[1]Revised 6/18/2014. Thanks to Kris Jenssen and Jan Koch for corrections. Supported in part by NSF Grant #DMS-1312342.

Abstract. These are notes from a two-quarter class on PDEs that are heavily based on the book *Partial Differential Equations* by L. C. Evans, together with other sources that are mostly listed in the Bibliography. The notes cover roughly Chapter 2 and Chapters 5–7 in Evans. There is no claim to any originality in the notes, but I hope — for some readers at least — they will provide a useful supplement.

Contents

CHAPTER 1

Preliminaries

In this chapter, we collect various definitions and theorems for future use. Proofs may be found in the references *e.g.* [**4, 11, 24, 37, 42, 44**].

1.1. Euclidean space

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. We denote the Euclidean norm of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

and the inner product of vectors $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We denote Lebesgue measure on $\mathbb{R}^n$ by dx, and the Lebesgue measure of a set $E \subset \mathbb{R}^n$ by $|E|$.

If E is a subset of $\mathbb{R}^n$, we denote the complement by $E^c = \mathbb{R}^n \setminus E$, the closure by $\overline{E}$, the interior by E° and the boundary by $\partial E = \overline{E} \setminus E^\circ$. The characteristic function $\chi_E : \mathbb{R}^n \to \mathbb{R}$ of E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

A set E is bounded if $\{|x| : x \in E\}$ is bounded in $\mathbb{R}$. A set is connected if it is not the disjoint union of two nonempty relatively open subsets. We sometimes refer to a connected open set as a domain.

We say that a (nonempty) open set Ω' is compactly contained in an open set Ω, written $\Omega' \Subset \Omega$, if $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact. If $\Omega' \Subset \Omega$, then

$$\operatorname{dist}(\Omega', \partial\Omega) = \inf\{|x - y| : x \in \Omega', y \in \partial\Omega\} > 0.$$

1.2. Spaces of continuous functions

Let Ω be an open set in $\mathbb{R}^n$. We denote the space of continuous functions $u : \Omega \to \mathbb{R}$ by $C(\Omega)$; the space of functions with continuous partial derivatives in Ω of order less than or equal to $k \in \mathbb{N}$ by $C^k(\Omega)$; and the space of functions with continuous derivatives of all orders by $C^\infty(\Omega)$. Functions in these spaces need not be bounded even if Ω is bounded; for example, $(1/x) \in C^\infty(0, 1)$.

If Ω is a bounded open set in $\mathbb{R}^n$, we denote by $C(\overline{\Omega})$ the space of continuous functions $u : \overline{\Omega} \to \mathbb{R}$. This is a Banach space with respect to the maximum, or supremum, norm

$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|.$$

We denote the support of a continuous function $u : \Omega \to \mathbb{R}^n$ by

$$\operatorname{supp} u = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We denote by $C_c(\Omega)$ the space of continuous functions whose support is compactly contained in Ω, and by $C_c^\infty(\Omega)$ the space of functions with continuous derivatives of all orders and compact support in Ω. We will sometimes refer to such functions as test functions.

The completion of $C_c(\mathbb{R}^n)$ with respect to the uniform norm is the space $C_0(\mathbb{R}^n)$ of continuous functions that approach zero at infinity. (Note that in many places the notation C_0 and C_0^∞ is used to denote the spaces of compactly supported functions that we denote by C_c and C_c^∞.)

If Ω is bounded, then we say that a function $u : \overline{\Omega} \to \mathbb{R}$ belongs to $C^k(\overline{\Omega})$ if it is continuous and its partial derivatives of order less than or equal to k are uniformly continuous in Ω, in which case they extend to continuous functions on $\overline{\Omega}$. The space $C^k(\overline{\Omega})$ is a Banach space with respect to the norm

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha|\le k} \sup_{\Omega} |\partial^\alpha u|$$

where we use the multi-index notation for partial derivatives explained in Section 1.8. This norm is finite because the derivatives $\partial^\alpha u$ are continuous functions on the compact set $\overline{\Omega}$.

A vector field $X : \Omega \to \mathbb{R}^m$ belongs to $C^k(\overline{\Omega})$ if each of its components belongs to $C^k(\overline{\Omega})$.

1.3. Hölder spaces

The definition of continuity is not a quantitative one, because it does not say how rapidly the values $u(y)$ of a function approach its value $u(x)$ as $y \to x$. The modulus of continuity $\omega : [0,\infty] \to [0,\infty]$ of a general continuous function u, satisfying

$$|u(x) - u(y)| \le \omega\left(|x-y|\right),$$

may decrease arbitrarily slowly. As a result, despite their simple and natural appearance, spaces of continuous functions are often not suitable for the analysis of PDEs, which is almost always based on quantitative estimates.

A straightforward and useful way to strengthen the definition of continuity is to require that the modulus of continuity is proportional to a power $|x-y|^\alpha$ for some exponent $0 < \alpha \le 1$. Such functions are said to be Hölder continuous, or Lipschitz continuous if $\alpha = 1$. Roughly speaking, one can think of Hölder continuous functions with exponent α as functions with bounded fractional derivatives of the the order α.

Definition 1.1. Suppose that Ω is an open set in $\mathbb{R}^n$ and $0 < \alpha \le 1$. A function $u : \Omega \to \mathbb{R}$ is uniformly Hölder continuous with exponent α in Ω if the quantity

$$[u]_{\alpha,\Omega} = \sup_{\substack{x,y\,\in\,\Omega\\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \tag{1.1}$$

is finite. A function $u : \Omega \to \mathbb{R}$ is locally uniformly Hölder continuous with exponent α in Ω if $[u]_{\alpha,\Omega'}$ is finite for every $\Omega' \Subset \Omega$. We denote by $C^{0,\alpha}(\Omega)$ the space of locally uniformly Hölder continuous functions with exponent α in Ω. If Ω is bounded, we denote by $C^{0,\alpha}\left(\overline{\Omega}\right)$ the space of uniformly Hölder continuous functions with exponent α in Ω.

We typically use Greek letters such as α, β both for Hölder exponents and multi-indices; it should be clear from the context which they denote.

When α and Ω are understood, we will abbreviate 'u is (locally) uniformly Hölder continuous with exponent α in Ω' to 'u is (locally) Hölder continuous.' If u is Hölder continuous with exponent one, then we say that u is Lipschitz continuous. There is no purpose in considering Hölder continuous functions with exponent greater than one, since any such function is differentiable with zero derivative and therefore is constant.

The quantity $[u]_{\alpha,\Omega}$ is a semi-norm, but it is not a norm since it is zero for constant functions. The space $C^{0,\alpha}\left(\overline{\Omega}\right)$, where Ω is bounded, is a Banach space with respect to the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\Omega} |u| + [u]_{\alpha,\Omega}\,.$$

Example 1.2. For $0 < \alpha < 1$, define $u(x) : (0,1) \to \mathbb{R}$ by $u(x) = |x|^\alpha$. Then $u \in C^{0,\alpha}([0,1])$, but $u \notin C^{0,\beta}([0,1])$ for $\alpha < \beta \leq 1$.

Example 1.3. The function $u(x) : (-1,1) \to \mathbb{R}$ given by $u(x) = |x|$ is Lipschitz continuous, but not continuously differentiable. Thus, $u \in C^{0,1}([-1,1])$, but $u \notin C^1([-1,1])$.

We may also define spaces of continuously differentiable functions whose kth derivative is Hölder continuous.

Definition 1.4. If Ω is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, and $0 < \alpha \leq 1$, then $C^{k,\alpha}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$ with continuous partial derivatives in Ω of order less than or equal to k whose kth partial derivatives are locally uniformly Hölder continuous with exponent α in Ω. If the open set Ω is bounded, then $C^{k,\alpha}\left(\overline{\Omega}\right)$ consists of functions with uniformly continuous partial derivatives in Ω of order less than or equal to k whose kth partial derivatives are uniformly Hölder continuous with exponent α in Ω.

The space $C^{k,\alpha}\left(\overline{\Omega}\right)$ is a Banach space with respect to the norm

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \sum_{|\beta|\leq k} \sup_{\Omega} \left|\partial^\beta u\right| + \sum_{|\beta|=k} \left[\partial^\beta u\right]_{\alpha,\Omega}$$

1.4. L^p spaces

As before, let Ω be an open set in $\mathbb{R}^n$ (or, more generally, a Lebesgue-measurable set).

Definition 1.5. For $1 \leq p < \infty$, the space $L^p(\Omega)$ consists of the Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f|^p \, dx < \infty,$$

and $L^\infty(\Omega)$ consists of the essentially bounded functions.

These spaces are Banach spaces with respect to the norms

$$\|f\|_p = \left(\int_\Omega |f|^p \, dx\right)^{1/p}, \qquad \|f\|_\infty = \sup_\Omega |f|$$

where sup denotes the essential supremum,

$$\sup_\Omega f = \inf \{M \in \mathbb{R} : f \le M \text{ almost everywhere in } \Omega\}.$$

Strictly speaking, elements of the Banach space L^p are equivalence classes of functions that are equal almost everywhere, but we identify a function with its equivalence class unless we need to refer to the pointwise values of a specific representative. For example, we say that a function $f \in L^p(\Omega)$ is continuous if it is equal almost everywhere to a continuous function, and that it has compact support if it is equal almost everywhere to a function with compact support.

Next we summarize some fundamental inequalities for integrals, in addition to Minkowski's inequality which is implicit in the statement that $\|\cdot\|_{L^p}$ is a norm for $p \ge 1$. First, we recall the definition of a convex function.

Definition 1.6. A set $C \subset \mathbb{R}^n$ is convex if $\lambda x + (1-\lambda)y \in C$ for every $x, y \in C$ and every $\lambda \in [0, 1]$. A function $\phi : C \to \mathbb{R}$ is convex if its domain C is convex and

$$\phi(\lambda x + (1-\lambda)y) \le \lambda \phi(x) + (1-\lambda)\phi(y)$$

for every $x, y \in C$ and every $\lambda \in [0, 1]$.

Jensen's inequality states that the value of a convex function at a mean is less than or equal to the mean of the values of the convex function.

Theorem 1.7. *Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, Ω is a set in $\mathbb{R}^n$ with finite Lebesgue measure, and $f \in L^1(\Omega)$. Then*

$$\phi\left(\frac{1}{|\Omega|}\int_\Omega f\,dx\right) \le \frac{1}{|\Omega|}\int_\Omega \phi \circ f\,dx.$$

To state the next inequality, we first define the Hölder conjugate of an exponent p. We denote it by p' to distinguish it from the Sobolev conjugate p^* which we will introduce later on.

Definition 1.8. The Hölder conjugate of $p \in [1, \infty]$ is the quantity $p' \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the convention that $1/\infty = 0$.

The following result is called Hölder's inequality.[1] The special case when $p = p' = 1/2$ is the Cauchy-Schwartz inequality.

Theorem 1.9. *If $1 \le p \le \infty$, $f \in L^p(\Omega)$, and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_{p'}.$$

Repeated application of this inequality gives the following generalization.

Theorem 1.10. *If $1 \le p_i \le \infty$ for $1 \le i \le N$ satisfy*

$$\sum_{i=1}^N \frac{1}{p_i} = 1$$

[1] In retrospect, it might have been better to use $L^{1/p}$ spaces instead of L^p spaces, just as it would've been better to use inverse temperature instead of temperature, with absolute zero corresponding to infinite coldness.

and $f_i \in L^{p_i}(\Omega)$ *for* $1 \leq i \leq N$*, then* $f = \prod_{i=1}^N f_i \in L^1(\Omega)$ *and*

$$\|f\|_1 \leq \prod_{i=1}^{N} \|f_i\|_{p_i}.$$

Suppose that Ω has finite measure and $1 \leq q \leq p$. If $f \in L^p(\Omega)$, an application of Hölder's inequality to $f = 1 \cdot f$, shows that $f \in L^q(\Omega)$ and

$$\|f\|_q \leq |\Omega|^{1/q-1/p}\|f\|_p.$$

Thus, the embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. This result is not true if the measure of Ω is infinite, but in general we have the following interpolation result.

LEMMA 1.11. *If* $1 \leq p \leq q \leq r$*, then* $L^p(\Omega) \cap L^r(\Omega) \hookrightarrow L^q(\Omega)$ *and*

$$\|f\|_q \leq \|f\|_p^{\theta}\|f\|_r^{1-\theta}$$

where $0 \leq \theta \leq 1$ *is given by*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

PROOF. Assume without loss of generality that $f \geq 0$. Using Hölder's inequality with exponents $1/\sigma$ and $1/(1-\sigma)$, we get

$$\int f^q\,dx = \int f^{\theta q} f^{(1-\theta)q}\,dx \leq \left(\int f^{\theta q/\sigma}\,dx\right)^{\sigma}\left(\int f^{(1-\theta)q/(1-\sigma)}\,dx\right)^{1-\sigma}.$$

Choosing $\sigma/\theta = q/p$, in which case $(1-\sigma)/(1-\theta) = q/r$, we get

$$\int f^q\,dx \leq \left(\int f^p\,dx\right)^{q\theta/p}\left(\int f^r\,dx\right)^{q(1-\theta)/r}$$

and the result follows. □

It is often useful to consider local L^p spaces consisting of functions that have finite integral on compact sets.

DEFINITION 1.12. The space $L^p_{\mathrm{loc}}(\Omega)$, where $1 \leq p \leq \infty$, consists of functions $f : \Omega \to \mathbb{R}$ such that $f \in L^p(\Omega')$ for every open set $\Omega' \Subset \Omega$. A sequence of functions $\{f_n\}$ converges to f in $L^p_{\mathrm{loc}}(\Omega)$ if $\{f_n\}$ converges to f in $L^p(\Omega')$ for every open set $\Omega' \Subset \Omega$.

If $p < q$, then $L^q_{\mathrm{loc}}(\Omega) \hookrightarrow L^p_{\mathrm{loc}}(\Omega)$ even if the measure of Ω is infinite. Thus, $L^1_{\mathrm{loc}}(\Omega)$ is the 'largest' space of integrable functions on Ω.

EXAMPLE 1.13. Consider $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{|x|^a}$$

where $a \in \mathbb{R}$. Then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if and only if $a < n$. To prove this, let

$$f^\epsilon(x) = \begin{cases} f(x) & \text{if } |x| > \epsilon, \\ 0 & \text{if } |x| \leq \epsilon. \end{cases}$$

Then $\{f^\epsilon\}$ is monotone increasing and converges pointwise almost everywhere to f as $\epsilon \to 0^+$. For any $R > 0$, the monotone convergence theorem implies that

$$\begin{aligned}\int_{B_R(0)} f\,dx &= \lim_{\epsilon\to 0^+} \int_{B_R(0)} f^\epsilon\,dx \\ &= \lim_{\epsilon\to 0^+} \int_\epsilon^R r^{n-a-1}\,dr \\ &= \begin{cases} \infty & \text{if } n-a \le 0, \\ (n-a)^{-1}R^{n-a} & \text{if } n-a > 0, \end{cases}\end{aligned}$$

which proves the result. The function f does not belong to $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ for any value of a, since the integral of f^p diverges at infinity whenever it converges at zero.

1.5. Compactness

Compactness results play a central role in the analysis of PDEs. Typically, we construct a sequence of approximate solutions of a PDE and show that they belong to a compact set. We then extract a convergent subsequence of approximate solutions and attempt to show that their limit is a solution of the original PDE. There are two main types of compactness — weak and strong compactness. We begin with criteria for strong compactness.

A subset F of a metric space X is precompact if the closure of F is compact; equivalently, F is precompact if every sequence in F has a subsequence that converges in X. The Arzelà-Ascoli theorem gives a basic criterion for compactness in function spaces: namely, a set of continuous functions on a compact metric space is precompact if and only if it is bounded and equicontinuous. We state the result explicitly for the spaces of interest here.

Theorem 1.14. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$. A subset $\mathcal{F}$ of $C\left(\overline{\Omega}\right)$, equipped with the maximum norm, is precompact if and only if:*

(1) *there exists a constant M such that*

$$\|f\|_\infty \le M \qquad \textit{for all } f \in \mathcal{F};$$

(2) *for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, x+h \in \overline{\Omega}$ and $|h| < \delta$ then*

$$|f(x+h) - f(x)| < \epsilon \qquad \textit{for all } f \in \mathcal{F}.$$

The following theorem (known variously as the Riesz-Tamarkin, or Kolmogorov-Riesz, or Fréchet-Kolmogorov theorem) gives conditions analogous to the ones in the Arzelà-Ascoli theorem for a set to be precompact in $L^p(\mathbb{R}^n)$, namely that the set is bounded, 'tight,' and L^p-equicontinuous. For a proof, see [**44**].

Theorem 1.15. *Let $1 \le p < \infty$. A subset $\mathcal{F}$ of $L^p(\mathbb{R}^n)$ is precompact if and only if:*

(1) *there exists M such that*

$$\|f\|_{L^p} \le M \qquad \textit{for all } f \in \mathcal{F};$$

(2) *for every $\epsilon > 0$ there exists R such that*

$$\left(\int_{|x|>R} |f(x)|^p\,dx\right)^{1/p} < \epsilon \qquad \textit{for all } f \in \mathcal{F}.$$

(3) *for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $|h| < \delta$,

$$\left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, dx\right)^{1/p} < \epsilon \qquad \textit{for all } f \in \mathcal{F}.$$

The 'tightness' condition (2) prevents the functions from escaping to infinity.

EXAMPLE 1.16. Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n = \chi_{(n,n+1)}$. The set $\{f_n : n \in \mathbb{N}\}$ is bounded and equicontinuous in $L^p(\mathbb{R})$ for any $1 \leq p < \infty$, but it is not precompact since $\|f_m - f_n\|_p = 2$ if $m \neq n$, nor is it tight since

$$\int_R^\infty |f_n|^p \, dx = 1 \qquad \text{for all } n \geq R.$$

The equicontinuity conditions in the hypotheses of these theorems for strong compactness are not always easy to verify; typically, one does so by obtaining a uniform estimate for the derivatives of the functions, as in the Sobolev-Rellich embedding theorems.

As we explain next, weak compactness is easier to verify, since we only need to show that the functions themselves are bounded. On the other hand, we get subsequences that converge weakly and not necessarily strongly. This can create difficulties, especially for nonlinear problems, since nonlinear functions are not continuous with respect to weak convergence.

Let X be a real Banach space and X^* (which we also denote by X') the dual space of bounded linear functionals on X. We denote the duality pairing between X^* and X by $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$.

DEFINITION 1.17. A sequence $\{x_n\}$ in X converges weakly to $x \in X$, written $x_n \rightharpoonup x$, if $\langle \omega, x_n \rangle \to \langle \omega, x \rangle$ for every $\omega \in X^*$. A sequence $\{\omega_n\}$ in X^* converges weak-star to $\omega \in X^*$, written $\omega_n \overset{*}{\rightharpoonup} \omega$ if $\langle \omega_n, x \rangle \to \langle \omega, x \rangle$ for every $x \in X$.

If X is reflexive, meaning that $X^{**} = X$, then weak and weak-star convergence are equivalent.

EXAMPLE 1.18. If $\Omega \subset \mathbb{R}^n$ is an open set and $1 \leq p < \infty$, then $L^p(\Omega)^* = L^{p'}(\Omega)$. Thus a sequence of functions $f_n \in L^p(\Omega)$ converges weakly to $f \in L^p(\Omega)$ if

$$\int_\Omega f_n g \, dx \to \int_\Omega f g \, dx \qquad \text{for every } g \in L^{p'}(\Omega). \tag{1.2}$$

If $p = \infty$ and $p' = 1$, then $L^\infty(\Omega)^* \neq L^1(\Omega)$ but $L^\infty(\Omega) = L^1(\Omega)^*$. In that case, (1.2) defines weak-star convergence in $L^\infty(\Omega)$.

A subset E of a Banach space X is (sequentially) weakly, or weak-star, precompact if every sequence in E has a subsequence that converges weakly, or weak-star, in X. The following Banach-Alagolu theorem characterizes weak-star precompact subsets of a Banach space; it may be thought of as generalization of the Heine-Borel theorem to infinite-dimensional spaces.

THEOREM 1.19. *A subset of a Banach space is weak-star precompact if and only if it is bounded.*

If X is reflexive, then bounded sets are weakly precompact. This result applies, in particular, to Hilbert spaces.

Example 1.20. Let $\mathcal{H}$ be a separable Hilbert space with inner-product $(\cdot,\cdot)$ and orthonormal basis $\{e_n : n \in \mathbb{N}\}$. The sequence $\{e_n\}$ is bounded in $\mathcal{H}$, but it has no strongly convergent subsequence since $\|e_n - e_m\| = \sqrt{2}$ for every $n \neq m$. On the other hand, the sequence converges weakly in $\mathcal{H}$ to zero: if $x = \sum x_n e_n \in \mathcal{H}$ then $(x, e_n) = x_n \to 0$ as $n \to \infty$ since $\|x\|^2 = \sum |x_n|^2 < \infty$.

1.6. Averages

For $x \in \mathbb{R}^n$ and $r > 0$, let

$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$$

denote the open ball centered at x with radius r, and

$$\partial B_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\}$$

the corresponding sphere.

The volume of the unit ball in $\mathbb{R}^n$ is given by

$$\alpha_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$$

where Γ is the Gamma function, which satisfies

$$\Gamma(1/2) = \sqrt{\pi}, \qquad \Gamma(1) = 1, \qquad \Gamma(x+1) = x\Gamma(x).$$

Thus, for example, $\alpha_2 = \pi$ and $\alpha_3 = 4\pi/3$. An integration with respect to polar coordinates shows that the area of the $(n-1)$-dimensional unit sphere is $n\alpha_n$.

We denote the average of a function $f \in L^1_{\mathrm{loc}}(\Omega)$ over a ball $B_r(x) \Subset \Omega$, or the corresponding sphere $\partial B_r(x)$, by

$$\fint_{B_r(x)} f\, dx = \frac{1}{\alpha_n r^n}\int_{B_r(x)} f\, dx, \qquad \fint_{\partial B_r(x)} f\, dS = \frac{1}{n\alpha_n r^{n-1}}\int_{\partial B_r(x)} f\, dS. \tag{1.3}$$

If f is continuous at x, then

$$\lim_{r\to 0^+}\fint_{B_r(x)} f\, dx = f(x).$$

The following result, called the Lebesgue differentiation theorem, implies that the averages of a locally integrable function converge pointwise almost everywhere to the function as the radius r shrinks to zero.

Theorem 1.21. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then*

$$\lim_{r\to 0^+}\fint_{B_r(x)} |f(y) - f(x)|\, dx = 0 \tag{1.4}$$

pointwise almost everywhere for $x \in \mathbb{R}^n$.

A point $x \in \mathbb{R}^n$ for which (1.4) holds is called a Lebesgue point of f. For a proof of this theorem (using the Wiener covering lemma and the Hardy-Littlewood maximal function) see Folland [**11**] or Taylor [**42**].

1.7. Convolutions

DEFINITION 1.22. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are measurable function, we define the convolution $f * g : \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy$$

provided that the integral converges for x pointwise almost everywhere in $\mathbb{R}^n$.

When defined, the convolution product is both commutative and associative,

$$f * g = g * f, \qquad f * (g * h) = (f * g) * h.$$

In many respects, the convolution of two functions inherits the best properties of both functions.

If $f, g \in C_c(\mathbb{R}^n)$, then their convolution also belongs to $C_c(\mathbb{R}^n)$ and

$$\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g.$$

If $f \in C_c(\mathbb{R}^n)$ and $g \in C(\mathbb{R}^n)$, then $f * g \in C(\mathbb{R}^n)$ is defined, however rapidly g grows at infinity, but typically it does not have compact support. If neither f nor g have compact support, then we need some conditions on their growth or decay at infinity to ensure that the convolution exists. The following result, called Young's inequality, gives conditions for the convolution of L^p functions to exist and estimates its norm.

THEOREM 1.23. *Suppose that* $1 \le p, q, r \le \infty$ *and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

If $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then* $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

The following special cases are useful to keep in mind.

EXAMPLE 1.24. If $p = q = 2$, or more generally if $q = p'$, then $r = \infty$. In this case, the result follows from the Cauchy-Schwartz inequality, since for all $x \in \mathbb{R}^n$

$$\left| \int f(x - y)g(y)\, dx \right| \le \|f\|_{L^2} \|g\|_{L^2}.$$

Moreover, a density argument shows that $f * g \in C_0(\mathbb{R}^n)$: Choose $f_k, g_k \in C_c(\mathbb{R}^n)$ such that $f_k \to f$, $g_k \to g$ in $L^2(\mathbb{R}^n)$, then $f_k * g_k \in C_c(\mathbb{R}^n)$ and $f_k * g_k \to f * g$ uniformly. A similar argument is used in the proof of the Riemann-Lebesgue lemma that $\hat{f} \in C_0(\mathbb{R}^n)$ if $f \in L^1(\mathbb{R}^n)$.

EXAMPLE 1.25. If $p = q = 1$, then $r = 1$, and the result follows directly from Fubini's theorem, since

$$\int \left| \int f(x - y)g(y)\, dy \right| dx \le \int |f(x - y)g(y)|\, dxdy = \left(\int |f(x)|\, dx \right) \left(\int |g(y)|\, dy \right).$$

Thus, the space $L^1(\mathbb{R}^n)$ is an algebra under the convolution product. The Fourier transform maps the convolution product of two L^1-functions to the pointwise product of their Fourier transforms.

EXAMPLE 1.26. If $q = 1$, then $p = r$. Thus, convolution with an integrable function $k \in L^1(\mathbb{R}^n)$ is a bounded linear map $f \mapsto k * f$ on $L^p(\mathbb{R}^n)$.

1.8. Derivatives and multi-index notation

We define the derivative of a scalar field $u : \Omega \to \mathbb{R}$ by

$$Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right).$$

We will also denote the ith partial derivative by $\partial_i u$, the ijth derivative by $\partial_{ij} u$, and so on. The divergence of a vector field $X = (X_1, X_2, \dots, X_n) : \Omega \to \mathbb{R}^n$ is

$$\operatorname{div} X = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \dots + \frac{\partial X_n}{\partial x_n}.$$

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denote the non-negative integers. An n-dimensional multi-index is a vector $\alpha \in \mathbb{N}_0^n$, meaning that

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \qquad \alpha_i = 0, 1, 2, \dots.$$

We write

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \qquad \alpha! = \alpha_1!\alpha_2! \dots \alpha_n!.$$

We define derivatives and powers of order α by

$$\partial^\alpha = \frac{\partial}{\partial x^{\alpha_1}} \frac{\partial}{\partial x^{\alpha_2}} \dots \frac{\partial}{\partial x^{\alpha_n}}, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are multi-indices, we define the multi-index $(\alpha + \beta)$ by

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n).$$

We denote by $\chi_n(k)$ the number of multi-indices $\alpha \in \mathbb{N}_0^n$ with order $0 \leq |\alpha| \leq k$, and by $\tilde{\chi}_n(k)$ the number of multi-indices with order $|\alpha| = k$. Then

$$\chi_n(k) = \frac{(n+k)!}{n!k!}, \qquad \tilde{\chi}_n(k) = \frac{(n+k-1)!}{(n-1)!k!}$$

1.8.1. Taylor's theorem for functions of several variables. The multi-index notation provides a compact way to write the multinomial theorem and the Taylor expansion of a function of several variables. The multinomial expansion of a power is

$$(x_1 + x_2 + \dots + x_n)^k = \sum_{\alpha_1 + \dots \alpha_n = k} \binom{k}{\alpha_1 \alpha_2 \dots \alpha_n} x_i^{\alpha_i} = \sum_{|\alpha| = k} \binom{k}{\alpha} x^\alpha$$

where the multinomial coefficient of a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of order $|\alpha| = k$ is given by

$$\binom{k}{\alpha} = \binom{k}{\alpha_1 \alpha_2 \dots \alpha_n} = \frac{k!}{\alpha_1!\alpha_2! \dots \alpha_n!}.$$

THEOREM 1.27. *Suppose that* $u \in C^k(B_r(x))$ *and* $h \in B_r(0)$. *Then*

$$u(x+h) = \sum_{|\alpha| \leq k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x, h)$$

where the remainder is given by

$$R_k(x, h) = \sum_{|\alpha| = k} \frac{\partial^\alpha u(x + \theta h)}{\alpha!} h^\alpha$$

for some $0 < \theta < 1$.

PROOF. Let $f(t) = u(x + th)$ for $0 \le t \le 1$. Taylor's theorem for a function of a single variable implies that

$$f(1) = \sum_{j=0}^{k-1} \frac{1}{j!} \frac{d^j f}{dt^j}(0) + \frac{1}{k!} \frac{d^k f}{dt^k}(\theta)$$

for some $0 < \theta < 1$. By the chain rule,

$$\frac{df}{dt} = Du \cdot h = \sum_{i=1}^{n} h_i \partial_i u,$$

and the multinomial theorem gives

$$\frac{d^k}{dt^k} = \left(\sum_{i=1}^{n} h_i \partial_i \right)^k = \sum_{|\alpha|=k} \binom{n}{\alpha} h^\alpha \partial^\alpha.$$

Using this expression to rewrite the Taylor series for f in terms of u, we get the result. □

A function $u : \Omega \to \mathbb{R}$ is real-analytic in an open set Ω if it has a power-series expansion that converges to the function in a ball of non-zero radius about every point of its domain. We denote by $C^\omega(\Omega)$ the space of real-analytic functions on Ω. A real-analytic function is C^∞, since its Taylor series can be differentiated term-by-term, but a C^∞ function need not be real-analytic. For example, see (1.5) below.

1.9. Mollifiers

The function

$$\eta(x) = \begin{cases} C \exp\left[-1/(1 - |x|^2)\right] & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases} \tag{1.5}$$

belongs to $C_c^\infty(\mathbb{R}^n)$ for any constant C. We choose C so that

$$\int_{\mathbb{R}^n} \eta \, dx = 1$$

and for any $\epsilon > 0$ define the function

$$\eta^\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right). \tag{1.6}$$

Then η^ϵ is a C^∞-function with integral equal to one whose support is the closed ball $\overline{B}_\epsilon(0)$. We refer to (1.6) as the 'standard mollifier.'

We remark that $\eta(x)$ in (1.5) is not real-analytic when $|x| = 1$. All of its derivatives are zero at those points, so the Taylor series converges to zero in any neighborhood, not to the original function. The only function that is real-analytic with compact support is the zero function. In rough terms, an analytic function is a single 'organic' entity: its values in, for example, a single open ball determine its values everywhere in a maximal domain of analyticity (which in the case of one complex variable is a Riemann surface) through analytic continuation. The behavior of a C^∞-function at one point is, however, completely unrelated to its behavior at another point.

Suppose that $f \in L^1_{\text{loc}}(\Omega)$ is a locally integrable function. For $\epsilon > 0$, let

$$\Omega^\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\} \tag{1.7}$$

and define $f^\epsilon : \Omega^\epsilon \to \mathbb{R}$ by

$$f^\epsilon(x) = \int_\Omega \eta^\epsilon(x-y) f(y)\, dy \tag{1.8}$$

where η^ϵ is the mollifier in (1.6). We define f^ϵ for $x \in \Omega^\epsilon$ so that $B_\epsilon(x) \subset \Omega$ and we have room to average f. If $\Omega = \mathbb{R}^n$, we have simply $\Omega^\epsilon = \mathbb{R}^n$. The function f^ϵ is a smooth approximation of f.

THEOREM 1.28. *Suppose that $f \in L^p_{\mathrm{loc}}(\Omega)$ for $1 \le p < \infty$, and $\epsilon > 0$. Define $f^\epsilon : \Omega^\epsilon \to \mathbb{R}$ by (1.8). Then: (a) $f^\epsilon \in C^\infty(\Omega^\epsilon)$ is smooth; (b) $f^\epsilon \to f$ pointwise almost everywhere in Ω as $\epsilon \to 0^+$; (c) $f^\epsilon \to f$ in $L^p_{\mathrm{loc}}(\Omega)$ as $\epsilon \to 0^+$.*

PROOF. The smoothness of f^ϵ follows by differentiation under the integral sign

$$\partial^\alpha f^\epsilon(x) = \int_\Omega \partial^\alpha \eta^\epsilon(x-y) f(y)\, dy$$

which may be justified by use of the dominated convergence theorem. The pointwise almost everywhere convergence (at every Lebesgue point of f) follows from the Lebesgue differentiation theorem. The convergence in L^p_{loc} follows by the approximation of f by a continuous function (for which the result is easy to prove) and the use of Young's inequality, since $\|\eta^\epsilon\|_{L^1} = 1$ is bounded independently of ϵ. □

One consequence of this theorem is that the space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$. Note that this is not true when $p = \infty$, since the uniform limit of smooth test functions is continuous.

1.9.1. Cutoff functions.

THEOREM 1.29. *Suppose that $\Omega' \Subset \Omega$ are open sets in $\mathbb{R}^n$. Then there is a function $\phi \in C_c^\infty(\Omega)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on Ω'.*

PROOF. Let $\delta = \operatorname{dist}(\Omega', \partial\Omega)$ and define

$$\Omega'' = \{x \in \Omega : \operatorname{dist}(x, \Omega') < \delta/2\}.$$

Let χ be the characteristic function of Ω'', and define $\phi = \eta^{\delta/4} * \chi$ where η^ϵ is the standard mollifier. Then one may verify that ϕ has the required properties. □

We refer to a function with the properties in this theorem as a cutoff function.

EXAMPLE 1.30. If $0 < r < R$ and $\Omega' = B_r(0)$, $\Omega = B_R(0)$ are balls in $\mathbb{R}^n$, then the corresponding cut-off function ϕ satisfies

$$|D\phi| \le \frac{C}{R-r}$$

where C is a constant that is independent of r, R.

1.9.2. Partitions of unity. Partitions of unity allow us to piece together global results from local results.

THEOREM 1.31. *Suppose that K is a compact set in $\mathbb{R}^n$ which is covered by a finite collection $\{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ of open sets. Then there exists a collection of functions $\{\eta_1, \eta_2, \ldots, \eta_N\}$ such that $0 \le \eta_i \le 1$, $\eta_i \in C_c^\infty(\Omega_i)$, and $\sum_{i=1}^N \eta_i = 1$ on K.*

We call $\{\eta_i\}$ a partition of unity subordinate to the cover $\{\Omega_i\}$. To prove this result, we use Urysohn's lemma to construct a collection of continuous functions with the desired properties, then use mollification to obtain a collection of smooth functions.

1.10. Boundaries of open sets

When we analyze solutions of a PDE in the interior of their domain of definition, we can often consider domains that are arbitrary open sets and analyze the solutions in a sufficiently small ball. In order to analyze the behavior of solutions at a boundary, however, we typically need to assume that the boundary has some sort of smoothness. In this section, we define the smoothness of the boundary of an open set. We also explain briefly how one defines analytically the normal vector-field and the surface area measure on a smooth boundary.

In general, the boundary of an open set may be complicated. For example, it can have nonzero Lebesgue measure.

EXAMPLE 1.32. Let $\{q_i : i \in \mathbb{N}\}$ be an enumeration of the rational numbers $q_i \in (0,1)$. For each $i \in \mathbb{N}$, choose an open interval $(a_i, b_i) \subset (0,1)$ that contains q_i, and let

$$\Omega = \bigcup_{i\in\mathbb{N}} (a_i, b_i).$$

The Lebesgue measure of $|\Omega| > 0$ is positive, but we can make it as small as we wish; for example, choosing $b_i - a_i = \epsilon 2^{-i}$, we get $|\Omega| \leq \epsilon$. One can check that $\partial\Omega = [0,1] \setminus \Omega$. Thus, if $|\Omega| < 1$, then $\partial\Omega$ has nonzero Lebesgue measure.

Moreover, an open set, or domain, need not lie on one side of its boundary (we say that Ω lies on one side of its boundary if $\overline{\Omega}^\circ = \Omega$), and corners, cusps, or other singularities in the boundary cause analytical difficulties.

EXAMPLE 1.33. The unit disc in $\mathbb{R}^2$ with the nonnegative x-axis removed,

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\} \setminus \left\{(x,0) \in \mathbb{R}^2 : 0 \leq x < 1\right\},$$

does not lie on one side of its boundary.

In rough terms, the boundary of an open set is smooth if it can be 'flattened out' locally by a smooth map.

DEFINITION 1.34. Suppose that $k \in \mathbb{N}$. A map $\phi : U \to V$ between open sets U, V in $\mathbb{R}^n$ is a C^k-diffeomorphism if it one-to-one, onto, and ϕ and ϕ^{-1} have continuous derivatives of order less than or equal to k.

Note that the derivative $D\phi(x) : \mathbb{R}^n \to \mathbb{R}^n$ of a diffeomorphism $\phi : U \to V$ is an invertible linear map for every $x \in U$, with $[D\phi(x)]^{-1} = (D\phi^{-1})(\phi(x))$.

DEFINITION 1.35. Let Ω be a bounded open set in $\mathbb{R}^n$ and $k \in \mathbb{N}$. We say that the boundary $\partial\Omega$ is C^k, or that Ω is C^k for short, if for every $x \in \overline{\Omega}$ there is an open neighborhood $U \subset \mathbb{R}^n$ of x, an open set $V \subset \mathbb{R}^n$, and a C^k-diffeomorphism $\phi : U \to V$ such that

$$\phi(U \cap \Omega) = V \cap \{y_n > 0\}, \qquad \phi(U \cap \partial\Omega) = V \cap \{y_n = 0\}$$

where $(y_1, \ldots, y_n)$ are coordinates in the image space $\mathbb{R}^n$.

If ϕ is a C^∞-diffeomorphism, then we say that the boundary is C^∞, with an analogous definition of a Lipschitz or analytic boundary.

In other words, the definition says that a C^k open set in $\mathbb{R}^n$ is an n-dimensional C^k-manifold with boundary. The maps ϕ in Definition 1.35 are coordinate charts for the manifold. It follows from the definition that Ω lies on one side of its boundary and that $\partial\Omega$ is an oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ without boundary. The standard orientation is given by the outward-pointing normal (see below).

Example 1.36. The open set

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : x > 0,\ y > \sin(1/x)\right\}$$

lies on one side of its boundary, but the boundary is not C^1 since there is no coordinate chart of the required form for the boundary points $\{(x, 0) : -1 \leq x \leq 1\}$.

1.10.1. Open sets in the plane. A simple closed curve, or Jordan curve, Γ is a set in the plane that is homeomorphic to a circle. That is, $\Gamma = \gamma(\mathbb{T})$ is the image of a one-to-one continuous map $\gamma : \mathbb{T} \to \mathbb{R}^2$ with continuous inverse $\gamma^{-1} : \Gamma \to \mathbb{T}$. (The requirement that the inverse is continuous follows from the other assumptions.) According to the Jordan curve theorem, a Jordan curve divides the plane into two disjoint connected open sets, so that $\mathbb{R}^2 \setminus \Gamma = \Omega_1 \cup \Omega_2$. One of the sets (the 'interior') is bounded and simply connected. The interior region of a Jordan curve is called a Jordan domain.

Example 1.37. The slit disc Ω in Example 1.33 is not a Jordan domain. For example, its boundary separates into two nonempty connected components when the point $(1, 0)$ is removed, but the circle remains connected when any point is removed, so $\partial\Omega$ cannot be homeomorphic to the circle.

Example 1.38. The interior Ω of the Koch, or 'snowflake,' curve is a Jordan domain. The Hausdorff dimension of its boundary is strictly greater than one. It is interesting to note that, despite the irregular nature of its boundary, this domain has the property that every function in $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and $1 \leq p < \infty$ can be extended to a function in $W^{k,p}(\mathbb{R}^2)$.

If $\gamma : \mathbb{T} \to \mathbb{R}^2$ is one-to-one, C^1, and $|D\gamma| \neq 0$, then the image of γ is the C^1 boundary of the open set which it encloses. The condition that γ is one-to-one is necessary to avoid self-intersections (for example, a figure-eight curve), and the condition that $|D\gamma| \neq 0$ is necessary in order to ensure that the image is a C^1-submanifold of $\mathbb{R}^2$.

Example 1.39. The curve $\gamma : t \mapsto \left(t^2, t^3\right)$ is not C^1 at $t = 0$ where $D\gamma(0) = 0$.

1.10.2. Parametric representation of a boundary. If Ω is an open set in $\mathbb{R}^n$ with C^k-boundary and ϕ is a chart on a neighborhood U of a boundary point, as in Definition 1.35, then we can define a local chart

$$\Phi = (\Phi_1, \Phi_2, \dots, \Phi_{n-1}) : U \cap \partial\Omega \subset \mathbb{R}^n \to W \subset \mathbb{R}^{n-1}$$

for the boundary $\partial\Omega$ by $\Phi = (\phi_1, \phi_2, \dots, \phi_{n-1})$. Thus, $\partial\Omega$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

The boundary is parameterized locally by $x_i = \Psi_i\left(y_1, y_2, \dots, y_{n-1}\right)$ where $1 \leq i \leq n$ and $\Psi = \Phi^{-1} : W \to U \cap \partial\Omega$. The $(n-1)$-dimensional tangent space of $\partial\Omega$ is spanned by the vectors

$$\frac{\partial\Psi}{\partial y_1}, \frac{\partial\Psi}{\partial y_2}, \dots, \frac{\partial\Psi}{\partial y_{n-1}}.$$

The outward unit normal $\nu : \partial\Omega \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$ is orthogonal to this tangent space, and it is given locally by

$$\nu = \frac{\tilde{\nu}}{|\tilde{\nu}|}, \qquad \tilde{\nu} = \frac{\partial\Psi}{\partial y_1} \wedge \frac{\partial\Psi}{\partial y_2} \wedge \cdots \wedge \frac{\partial\Psi}{\partial y_{n-1}},$$

$$\tilde{\nu}_i = \begin{vmatrix} \partial\Psi_1/\partial y_1 & \partial\Psi_1/\partial\Psi_2 & \dots & \partial\Psi_1/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_{i-1}/\partial y_1 & \partial\Psi_{i-1}/\partial y_2 & \dots & \partial\Psi_{i-1}/\partial y_{n-1} \\ \partial\Psi_{i+1}/\partial y_1 & \partial\Psi_{i+1}/\partial y_2 & \dots & \partial\Psi_{i+1}/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_n/\partial y_1 & \partial\Psi_n/\partial y_2 & \dots & \partial\Psi_n/\partial y_{n-1} \end{vmatrix}.$$

EXAMPLE 1.40. For a three-dimensional region with two-dimensional boundary, the outward unit normal is

$$\nu = \frac{(\partial\Psi/\partial y_1) \times (\partial\Psi/\partial y_2)}{|(\partial\Psi/\partial y_1) \times (\partial\Psi/\partial y_2)|}.$$

The restriction of the Euclidean metric on $\mathbb{R}^n$ to the tangent space of the boundary gives a Riemannian metric on the boundary whose volume form defines the surface measure dS. Explicitly, the pull-back of the Euclidean metric

$$\sum_{i=1}^{n} dx_i^2$$

to the boundary under the mapping $x = \Psi(y)$ is the metric

$$\sum_{i=1}^{n} \sum_{p,q=1}^{n-1} \frac{\partial\Psi_i}{\partial y_p} \frac{\partial\Psi_i}{\partial y_q} \, dy_p dy_q.$$

The volume form associated with a Riemannian metric $\sum h_{pq} \, dy_p dy_q$ is

$$\sqrt{\det h} \, dy_1 dy_2 \dots dy_{n-1}.$$

Thus the surface measure on $\partial\Omega$ is given locally by

$$dS = \sqrt{\det\left(D\Psi^t D\Psi\right)} \, dy_1 dy_2 \dots dy_{n-1}$$

where $D\Psi$ is the derivative of the parametrization,

$$D\Psi = \begin{pmatrix} \partial\Psi_1/\partial y_1 & \partial\Psi_1/\partial y_2 & \dots & \partial\Psi_1/\partial y_{n-1} \\ \partial\Psi_2/\partial y_1 & \partial\Psi_2/\partial y_2 & \dots & \partial\Psi_2/\partial y_{n-1} \\ \dots & \dots & \dots & \dots \\ \partial\Psi_n/\partial y_1 & \partial\Psi_n/\partial y_2 & \dots & \partial\Psi_n/\partial y_{n-1} \end{pmatrix}.$$

These local expressions may be combined to give a global definition of the surface integral by means of a partition of unity.

EXAMPLE 1.41. In the case of a two-dimensional surface with metric

$$ds^2 = E \, dy_1^2 + 2F \, dy_1 dy_2 + G \, dy_2^2,$$

the element of surface area is

$$dS = \sqrt{EG - F^2} \, dy_1 dy_2.$$

EXAMPLE 1.42. The two-dimensional sphere

$$\mathbb{S}^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

is a C^∞ submanifold of $\mathbb{R}^3$. A local C^∞-parametrization of

$$U = \mathbb{S}^2 \setminus \left\{(x, 0, z) \in \mathbb{R}^3 : x \geq 0\right\}$$

is given by $\Psi : W \subset \mathbb{R}^2 \to U \subset \mathbb{S}^2$ where

$$\begin{aligned}
\Psi(\theta, \phi) &= (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \\
W &= \left\{(\theta, \phi) \in \mathbb{R}^3 : 0 < \theta < 2\pi,\ 0 < \phi < \pi\right\}.
\end{aligned}$$

The metric on the sphere is

$$\Psi^*\left(dx^2 + dy^2 + dz^2\right) = \sin^2\phi\, d\theta^2 + d\phi^2$$

and the corresponding surface area measure is

$$dS = \sin\phi\, d\theta d\phi.$$

The integral of a continuous function $f(x, y, z)$ over the sphere that is supported in U is then given by

$$\int_{\mathbb{S}^2} f\, dS = \int_W f\left(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi\right) \sin\phi\, d\theta d\phi.$$

We may use similar rotated charts to cover the points with $x \geq 0$ and $y = 0$.

1.10.3. Representation of a boundary as a graph. An alternative, and computationally simpler, way to represent the boundary of a smooth open set is as a graph. After rotating coordinates, if necessary, we may assume that the nth component of the normal vector to the boundary is nonzero. If $k \geq 1$, the implicit function theorem implies that we may represent a C^k-boundary as a graph

$$x_n = h\left(x_1, x_2, \ldots, x_{n-1}\right)$$

where $h : W \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is in $C^k(W)$ and Ω is given locally by $x_n < h(x_1, \ldots, x_{n-1})$. If the boundary is only Lipschitz, then the implicit function theorem does not apply, and it is not always possible to represent a Lipschitz boundary locally as the region lying below the graph of a Lipschitz continuous function.

If $\partial\Omega$ is C^1, then the outward normal ν is given in terms of h by

$$\nu = \frac{1}{\sqrt{1 + |Dh|^2}}\left(-\frac{\partial h}{\partial x_1}, -\frac{\partial h}{\partial x_2}, \ldots, -\frac{\partial h}{\partial x_{n-1}}, 1\right)$$

and the surface area measure on $\partial\Omega$ is given by

$$dS = \sqrt{1 + |Dh|^2}\, dx_1 dx_2 \ldots dx_{n-1}.$$

EXAMPLE 1.43. Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and $\partial\Omega$ the unit sphere. The upper hemisphere

$$H = \{x \in \partial\Omega : x_n > 0\}$$

is the graph of $x_n = h(x')$ where $h : D \to \mathbb{R}$ is given by

$$h(x') = \sqrt{1 - |x'|^2}, \qquad D = \left\{x' \in \mathbb{R}^{n-1} : |x'| < 1\right\}$$

and we write $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. The surface measure on H is

$$dS = \frac{1}{\sqrt{1 - |x'|^2}}\, dx'$$

and the surface integral of a function $f(x)$ over H is given by

$$\int_H f\, dS = \int_D \frac{f\left(x', h(x')\right)}{\sqrt{1 - |x'|^2}}\, dx'.$$

The integral of a function over $\partial\Omega$ may be computed in terms of such integrals by use of a partition of unity subordinate to an atlas of hemispherical charts.

1.11. Change of variables

We state a theorem for a C^1 change of variables in the Lebesgue integral. A special case is the change of variables from Cartesian to polar coordinates. For proofs, see [**11, 42**].

THEOREM 1.44. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $\phi : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism of Ω onto its image $\phi(\Omega)$. If $f : \phi(\Omega) \to \mathbb{R}$ is a nonnegative Lebesgue measurable function or an integrable function, then*

$$\int_{\phi(\Omega)} f(y)\, dy = \int_\Omega f \circ \phi(x) \left|\det D\phi(x)\right|\, dx.$$

We define polar coordinates in $\mathbb{R}^n \setminus \{0\}$ by $x = ry$, where $r = |x| > 0$ and $y \in \partial B_1(0)$ is a point on the unit sphere. In these coordinates, Lebesgue measure has the representation

$$dx = r^{n-1} dr dS(y)$$

where $dS(y)$ is the surface area measure on the unit sphere. We have the following result for integration in polar coordinates.

PROPOSITION 1.45. *If $f : \mathbb{R}^n \to \mathbb{R}$ is integrable, then*

$$\begin{aligned}\int f\, dx &= \int_0^\infty \left[\int_{\partial B_1(0)} f\left(x + ry\right)\, dS(y)\right] r^{n-1}\, dr \\ &= \int_{\partial B_1(0)} \left[\int_0^\infty f\left(x + ry\right)\, r^{n-1}\, dr\right] dS(y).\end{aligned}$$

1.12. Divergence theorem

We state the divergence (or Gauss-Green) theorem.

THEOREM 1.46. *Let $X : \overline{\Omega} \to \mathbb{R}^n$ be a $C^1(\overline{\Omega})$-vector field, and $\Omega \subset \mathbb{R}^n$ a bounded open set with C^1-boundary $\partial\Omega$. Then*

$$\int_\Omega \operatorname{div} X\, dx = \int_{\partial\Omega} X \cdot \nu\, dS.$$

To prove the theorem, we prove it for functions that are compactly supported in a half-space, show that it remains valid under a C^1 change of coordinates with the divergence defined in an appropriately invariant way, and then use a partition of unity to add the results together.

In particular, if $u, v \in C^1(\overline{\Omega})$, then an application of the divergence theorem to the vector field $X = (0, 0, \dots, uv, \dots, 0)$, with ith component uv, gives the integration by parts formula

$$\int_\Omega u\,(\partial_i v)\;dx = -\int_\Omega (\partial_i u)\,v\,dx + \int_{\partial\Omega} uv\nu_i\,dS.$$

The statement in Theorem 1.46 is, perhaps, the natural one from the perspective of smooth differential geometry. The divergence theorem, however, remains valid under weaker assumptions than the ones in Theorem 1.46. For example, it applies to a cube, whose boundary is not C^1, as well as to other sets with piecewise smooth boundaries.

From the perspective of geometric measure theory, a general form of the divergence theorem holds for Lipschitz vector fields (vector fields whose weak derivative belongs to L^∞) and sets of finite perimeter (sets whose characteristic function has bounded variation). The surface integral is taken over a measure-theoretic boundary with respect to $(n-1)$-dimensional Hausdorff measure, and a measure-theoretic normal exists almost everywhere on the boundary with respect to this measure [**10, 45**].

1.13. Gronwall's inequality

In estimating some norm of a solution of a PDE, we are often led to a differential inequality for the norm from which we want to deduce an inequality for the norm itself. Gronwall's inequality allows one to do this: roughly speaking, it states that a solution of a differential inequality is bounded by the solution of the corresponding differential equality. There are both linear and nonlinear versions of Gronwall's inequality. We state only the simplest version of the linear inequality.

LEMMA 1.47. *Suppose that $u : [0, T] \to [0, \infty)$ is a nonnegative, absolutely continuous function such that*

$$\frac{du}{dt} \le Cu, \qquad u(0) = u_0. \tag{1.9}$$

for some constants C, $u_0 \ge 0$. Then

$$u(t) \le u_0 e^{Ct} \qquad \text{for } 0 \le t \le T.$$

PROOF. Let $v(t) = e^{-Ct}u(t)$. Then

$$\frac{dv}{dt} = e^{-Ct}\left[\frac{du}{dt} - Cu(t)\right] \le 0.$$

If follows that

$$v(t) - u_0 = \int_0^t \frac{dv}{ds}\,ds \le 0,$$

or $e^{-Ct}u(t) \le u_0$, which proves the result. □

In particular, if $u_0 = 0$, it follows that $u(t) = 0$. We can alternatively write (1.9) in the integral form

$$u(t) \le u_0 + C\int_0^t u(s)\,ds.$$

CHAPTER 2

Laplace's equation

> There can be but one option as to the beauty and utility of this analysis by Laplace; but the manner in which it has hitherto been presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.[1]

Laplace's equation is

$$\Delta u = 0$$

where the Laplacian Δ is defined in Cartesian coordinates by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We may also write $\Delta = \operatorname{div} D$. The Laplacian Δ is invariant under translations (it has constant coefficients) and orthogonal transformations of $\mathbb{R}^n$. A solution of Laplace's equation is called a harmonic function.

Laplace's equation is a linear, scalar equation. It is the prototype of an elliptic partial differential equation, and many of its qualitative properties are shared by more general elliptic PDEs. The non-homogeneous version of Laplace's equation

$$-\Delta u = f$$

is called Poisson's equation. It is convenient to include a minus sign here because Δ is a negative definite operator.

The Laplace and Poisson equations, and their generalizations, arise in many different contexts.

(1) Potential theory *e.g.* in the Newtonian theory of gravity, electrostatics, heat flow, and potential flows in fluid mechanics.
(2) Riemannian geometry *e.g.* the Laplace-Beltrami operator.
(3) Stochastic processes *e.g.* the stationary Kolmogorov equation for Brownian motion.
(4) Complex analysis *e.g.* the real and imaginary parts of an analytic function of a single complex variable are harmonic.

As with any PDE, we typically want to find solutions of the Laplace or Poisson equation that satisfy additional conditions. For example, if Ω is a bounded domain in $\mathbb{R}^n$, then the classical Dirichlet problem for Poisson's equation is to find a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

[1] Kelvin and Tait, *Treatise on Natural Philosophy*, 1879

where $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ are given functions. The classical Neumann problem is to find a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ and

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= g \qquad \text{on } \partial\Omega.
\end{aligned}
\tag{2.2}
$$

Here, 'classical' refers to the requirement that the functions and derivatives appearing in the problem are defined pointwise as continuous functions. Dirichlet boundary conditions specify the function on the boundary, while Neumann conditions specify the normal derivative. Other boundary conditions, such as mixed (or Robin) and oblique-derivative conditions are also of interest. Also, one may impose different types of boundary conditions on different parts of the boundary (*e.g.* Dirichlet on one part and Neumann on another).

Here, we mostly follow Evans [**9**] (§2.2), Gilbarg and Trudinger [**17**], and Han and Lin [**23**].

2.1. Mean value theorem

Harmonic functions have the following mean-value property which states that the average value (1.3) of the function over a ball or sphere is equal to its value at the center.

THEOREM 2.1. *Suppose that $u \in C^2(\Omega)$ is harmonic in an open set Ω and $B_r(x) \Subset \Omega$. Then*

$$
u(x) = \fint_{B_r(x)} u\, dx, \qquad u(x) = \fint_{\partial B_r(x)} u\, dS.
\tag{2.3}
$$

PROOF. If $u \in C^2(\Omega)$ and $B_r(x) \Subset \Omega$, then the divergence theorem (Theorem 1.46) implies that

$$
\begin{aligned}
\int_{B_r(x)} \Delta u\, dx &= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\, dS \\
&= r^{n-1} \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x + ry)\, dS(y) \\
&= r^{n-1} \frac{\partial}{\partial r} \left[\int_{\partial B_1(0)} u(x + ry)\, dS(y) \right].
\end{aligned}
$$

Dividing this equation by $\alpha_n r^n$, we find that

$$
\fint_{B_r(x)} \Delta u\, dx = \frac{n}{r} \frac{\partial}{\partial r} \left[\fint_{\partial B_r(x)} u\, dS \right].
\tag{2.4}
$$

It follows that if u is harmonic, then its mean value over a sphere centered at x is independent of r. Since the mean value integral at $r = 0$ is equal to $u(x)$, the mean value property for spheres follows.

The mean value property for the ball follows from the mean value property for spheres by radial integration. □

The mean value property characterizes harmonic functions and has a remarkable number of consequences. For example, harmonic functions are smooth because local averages over a ball vary smoothly as the ball moves. We will prove this result by mollification, which is a basic technique in the analysis of PDEs.

THEOREM 2.2. *Suppose that $u \in C(\Omega)$ has the mean-value property (2.3). Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.*

PROOF. Let $\eta^\epsilon(x) = \tilde{\eta}^\epsilon(|x|)$ be the standard, radially symmetric mollifier (1.6). If $B_\epsilon(x) \Subset \Omega$, then, using Proposition 1.45 together with the facts that the average of u over each sphere centered at x is equal to $u(x)$ and the integral of η^ϵ is one, we get

$$\begin{aligned}
(\eta^\epsilon * u)(x) &= \int_{B_\epsilon(0)} \eta^\epsilon(y) u(x-y)\, dy \\
&= \int_0^\epsilon \left[\int_{\partial B_1(0)} \eta^\epsilon(rz) u(x - rz)\, dS(z) \right] r^{n-1}\, dr \\
&= n\alpha_n \int_0^\epsilon \left[\fint_{\partial B_r(x)} u\, dS \right] \tilde{\eta}^\epsilon(r) r^{n-1}\, dr \\
&= n\alpha_n u(x) \int_0^\epsilon \tilde{\eta}^\epsilon(r) r^{n-1}\, dr \\
&= u(x) \int \eta^\epsilon(y)\, dy \\
&= u(x).
\end{aligned}$$

Thus, u is smooth since $\eta^\epsilon * u$ is smooth.

If u has the mean value property, then (2.4) shows that

$$\int_{B_r(x)} \Delta u\, dx = 0$$

for every ball $B_r(x) \Subset \Omega$. Since Δu is continuous, it follows that $\Delta u = 0$ in Ω. □

Theorems 2.1–2.2 imply that any C^2-harmonic function is C^∞. The assumption that $u \in C^2(\Omega)$ is, if fact, unnecessary: Weyl showed that if a distribution $u \in \mathcal{D}'(\Omega)$ is harmonic in Ω, then $u \in C^\infty(\Omega)$.

Note that these results say nothing about the behavior of u at the boundary of Ω, which can be nasty. The reverse implication of this observation is that the Laplace equation can take rough boundary data and immediately smooth it to an analytic function in the interior.

EXAMPLE 2.3. Consider the meromorphic function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z}.$$

The real and imaginary parts of f

$$u(x,y) = \frac{x}{x^2 + y^2}, \qquad v(x,y) = -\frac{y}{x^2 + y^2}$$

are harmonic and C^∞ in, for example, the open unit disc

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1 \right\}$$

but both are unbounded as $(x,y) \to (0,0) \in \partial\Omega$.

The boundary behavior of harmonic functions can be much worse than in this example. If $\Omega \subset \mathbb{R}^n$ is any open set, then there exists a harmonic function in Ω such that

$$\liminf_{x\to\xi} u(x) = -\infty, \qquad \limsup_{x\to\xi} u(x) = \infty$$

for all $\xi \in \partial\Omega$. One can construct such a function as a sum of harmonic functions, converging uniformly on compact subsets of Ω, whose terms have singularities on a dense subset of points on $\partial\Omega$.

It is interesting to contrast this result with the the corresponding behavior of holomorphic functions of several variables. An open set $\Omega \subset \mathbb{C}^n$ is said to be a domain of holomorphy if there exists a holomorphic function $f : \Omega \to \mathbb{C}$ which cannot be extended to a holomorphic function on a strictly larger open set. Every open set in $\mathbb{C}$ is a domain of holomorphy, but when $n \geq 2$ there are open sets in $\mathbb{C}^n$ that are not domains of holomorphy, meaning that every holomorphic function on those sets can be extended to a holomorphic function on a larger open set.

2.1.1. Subharmonic and superharmonic functions. The mean value property has an extension to functions that are not necessarily harmonic but whose Laplacian does not change sign.

Definition 2.4. Suppose that Ω is an open set. A function $u \in C^2(\Omega)$ is subharmonic if $\Delta u \geq 0$ in Ω and superharmonic if $\Delta u \leq 0$ in Ω.

A function u is superharmonic if and only if $-u$ is subharmonic, and a function is harmonic if and only if it is both subharmonic and superharmonic. A suitable modification of the proof of Theorem 2.1 gives the following mean value inequality.

Theorem 2.5. *Suppose that Ω is an open set, $B_r(x) \Subset \Omega$, and $u \in C^2(\Omega)$. If u is subharmonic in Ω, then*

$$u(x) \leq \fint_{B_r(x)} u\,dx, \qquad u(x) \leq \fint_{\partial B_r(x)} u\,dS. \tag{2.5}$$

If u is superharmonic in Ω, then

$$u(x) \geq \fint_{B_r(x)} u\,dx, \qquad u(x) \geq \fint_{\partial B_r(x)} u\,dS. \tag{2.6}$$

It follows from these inequalities that the value of a subharmonic (or superharmonic) function at the center of a ball is less (or greater) than or equal to the value of a harmonic function with the same values on the boundary. Thus, the graphs of subharmonic functions lie below the graphs of harmonic functions and the graphs of superharmonic functions lie above, which explains the terminology. The direction of the inequality ($-\Delta u \leq 0$ for subharmonic functions and $-\Delta u \geq 0$ for superharmonic functions) is more natural when the inequality is stated in terms of the positive operator $-\Delta$.

Example 2.6. The function $u(x) = |x|^4$ is subharmonic in $\mathbb{R}^n$ since $\Delta u = 4(n+2)|x|^2 \geq 0$. The function is equal to the constant harmonic function $U(x) = 1$ on the sphere $|x| = 1$, and $u(x) \leq U(x)$ when $|x| \leq 1$.

2.2. Derivative estimates and analyticity

An important feature of Laplace equation is that we can estimate the derivatives of a solution in a ball in terms of the solution on a larger ball. This feature is closely connected with the smoothing properties of the Laplace equation.

THEOREM 2.7. *Suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any $1 \leq i \leq n$,*

$$|\partial_i u(x)| \leq \frac{n}{r} \max_{\overline{B}_r(x)} |u|.$$

PROOF. Since u is smooth, differentiation of Laplace's equation with respect to x_i shows that $\partial_i u$ is harmonic, so by the mean value property for balls and the divergence theorem

$$\partial_i u = \fint_{B_r(x)} \partial_i u \, dx = \frac{1}{\alpha_n r^n} \int_{\partial B_r(x)} u\nu_i \, dS.$$

Taking the absolute value of this equation and using the estimate

$$\left| \int_{\partial B_r(x)} u\nu_i \, dS \right| \leq n\alpha_n r^{n-1} \max_{\overline{B}_r(x)} |u|$$

we get the result. □

One consequence of Theorem 2.7 is that a bounded harmonic function on $\mathbb{R}^n$ is constant; this is an n-dimensional extension of Liouville's theorem for bounded entire functions.

COROLLARY 2.8. *If $u \in C^2(\mathbb{R}^n)$ is bounded and harmonic in $\mathbb{R}^n$, then u is constant.*

PROOF. If $|u| \leq M$ on $\mathbb{R}^n$, then Theorem 2.7 implies that

$$|\partial_i u(x)| \leq \frac{Mn}{r}$$

for any $r > 0$. Taking the limit as $r \to \infty$, we conclude that $Du = 0$, so u is constant. □

Next we extend the estimate in Theorem 2.7 to higher-order derivatives. We use a somewhat tricky argument that gives sharp enough estimates to prove analyticity.

THEOREM 2.9. *Suppose that $u \in C^2(\Omega)$ is harmonic in the open set Ω and $B_r(x) \Subset \Omega$. Then for any multi-index $\alpha \in \mathbb{N}_0^n$ of order $k = |\alpha|$*

$$|\partial^\alpha u(x)| \leq \frac{n^k e^{k-1} k!}{r^k} \max_{\overline{B}_r(x)} |u|.$$

PROOF. We prove the result by induction on $|\alpha| = k$. From Theorem 2.7, the result is true when $k = 1$. Suppose that the result is true when $|\alpha| = k$. If $|\alpha| = k+1$, we may write $\partial^\alpha = \partial_i \partial^\beta$ where $1 \leq i \leq n$ and $|\beta| = k$. For $0 < \theta < 1$, let

$$\rho = (1-\theta)r.$$

Then, since $\partial^\beta u$ is harmonic and $B_\rho(x) \Subset \Omega$, Theorem 2.7 implies that

$$|\partial^\alpha u(x)| \leq \frac{n}{\rho} \max_{\overline{B}_\rho(x)} \left|\partial^\beta u\right|.$$

Suppose that $y \in B_\rho(x)$. Then $B_{r-\rho}(y) \subset B_r(x)$, and using the induction hypothesis we get

$$\left|\partial^\beta u(y)\right| \leq \frac{n^k e^{k-1} k!}{(r-\rho)^k} \max_{\overline{B}_{r-\rho}(y)} |u| \leq \frac{n^k e^{k-1} k!}{r^k \theta^k} \max_{\overline{B}_r(x)} |u|.$$

It follows that

$$|\partial^\alpha u(x)| \leq \frac{n^{k+1} e^{k-1} k!}{r^{k+1} \theta^k (1-\theta)} \max_{\overline{B}_r(x)} |u|.$$

Choosing $\theta = k/(k+1)$ and using the inequality

$$\frac{1}{\theta^k(1-\theta)} = \left(1 + \frac{1}{k}\right)^k (k+1) \leq e(k+1)$$

we get

$$|\partial^\alpha u(x)| \leq \frac{n^{k+1} e^k (k+1)!}{r^{k+1}} \max_{\overline{B}_r(x)} |u|.$$

The result follows by induction. □

A consequence of this estimate is that the Taylor series of u converges to u near any point. Thus, we have the following result.

THEOREM 2.10. *If $u \in C^2(\Omega)$ is harmonic in an open set Ω then u is real-analytic in Ω.*

PROOF. Suppose that $x \in \Omega$ and choose $r > 0$ such that $B_{2r}(x) \Subset \Omega$. Since $u \in C^\infty(\Omega)$, we may expand it in a Taylor series with remainder of any order $k \in \mathbb{N}$ to get

$$u(x+h) = \sum_{|\alpha| \leq k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h),$$

where we assume that $|h| < r$. From Theorem 1.27, the remainder is given by

$$R_k(x,h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x+\theta h)}{\alpha!} h^\alpha \tag{2.7}$$

for some $0 < \theta < 1$.

To estimate the remainder, we use Theorem 2.9 to get

$$|\partial^\alpha u(x+\theta h)| \leq \frac{n^k e^{k-1} k!}{r^k} \max_{\overline{B}_r(x+\theta h)} |u|.$$

Since $|h| < r$, we have $B_r(x+\theta h) \subset B_{2r}(x)$, so for any $0 < \theta < 1$ we have

$$\max_{\overline{B}_r(x+\theta h)} |u| \leq M, \qquad M = \max_{\overline{B}_{2r}(x)} |u|.$$

It follows that

$$|\partial^\alpha u(x+\theta h)| \leq \frac{M n^k e^{k-1} k!}{r^k}. \tag{2.8}$$

Since $|h^\alpha| \leq |h|^k$ when $|\alpha| = k$, we get from (2.7) and (2.8) that

$$|R_k(x,h)| \leq \frac{M n^k e^{k-1} |h|^k k!}{r^k} \left(\sum_{|\alpha|=k} \frac{1}{\alpha!}\right).$$

The multinomial expansion

$$n^k = (1 + 1 + \cdots + 1)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} = \sum_{|\alpha|=k} \frac{k!}{\alpha!}$$

shows that

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{n^k}{k!}.$$

Therefore, we have

$$|R_k(x, h)| \le \frac{M}{e} \left(\frac{n^2 e |h|}{r} \right)^k.$$

Thus $R_k(x, h) \to 0$ as $k \to \infty$ if

$$|h| < \frac{r}{n^2 e},$$

meaning that the Taylor series of u at any $x \in \Omega$ converges to u in a ball of non-zero radius centered at x. □

It follows that, as for analytic functions, the global values of a harmonic function is determined its values in arbitrarily small balls (or by the germ of the function at a single point).

Corollary 2.11. *Suppose that u, v are harmonic in a connected open set $\Omega \subset \mathbb{R}^n$ and $\partial^\alpha u(\bar{x}) = \partial^\alpha v(\bar{x})$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ at some point $\bar{x} \in \Omega$. Then $u = v$ in Ω.*

Proof. Let

$$F = \{x \in \Omega : \partial^\alpha u(x) = \partial^\alpha v(x) \text{ for all } \alpha \in \mathbb{N}_0^n\}.$$

Then $F \neq \emptyset$, since $\bar{x} \in F$, and F is closed in Ω, since

$$F = \bigcap_{\alpha \in \mathbb{N}_0^n} [\partial^\alpha (u - v)]^{-1}(0)$$

is an intersection of relatively closed sets. Theorem 2.10 implies that if $x \in F$, then the Taylor series of u, v converge to the same value in some ball centered at x. Thus u, v and all of their partial derivatives are equal in this ball, so F is open. Since Ω is connected, it follows that $F = \Omega$. □

A physical explanation of this property is that Laplace's equation describes an equilibrium solution obtained from a time-dependent solution in the limit of infinite time. For example, in heat flow, the equilibrium is attained as the result of thermal diffusion across the entire domain, while an electrostatic field is attained only after all non-equilibrium electric fields propagate away as electromagnetic radiation. In this infinite-time limit, a change in the field near any point influences the field everywhere else, and consequently complete knowledge of the solution in an arbitrarily small region carries information about the solution in the entire domain.

Although, in principle, a harmonic function function is globally determined by its local behavior near any point, the reconstruction of the global behavior is sensitive to small errors in the local behavior.

EXAMPLE 2.12. Let $\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, \, y \in \mathbb{R}\right\}$ and consider for $n \in \mathbb{N}$ the function

$$u_n(x, y) = ne^{-nx} \sin ny,$$

which is harmonic. Then

$$\partial_y^k u_n(x, 1) = (-1)^k n^{k+1} e^{-n} \sin nx$$

converges uniformly to zero as $n \to \infty$ for any $k \in \mathbb{N}_0$. Thus, u_n and any finite number of its derivatives are arbitrarily close to zero at $x = 1$ when n is sufficiently large. Nevertheless, $u_n(0, y) = n \sin(ny)$ is arbitrarily large at $y = 0$.

2.3. Maximum principle

The maximum principle states that a non-constant harmonic function cannot attain a maximum (or minimum) at an interior point of its domain. This result implies that the values of a harmonic function in a bounded domain are bounded by its maximum and minimum values on the boundary. Such maximum principle estimates have many uses, but they are typically available only for scalar equations, not systems of PDEs.

THEOREM 2.13. *Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is subharmonic and attains a global maximum value in Ω, then u is constant in Ω.*

PROOF. By assumption, u is bounded from above and attains its maximum in Ω. Let

$$M = \max_{\Omega} u,$$

and consider

$$F = u^{-1}\left(\{M\}\right) = \{x \in \Omega : u(x) = M\}.$$

Then F is nonempty and relatively closed in Ω since u is continuous. (A subset F is relatively closed in Ω if $F = \tilde{F} \cap \Omega$ where $\tilde{F}$ is closed in $\mathbb{R}^n$.) If $x \in F$ and $B_r(x) \Subset \Omega$, then the mean value inequality (2.5) for subharmonic functions implies that

$$\fint_{B_r(x)} [u(y) - u(x)] \, dy = \fint_{B_r(x)} u(y) \, dy - u(x) \geq 0.$$

Since u attains its maximum at x, we have $u(y) - u(x) \leq 0$ for all $y \in \Omega$, and it follows that $u(y) = u(x)$ in $B_r(x)$. Therefore F is open as well as closed. Since Ω is connected, and F is nonempty, we must have $F = \Omega$, so u is constant in Ω. □

If Ω is not connected, then u is constant in any connected component of Ω that contains an interior point where u attains a maximum value.

EXAMPLE 2.14. The function $u(x) = |x|^2$ is subharmonic in $\mathbb{R}^n$. It attains a global minimum in $\mathbb{R}^n$ at the origin, but it does not attain a global maximum in any open set $\Omega \subset \mathbb{R}^n$. It does, of course, attain a maximum on any bounded closed set $\overline{\Omega}$, but the attainment of a maximum at a boundary point instead of an interior point does not imply that a subharmonic function is constant.

It follows immediately that superharmonic functions satisfy a minimum principle, and harmonic functions satisfy a maximum and minimum principle.

THEOREM 2.15. *Suppose that Ω is a connected open set and $u \in C^2(\Omega)$. If u is harmonic and attains either a global minimum or maximum in Ω, then u is constant.*

PROOF. Any superharmonic function u that attains a minimum in Ω is constant, since $-u$ is subharmonic and attains a maximum. A harmonic function is both subharmonic and superharmonic. $\square$

EXAMPLE 2.16. The function

$$u(x,y) = x^2 - y^2$$

is harmonic in $\mathbb{R}^2$ (it's the real part of the analytic function $f(z) = z^2$). It has a critical point at 0, meaning that $Du(0) = 0$. This critical point is a saddle-point, however, not an extreme value. Note also that

$$\fint_{B_r(0)} u\, dxdy = \frac{1}{2\pi}\int_0^{2\pi} \left(\cos^2\theta - \sin^2\theta\right)\, d\theta = 0$$

as required by the mean value property.

One consequence of this property is that any nonconstant harmonic function is an open mapping, meaning that it maps opens sets to open sets. This is not true of smooth functions such as $x \mapsto |x|^2$ that attain an interior extreme value.

2.3.1. The weak maximum principle. Theorem 2.13 is an example of a strong maximum principle, because it states that a function which attains an interior maximum is a trivial constant function. This result leads to a weak maximum principle for harmonic functions, which states that the function is bounded inside a domain by its values on the boundary. A weak maximum principle does not exclude the possibility that a non-constant function attains an interior maximum (although it implies that an interior maximum value cannot exceed the maximum value of the function on the boundary).

THEOREM 2.17. *Suppose that Ω is a bounded, connected open set in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \qquad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

PROOF. Since u is continuous and $\overline{\Omega}$ is compact, u attains its global maximum and minimum on $\overline{\Omega}$. If u attains a maximum or minimum value at an interior point, then u is constant by Theorem 2.15, otherwise both extreme values are attained on the boundary. In either case, the result follows. $\square$

Let us give a second proof of this theorem that does not depend on the mean value property. Instead, we use an argument based on the non-positivity of the second derivative at an interior maximum. In the proof, we need to account for the possibility of degenerate maxima where the second derivative is zero.

PROOF. For $\epsilon > 0$, let

$$u^\epsilon(x) = u(x) + \epsilon|x|^2.$$

Then $\Delta u^\epsilon = 2n\epsilon > 0$ since u is harmonic. If u^ϵ attained a local maximum at an interior point, then $\Delta u^\epsilon \leq 0$ by the second derivative test. Thus u^ϵ has no interior maximum, and it attains its maximum on the boundary. If $|x| \leq R$ for all $x \in \Omega$, it follows that

$$\sup_{\Omega} u \leq \sup_{\Omega} u^\epsilon \leq \sup_{\partial\Omega} u^\epsilon \leq \sup_{\partial\Omega} u + \epsilon R^2.$$

Letting $\epsilon \to 0^+$, we get that $\sup_\Omega u \leq \sup_{\partial\Omega} u$. An application of the same argument to $-u$ gives $\inf_\Omega u \geq \inf_{\partial\Omega} u$, and the result follows. □

Subharmonic functions satisfy a maximum principle, $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, while superharmonic functions satisfy a minimum principle $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

The conclusion of Theorem 2.17 may also be stated as

$$\min_{\partial\Omega} u \leq u(x) \leq \max_{\partial\Omega} u \qquad \text{for all } x \in \Omega.$$

In physical terms, this means for example that the interior of a bounded region which contains no heat sources or sinks cannot be hotter than the maximum temperature on the boundary or colder than the minimum temperature on the boundary.

The maximum principle gives a uniqueness result for the Dirichlet problem for the Poisson equation.

Theorem 2.18. *Suppose that Ω is a bounded, connected open set in $\mathbb{R}^n$ and $f \in C(\Omega)$, $g \in C(\partial\Omega)$ are given functions. Then there is at most one solution of the Dirichlet problem (2.1) with $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Proof. Suppose that $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy (2.1). Let $v = u_1 - u_2$. Then $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω and $v = 0$ on $\partial\Omega$. The maximum principle implies that $v = 0$ in Ω, so $u_1 = u_2$, and a solution is unique. □

This theorem, of course, does not address the question of whether such a solution exists. In general, the stronger the conditions we impose upon a solution, the easier it is to show uniqueness and the harder it is to prove existence. When we come to prove an existence theorem, we will begin by showing the existence of weaker solutions *e.g.* solutions in $H^1(\Omega)$ instead of $C^2(\Omega)$. We will then show that these solutions are smooth under suitable assumptions on f, g, and Ω.

2.3.2. Hopf's proof of the maximum principle. Next, we give an alternative proof of the strong maximum principle Theorem 2.13 due to E. Hopf.[2] This proof does not use the mean value property and it works for other elliptic PDEs, not just the Laplace equation.

Proof. As before, let $M = \max_{\overline{\Omega}} u$ and define

$$F = \{x \in \Omega : u(x) = M\}.$$

Then F is nonempty by assumption, and it is relatively closed in Ω since u is continuous.

Now suppose, for contradiction, that $F \neq \Omega$. Then

$$G = \Omega \setminus F$$

is nonempty and open, and the boundary $\partial F \cap \Omega = \partial G \cap \Omega$ is nonempty (otherwise F, G are open and Ω is not connected).

Choose $y \in \partial G \cap \Omega$ and let $d = \operatorname{dist}(y, \partial\Omega) > 0$. There exist points in G that are arbitrarily close to y, so we may choose $x \in G$ such that $|x - y| < d/2$. If

[2]There were two Hopf's (at least): Eberhard Hopf (1902–1983) is associated with the Hopf maximum principle (1927), the Hopf bifurcation theorem, the Wiener-Hopf method in integral equations, and the Cole-Hopf transformation for solving Burgers equation; Heinz Hopf (1894–1971) is associated with the Hopf-Rinow theorem in Riemannian geometry, the Hopf fibration in topology, and Hopf algebras.

$r = \operatorname{dist}(x, F)$, it follows that $0 < r < d/2$, so $\overline{B}_r(x) \subset G$. Moreover, there exists at least one point $\bar{x} \in \partial B_r(x) \cap \partial G$ such that $u(\bar{x}) = M$.

We therefore have the following situation: u is subharmonic in an open set G where $u < M$, the ball $B_r(x)$ is contained in G, and $u(\bar{x}) = M$ for some point $\bar{x} \in \partial B_r(x) \cap \partial G$. The Hopf boundary point lemma, proved below, then implies that

$$\partial_\nu u(\bar{x}) > 0,$$

where ∂_ν is the outward unit normal derivative to the sphere $\partial B_r(x)$.

However, since $\bar{x}$ is an interior point of Ω and u attains its maximum value M there, we have $Du(\bar{x}) = 0$, so

$$\partial_\nu u(\bar{x}) = Du(\bar{x}) \cdot \nu = 0.$$

This contradiction proves the theorem. □

Before proving the Hopf lemma, we make a definition.

DEFINITION 2.19. An open set Ω satisfies the interior sphere condition at $\bar{x} \in \partial\Omega$ if there is an open ball $B_r(x)$ contained in Ω such that $\bar{x} \in \partial B_r(x)$

The interior sphere condition is satisfied by open sets with a C^2-boundary, but — as the following example illustrates — it need not be satisfied by open sets with a C^1-boundary, and in that case the conclusion of the Hopf lemma may not hold.

EXAMPLE 2.20. Let

$$u = \Re\left(\frac{z}{\log z}\right) = \frac{x \log r + y\theta}{\log^2 r + \theta^2}$$

where $\log z = \log r + i\theta$ with $-\pi/2 < \theta < \pi/2$. Define

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\, u(x, y) < 0\right\}.$$

Then u is harmonic in Ω, since $z/\log z$ is analytic in Ω, and $\partial\Omega$ is C^1 near the origin, with unit outward normal $(-1, 0)$ at the origin. The curvature of $\partial\Omega$, however, becomes infinite at the origin, and the interior sphere condition fails. Moreover, the normal derivative $\partial_\nu u(0, 0) = -u_x(0, 0) = 0$ vanishes at the origin, and it is not strictly positive as would be required by the Hopf lemma.

LEMMA 2.21. *Suppose that $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ is subharmonic in an open set Ω and $u(x) < M$ for every $x \in \Omega$. If $u(\bar{x}) = M$ for some $\bar{x} \in \partial\Omega$ and Ω satisfies the interior sphere condition at $\bar{x}$, then $\partial_\nu u(\bar{x}) > 0$, where ∂_ν is the derivative in the outward unit normal direction to a sphere that touches $\partial\Omega$ at $\bar{x}$.*

PROOF. We want to perturb u to $u^\epsilon = u + \epsilon v$ by a function ϵv with strictly negative normal derivative at $\bar{x}$, while preserving the conditions that $u^\epsilon(\bar{x}) = M$, u^ϵ is subharmonic, and $u^\epsilon < M$ near $\bar{x}$. This will imply that the normal derivative of u at $\bar{x}$ is strictly positive.

We first construct a suitable perturbing function v. Given a ball $B_R(x)$, we want $v \in C^2(\mathbb{R}^n)$ to have the following properties:

(1) $v = 0$ on $\partial B_R(x)$;
(2) $v = 1$ on $\partial B_{R/2}(x)$;
(3) $\partial_\nu v < 0$ on $\partial B_R(x)$;
(4) $\Delta v \geq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$.

We consider without loss of generality a ball $B_R(0)$ centered at 0. Thus, we want to construct a subharmonic function in the annular region $R/2 < |x| < R$ which is 1 on the inner boundary and 0 on the outer boundary, with strictly negative outward normal derivative.

The harmonic function that is equal to 1 on $|x| = R/2$ and 0 on $|x| = R$ is given by

$$u(x) = \frac{1}{2^{n-2} - 1} \left[\left(\frac{R}{|x|} \right)^{n-2} - 1 \right]$$

(We assume that $n \geq 3$ for simplicity.) Note that

$$\partial_\nu u = -\frac{n-2}{2^{n-2}-1} \frac{1}{R} < 0 \qquad \text{on } |x| = R,$$

so we have room to fit a subharmonic function beneath this harmonic function while preserving the negative normal derivative.

Explicitly, we look for a subharmonic function of the form

$$v(x) = c \left[e^{-\alpha|x|^2} - e^{-\alpha R^2} \right]$$

where c, α are suitable positive constants. We have $v(x) = 0$ on $|x| = R$, and choosing

$$c = \frac{1}{e^{-\alpha R^2/4} - e^{-\alpha R^2}},$$

we have $v(R/2) = 1$. Also, $c > 0$ for $\alpha > 0$. The outward normal derivative of v is the radial derivative, so

$$\partial_\nu v(x) = -2c\alpha|x|e^{-\alpha|x|^2} < 0 \qquad \text{on } |x| = R.$$

Finally, using the expression for the Laplacian in polar coordinates, we find that

$$\Delta v(x) = 2c\alpha \left[2\alpha|x|^2 - n \right] e^{-\alpha|x|^2}.$$

Thus, choosing $\alpha \geq 2n/R^2$, we get $\Delta v < 0$ for $R/2 < |x| < R$, and this gives a function v with the required properties.

By the interior sphere condition, there is a ball $B_R(x) \subset \Omega$ with $\bar{x} \in \partial B_R(x)$. Let

$$M' = \max_{\overline{B}_{R/2}(x)} u < M$$

and define $\epsilon = M - M' > 0$. Let

$$w = u + \epsilon v - M.$$

Then $w \leq 0$ on $\partial B_R(x)$ and $\partial B_{R/2}(x)$ and $\Delta w \geq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$. The maximum principle for subharmonic functions implies that $w \leq 0$ in $B_R(x) \setminus \overline{B}_{R/2}(x)$. Since $w(\bar{x}) = 0$, it follows that $\partial_\nu w(\bar{x}) \geq 0$. Therefore

$$\partial_\nu u(\bar{x}) = \partial_\nu w(\bar{x}) - \epsilon \partial_\nu v(\bar{x}) > 0,$$

which proves the result. □

2.4. Harnack's inequality

The maximum principle gives a basic pointwise estimate for solutions of Laplace's equation, and it has a natural physical interpretation. Harnack's inequality is another useful pointwise estimate, although its physical interpretation is less clear. It states that if a function is nonnegative and harmonic in a domain, then the ratio of the maximum and minimum of the function on a compactly supported subdomain is bounded by a constant that depends only on the domains. This inequality controls, for example, the amount by which a harmonic function can oscillate inside a domain in terms of the size of the function.

THEOREM 2.22. *Suppose that $\Omega' \Subset \Omega$ is a connected open set that is compactly contained an open set Ω. There exists a constant C, depending only on Ω and Ω', such that if $u \in C(\Omega)$ is a non-negative function with the mean value property, then*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u. \tag{2.9}$$

PROOF. First, we establish the inequality for a compactly contained open ball. Suppose that $x \in \Omega$ and $B_{4R}(x) \subset \Omega$, and let u be any non-negative function with the mean value property in Ω. If $y \in B_R(x)$, then,

$$u(y) = \fint_{B_R(y)} u\, dx \leq 2^n \fint_{B_{2R}(x)} u\, dx$$

since $B_R(y) \subset B_{2R}(x)$ and u is non-negative. Similarly, if $z \in B_R(x)$, then

$$u(z) = \fint_{B_{3R}(z)} u\, dx \geq \left(\frac{2}{3}\right)^n \fint_{B_{2R}(x)} u\, dx$$

since $B_{3R}(z) \supset B_{2R}(x)$. It follows that

$$\sup_{B_R(x)} u \leq 3^n \inf_{B_R(x)} u.$$

Suppose that $\Omega' \Subset \Omega$ and $0 < 4R < \operatorname{dist}(\Omega', \partial\Omega)$. Since $\overline{\Omega'}$ is compact, we may cover Ω' by a finite number of open balls of radius R, where the number N of such balls depends only on Ω' and Ω. Moreover, since Ω' is connected, for any $x, y \in \Omega$ there is a sequence of at most N overlapping balls $\{B_1, B_2, \dots, B_k\}$ such that $B_i \cap B_{i+1} \neq \emptyset$ and $x \in B_1$, $y \in B_k$. Applying the above estimate to each ball and combining the results, we obtain that

$$\sup_{\Omega'} u \leq 3^{nN} \inf_{\Omega'} u.$$

□

In particular, it follows from (2.9) that for any $x, y \in \Omega'$, we have

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y).$$

Harnack's inequality has strong consequences. For example, it implies that if $\{u_n\}$ is a decreasing sequence of harmonic functions in Ω and $\{u_n(x)\}$ is bounded for some $x \in \Omega$, then the sequence converges uniformly on compact subsets of Ω to a function that is harmonic in Ω. By contrast, the convergence of an arbitrary sequence of smooth functions at a single point in no way implies its convergence anywhere else, nor does uniform convergence of smooth functions imply that their limit is smooth.

It is useful to compare this situation with what happens for analytic functions in complex analysis. If $\{f_n\}$ is a sequence of analytic functions

$$f_n : \Omega \subset \mathbb{C} \to \mathbb{C}$$

that converges uniformly on compact subsets of Ω to a function f, then f is also analytic in Ω because uniform convergence implies that the Cauchy integral formula continues to hold for f, and differentiation of this formula implies that f is analytic.

2.5. Green's identities

Green's identities provide the main energy estimates for the Laplace and Poisson equations.

THEOREM 2.23. *If Ω is a bounded C^1 open set in $\mathbb{R}^n$ and $u, v \in C^2(\overline{\Omega})$, then*

$$\int_\Omega u\Delta v\, dx = -\int_\Omega Du \cdot Dv\, dx + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\, dS, \tag{2.10}$$

$$\int_\Omega u\Delta v\, dx = \int_\Omega v\Delta u\, dx + \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) dS. \tag{2.11}$$

PROOF. Integrating the identity

$$\operatorname{div}(uDv) = u\Delta v + Du \cdot Dv$$

over Ω and using the divergence theorem, we get (2.10). Integrating the identity

$$\operatorname{div}(uDv - vDu) = u\Delta v - v\Delta u,$$

we get (2.11). □

Equations (2.10) and (2.11) are Green's first and second identity, respectively. The second Green's identity implies that the Laplacian Δ is a formally self-adjoint differential operator.

Green's first identity provides a proof of the uniqueness of solutions of the Dirichlet problem based on estimates of L^2-norms of derivatives instead of maximum norms. Such integral estimates are called energy estimates, because in many (though not all) cases these integral norms may be interpreted physically as the energy of a solution.

THEOREM 2.24. *Suppose that Ω is a connected, bounded C^1 open set, $f \in C(\overline{\Omega})$, and $g \in C(\partial\Omega)$. If $u_1, u_2 \in C^2(\overline{\Omega})$ are solution of the Dirichlet problem (2.1), then $u_1 = u_2$; and if $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions of the Neumann problem (2.2), then $u_1 = u_2 + C$ where $C \in \mathbb{R}$ is a constant.*

PROOF. Let $w = u_1 - u_2$. Then $\Delta w = 0$ in Ω and either $w = 0$ or $\partial w/\partial\nu = 0$ on $\partial\Omega$. Setting $u = w$, $v = w$ in (2.10), it follows that the boundary integral and the integral $\int_\Omega w\Delta w\, dx$ vanish, so that

$$\int_\Omega |Dw|^2\, dx = 0.$$

Therefore $Dw = 0$ in Ω, so w is constant. For the Dirichlet problem, $w = 0$ on $\partial\Omega$ so the constant is zero, and both parts of the result follow. □

2.6. Fundamental solution

We define the fundamental solution or free-space Green's function $\Gamma : \mathbb{R}^n \to \mathbb{R}$ (not to be confused with the Gamma function!) of Laplace's equation by

$$
\begin{aligned}
\Gamma(x) &= \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}} && \text{if } n \geq 3, \\
\Gamma(x) &= -\frac{1}{2\pi} \log |x| && \text{if } n = 2.
\end{aligned}
\tag{2.12}
$$

The corresponding potential for $n = 1$ is

$$
\Gamma(x) = -\frac{1}{2}|x|, \tag{2.13}
$$

but we will consider only the multi-variable case $n \geq 2$. (Our sign convention for Γ is the same as Evans [**9**], but the opposite of Gilbarg and Trudinger [**17**].)

2.6.1. Properties of the solution. The potential $\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is smooth away from the origin. For $x \neq 0$, we compute that

$$
\partial_i \Gamma(x) = -\frac{1}{n\alpha_n} \frac{1}{|x|^{n-1}} \frac{x_i}{|x|}, \tag{2.14}
$$

and

$$
\partial_{ii} \Gamma(x) = \frac{1}{\alpha_n} \frac{x_i^2}{|x|^{n+2}} - \frac{1}{n\alpha_n} \frac{1}{|x|^n}.
$$

It follows that

$$
\Delta \Gamma = 0 \qquad \text{if } x \neq 0,
$$

so Γ is harmonic in any open set that does not contain the origin. The function Γ is homogeneous of degree $-n+2$, its first derivative is homogeneous of degree $-n+1$, and its second derivative is homogeneous of degree n.

From (2.14), we have for $x \neq 0$ that

$$
D\Gamma \cdot \frac{x}{|x|} = -\frac{1}{n\alpha_n} \frac{1}{|x|^{n-1}}
$$

Thus we get the following surface integral over a sphere centered at the origin with normal $\nu = x/|x|$:

$$
-\int_{\partial B_r(0)} D\Gamma \cdot \nu \, dS = 1. \tag{2.15}
$$

As follows from the divergence theorem and the fact that Γ is harmonic in $B_R(0) \setminus B_r(0)$, this integral does not depend on r. The surface integral is not zero, however, as it would be for a function that was harmonic everywhere inside $B_r(0)$, including at the origin. The normalization of the flux integral in (2.15) to one accounts for the choice of the multiplicative constant in the definition of Γ.

The function Γ is unbounded as $x \to 0$ with $\Gamma(x) \to \infty$. Nevertheless, Γ and $D\Gamma$ are locally integrable. For example, the local integrability of $\partial_i \Gamma$ in (2.14) follows from the estimate

$$
|\partial_i \Gamma(x)| \leq \frac{C_n}{|x|^{n-1}},
$$

since $|x|^{-a}$ is locally integrable on $\mathbb{R}^n$ when $a < n$ (see Example 1.13). The second partial derivatives of Γ are not locally integrable, however, since they are of the order $|x|^{-n}$ as $x \to 0$.

2.6.2. Physical interpretation. Suppose, as in electrostatics, that u is the potential due to a charge distribution with smooth density f, where $-\Delta u = f$, and $E = -Du$ is the electric field. By the divergence theorem, the flux of E through the boundary $\partial\Omega$ of an open set Ω is equal to the to charge inside the enclosed volume,

$$\int_{\partial\Omega} E \cdot \nu \, dS = \int_{\Omega} (-\Delta u) \, dx = \int_{\Omega} f \, dx.$$

Thus, since $\Delta\Gamma = 0$ for $x \neq 0$ and from (2.15) the flux of $-D\Gamma$ through any sphere centered at the origin is equal to one, we may interpret Γ as the potential due to a point charge located at the origin. In the sense of distributions, Γ satisfies the PDE

$$-\Delta\Gamma = \delta$$

where δ is the delta-function supported at the origin. We refer to such a solution as a Green's function of the Laplacian.

In three space dimensions, the electric field $E = -D\Gamma$ of the point charge is given by

$$E = -\frac{1}{4\pi}\frac{1}{|x|^2}\frac{x}{|x|},$$

corresponding to an inverse-square force directed away from the origin. For gravity, which is always attractive, the force has the opposite sign. This explains the connection between the Laplace and Poisson equations and Newton's inverse square law of gravitation.

As $|x| \to \infty$, the potential $\Gamma(x)$ approaches zero if $n \geq 3$, but $\Gamma(x) \to -\infty$ as $|x| \to \infty$ if $n = 2$. Physically, this corresponds to the fact that only a finite amount of energy is required to remove an object from a point source in three or more space dimensions (for example, to remove a rocket from the earth's gravitational field) but an infinite amount of energy is required to remove an object from a line source in two space dimensions.

We will use the point-source potential Γ to construct solutions of Poisson's equation for rather general right hand sides. The physical interpretation of the method is that we can obtain the potential of a general source by representing the source as a continuous distribution of point sources and superposing the corresponding point-source potential as in (2.24) below. This method, of course, depends crucially on the linearity of the equation.

2.7. The Newtonian potential

Consider the equation

$$-\Delta u = f \qquad \text{in } \mathbb{R}^n$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given function, which for simplicity we assume is smooth and compactly supported.

THEOREM 2.25. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and let*

$$u = \Gamma * f$$

where Γ is the fundamental solution (2.12). Then $u \in C^\infty(\mathbb{R}^n)$ and

$$-\Delta u = f. \tag{2.16}$$

PROOF. Since $f \in C_c^\infty(\mathbb{R}^n)$ and $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, Theorem 1.28 implies that $u \in C^\infty(\mathbb{R}^n)$ and

$$\Delta u = \Gamma * (\Delta f) \tag{2.17}$$

Our objective is to transfer the Laplacian across the convolution from f to Γ.

If $x \notin \operatorname{supp} f$, then we may choose a smooth open set Ω that contains $\operatorname{supp} f$ such that $x \notin \Omega$. Then $\Gamma(x-y)$ is a smooth, harmonic function of y in $\overline{\Omega}$ and f, Df are zero on $\partial\Omega$. Green's theorem therefore implies that

$$\Delta u(x) = \int_\Omega \Gamma(x-y)\Delta f(y)\, dy = \int_\Omega \Delta\Gamma(x-y) f(y)\, dy = 0,$$

which shows that $-\Delta u(x) = f(x)$.

If $x \in \operatorname{supp} f$, we must be careful about the non-integrable singularity in $\Delta\Gamma$. We therefore 'cut out' a ball of radius r about the singularity, apply Green's theorem to the resulting smooth integral, and then take the limit as $r \to 0^+$.

Let Ω be an open set that contains the support of f and define

$$\Omega_r(x) = \Omega \setminus B_r(x)\,. \tag{2.18}$$

Since Δf is bounded with compact support and Γ is locally integrable, the Lebesgue dominated convergence theorem implies that

$$\Gamma * (\Delta f)(x) = \lim_{r\to 0^+} \int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\, dy. \tag{2.19}$$

The potential $\Gamma(x-y)$ is a smooth, harmonic function of y in $\overline{\Omega_r(x)}$. Thus Green's identity (2.11) gives

$$\begin{aligned}
&\int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\, dy \\
&\quad= \int_{\partial\Omega} \left[\Gamma(x-y) D_y f(y)\cdot\nu(y) - D_y\Gamma(x-y)\cdot\nu(y) f(y)\right] dS(y) \\
&\quad- \int_{\partial B_r(x)} \left[\Gamma(x-y) D_y f(y)\cdot\nu(y) - D_y\Gamma(x-y)\cdot\nu(y) f(y)\right] dS(y)
\end{aligned}$$

where we use the radially outward unit normal on the boundary. The boundary terms on $\partial\Omega$ vanish because f and Df are zero there, so

$$\begin{aligned}
\int_{\Omega_r(x)} \Gamma(x-y)\Delta f(y)\, dy = &-\int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\, dS(y) \\
&+ \int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y) f(y)\, dS(y).
\end{aligned} \tag{2.20}$$

Since Df is bounded and $\Gamma(x) = O(|x|^{n-2})$ if $n \geq 3$, we have

$$\int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\, dS(y) = O(r) \qquad \text{as } r \to 0^+.$$

The integral is $O(r\log r)$ if $n = 2$. In either case,

$$\lim_{r\to 0^+} \int_{\partial B_r(x)} \Gamma(x-y) D_y f(y)\cdot\nu(y)\, dS(y) = 0. \tag{2.21}$$

For the surface integral in (2.20) that involves $D\Gamma$, we write

$$\begin{aligned}&\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)f(y)\,dS(y)\\&\qquad=\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\,[f(y)-f(x)]\,\,dS(y)\\&\qquad+f(x)\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\,dS(y).\end{aligned}$$

From (2.15),

$$\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)\,dS(y)=-1;$$

and, since f is smooth,

$$\int_{\partial B_r(x)} D_y\Gamma(x-y)\,[f(y)-f(x)]\,\,dS(y)=O\left(r^{n-1}\cdot\frac{1}{r^{n-1}}\cdot r\right)\to 0$$

as $r\to 0^+$. It follows that

$$\lim_{r\to 0^+}\int_{\partial B_r(x)} D_y\Gamma(x-y)\cdot\nu(y)f(y)\,dS(y)=-f(x). \tag{2.22}$$

Taking the limit of (2.20) as $r\to 0^+$ and using (2.21) and (2.22) in the result, we get

$$\lim_{r\to 0^+}\int_{\Omega_r(x)}\Gamma(x-y)\Delta f(y)\,dy=-f(x).$$

The use of this equation in (2.19) shows that

$$\Gamma*(\Delta f)=-f, \tag{2.23}$$

and the use of (2.23) in (2.17) gives (2.16). □

Equation (2.23) is worth noting: it provides a representation of a function $f\in C_c^\infty(\mathbb{R}^n)$ as a convolution of its Laplacian with the Newtonian potential.

The potential u associated with a source distribution f is given by

$$u(x)=\int\Gamma(x-y)f(y)\,dy. \tag{2.24}$$

We call u the Newtonian potential of f. We may interpret $u(x)$ as a continuous superposition of potentials proportional to $\Gamma(x-y)$ due to point sources of strength $f(y)\,dy$ located at y.

If $n\geq 3$, the potential $\Gamma*f(x)$ of a compactly supported, integrable function approaches zero as $|x|\to\infty$. We have

$$\Gamma*f(x)=\frac{1}{n(n-2)\alpha_n|x|^{n-2}}\int\left(\frac{|x|}{|x-y|}\right)^{n-2}f(y)\,dy,$$

and by the Lebesgue dominated convergence theorem,

$$\lim_{|x|\to\infty}\int\left(\frac{|x|}{|x-y|}\right)^{n-2}f(y)\,dy=\int f(y)\,dy.$$

Thus, the asymptotic behavior of the potential is the same as that of a point source whose charge is equal to the total charge of the source density f. If $n=2$, the potential, in general, grows logarithmically as $|x|\to\infty$.

If $n \geq 3$, Liouville's theorem (Corollary 2.8) implies that the Newtonian potential $\Gamma * f$ is the unique solution of $-\Delta u = f$ such that $u(x) \to 0$ as $x \to \infty$. (If u_1, u_2 are solutions, then $v = u_1 - u_2$ is harmonic in $\mathbb{R}^n$ and approaches 0 as $x \to \infty$; thus v is bounded and therefore constant, so $v = 0$.) If $n = 2$, then a similar argument shows that any solution of Poisson's equation such that $Du(x) \to 0$ as $|x| \to \infty$ differs from the Newtonian potential by a constant.

2.7.1. Second derivatives of the potential. In order to study the regularity of the Newtonian potential u in terms of f, we derive an integral representation for its second derivatives.

We write $\partial_i \partial_j = \partial_{ij}$, and let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

denote the Kronecker delta. In the following $\partial_i \Gamma(x-y)$ denotes the ith partial derivative of Γ evaluated at $x-y$, with similar notation for other derivatives. Thus,

$$\frac{\partial}{\partial y_i}\Gamma(x-y) = -\partial_i \Gamma(x-y).$$

THEOREM 2.26. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and $u = \Gamma * f$ where Γ is the Newtonian potential (2.12). If Ω is any smooth open set that contains the support of f, then*

$$\begin{aligned} \partial_{ij} u(x) = &\int_\Omega \partial_{ij}\Gamma(x-y)\left[f(y) - f(x)\right] dy \\ &- f(x)\int_{\partial\Omega} \partial_i \Gamma(x-y)\nu_j(y)\, dS(y). \end{aligned} \tag{2.25}$$

PROOF. As before, the result is straightforward to prove if $x \notin \operatorname{supp} f$. We choose $\Omega \supset \operatorname{supp} f$ such that $x \notin \overline{\Omega}$. Then Γ is smooth on $\overline{\Omega}$ so we may differentiate under the integral sign to get

$$\partial_{ij} u(x) = \int_\Omega \partial_{ij}\Gamma(x-y) f(y)\, dy.,$$

which is (2.25) with $f(x) = 0$.

If $x \in \operatorname{supp} f$, we follow a similar procedure to the one used in the proof of Theorem 2.25: We differentiate under the integral sign in the convolution $u = \Gamma * f$ on f, cut out a ball of radius r about the singularity in Γ, apply Greens' theorem, and let $r \to 0^+$.

In detail, define $\Omega_r(x)$ as in (2.18), where $\Omega \supset \operatorname{supp} f$ is a smooth open set. Since Γ is locally integrable, the Lebesgue dominated convergence theorem implies that

$$\partial_{ij} u(x) = \int_\Omega \Gamma(x-y)\partial_{ij} f(y)\, dy = \lim_{r\to 0^+}\int_{\Omega_r(x)} \Gamma(x-y)\partial_{ij} f(y)\, dy. \tag{2.26}$$

For $x \neq y$, we have the identity

$$\begin{aligned} &\Gamma(x-y)\partial_{ij} f(y) - \partial_{ij}\Gamma(x-y) f(y) \\ &\qquad = \frac{\partial}{\partial y_i}\left[\Gamma(x-y)\partial_j f(y)\right] + \frac{\partial}{\partial y_j}\left[\partial_i \Gamma(x-y) f(y)\right]. \end{aligned}$$

Thus, using Green's theorem, we get

$$\begin{aligned}\int_{\Omega_r(x)} \Gamma(x-y)\partial_{ij} f(y)\,dy &= \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y) f(y)\,dy \\ &\quad - \int_{\partial B_r(x)} \left[\Gamma(x-y)\partial_j f(y)\nu_i(y) + \partial_i\Gamma(x-y) f(y)\nu_j(y)\right]\,dS(y).\end{aligned} \tag{2.27}$$

In (2.27), ν denotes the radially outward unit normal vector on $\partial B_r(x)$, which accounts for the minus sign of the surface integral; the integral over the boundary $\partial\Omega$ vanishes because f is identically zero there.

We cannot take the limit of the integral over $\Omega_r(x)$ directly, since $\partial_{ij}\Gamma$ is not locally integrable. To obtain a limiting integral that is convergent, we write

$$\begin{aligned}&\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y) f(y)\,dy \\ &\qquad = \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy + f(x)\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\,dy \\ &\qquad = \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy \\ &\qquad\quad - f(x)\left[\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) - \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\right].\end{aligned}$$

Using this expression in (2.27) and using the result in (2.26), we get

$$\begin{aligned}\partial_{ij} u(x) = \lim_{r\to 0^+} &\int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy \\ &- f(x)\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) \\ &- \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\big[f(y)-f(x)\big]\nu_j(y)\,dS(y) \\ &- \int_{\partial B_r(x)} \Gamma(x-y)\partial_j f(y)\nu_i(y)\,dS(y).\end{aligned} \tag{2.28}$$

Since f is smooth, the function $y \mapsto \partial_{ij}\Gamma(x-y)\left[f(y)-f(x)\right]$ is integrable on Ω, and by the Lebesgue dominated convergence theorem

$$\lim_{r\to 0^+} \int_{\Omega_r(x)} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy = \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy.$$

We also have

$$\begin{aligned}&\lim_{r\to 0^+} \int_{\partial B_r(x)} \partial_i\Gamma(x-y)\big[f(y)-f(x)\big]\nu_j(y)\,dS(y) = 0, \\ &\lim_{r\to 0^+} \int_{\partial B_r(x)} \Gamma(x-y)\partial_j f(y)\nu_i(y)\,dS(y) = 0.\end{aligned}$$

Using these limits in (2.28), we get (2.25). □

Note that if $\Omega' \supset \Omega \supset \operatorname{supp} f$, then writing

$$\Omega' = \Omega \cup (\Omega' \setminus \Omega)$$

and using the divergence theorem, we get

$$\begin{aligned}
&\int_{\Omega'} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy - f(x)\int_{\partial\Omega'} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\\
&\quad= \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy\\
&\qquad - f(x)\left[\int_{\partial\Omega'} \partial_i\Gamma(x-y)\nu_j(x-y)\,dS(y) + \int_{\Omega'\setminus\Omega} \partial_{ij}\Gamma(x-y)\,dy\right]\\
&\quad= \int_{\Omega} \partial_{ij}\Gamma(x-y)\big[f(y)-f(x)\big]\,dy - f(x)\int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y).
\end{aligned}$$

Thus, the expression on the right-hand side of (2.25) does not depend on Ω provided that it contains the support of f. In particular, we can choose Ω to be a sufficiently large ball centered at x.

Corollary 2.27. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and $u = \Gamma * f$ where Γ is the Newtonian potential (2.12). Then*

$$\partial_{ij}u(x) = \int_{B_R(x)} \partial_{ij}\Gamma(x-y)\,[f(y)-f(x)]\,dy - \frac{1}{n}f(x)\delta_{ij} \tag{2.29}$$

where $B_R(x)$ is any open ball centered at x that contains the support of f.

Proof. In (2.25), we choose $\Omega = B_R(x) \supset \operatorname{supp} f$. From (2.14), we have

$$\begin{aligned}
\int_{\partial B_R(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) &= \int_{\partial B_R(x)} \frac{-(x_i-y_i)}{n\alpha_n|x-y|^n}\,\frac{y_j-x_j}{|y-x|}\,dS(y)\\
&= \int_{\partial B_R(0)} \frac{y_iy_j}{n\alpha_n|y|^{n+1}}\,dS(y)
\end{aligned}$$

If $i \ne j$, then y_iy_j is odd under a reflection $y_i \mapsto -y_i$, so this integral is zero. If $i = j$, then the value of the integral does not depend on i, since we may transform the i-integral into an i'-integral by a rotation. Therefore

$$\begin{aligned}
\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{y_i^2}{|y|^{n+1}}\,dS(y) &= \frac{1}{n}\sum_{i=1}^n\left(\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{y_i^2}{|y|^{n+1}}\,dS(y)\right)\\
&= \frac{1}{n}\frac{1}{n\alpha_n}\int_{\partial B_R(0)} \frac{1}{|y|^{n-1}}\,dS(y)\\
&= \frac{1}{n}.
\end{aligned}$$

It follows that

$$\int_{\partial B_R(x)} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) = \frac{1}{n}\delta_{ij}.$$

Using this result in (2.25), we get (2.29). □

2.7.2. Hölder estimates. We want to derive estimates of the derivatives of the Newtonian potential $u = \Gamma * f$ in terms of the source density f. We continue to assume that $f \in C_c^\infty(\mathbb{R}^n)$; the estimates extend by a density argument to any Hölder-continuous function f with compact support (or sufficiently rapid decay at infinity).

In one space dimension, a solution of the ODE

$$-u'' = f$$

is given in terms of the potential (2.13) by

$$u(x) = -\frac{1}{2}\int |x-y|\, f(y)\, dy.$$

If $f \in C_c(\mathbb{R})$, then obviously $u \in C^2(\mathbb{R})$ and $\max|u''| = \max|f|$.

In more than one space dimension, however, it is not possible estimate the maximum norm of the second derivative D^2u of the potential $u = \Gamma * f$ in terms of the maximum norm of f, and there exist functions $f \in C_c(\mathbb{R}^n)$ for which $u \notin C^2(\mathbb{R}^n)$.

Nevertheless, if we measure derivatives in an appropriate way, we gain two derivatives in solving the Laplace equation (and other second-order elliptic PDEs). The fact that in inverting the Laplacian we gain as many derivatives as the order of the PDE is the essential point of elliptic regularity theory; this does not happen for many other types of PDEs, such as hyperbolic PDEs.

In particular, if we measure derivatives in terms of their Hölder continuity, we can estimate the $C^{2,\alpha}$-norm of u in terms of the $C^{0,\alpha}$-norm of f. These Hölder estimates were used by Schauder[3] to develop a general existence theory for elliptic PDEs with Hölder continuous coefficients, typically referred to as the Schauder theory [**17**].

Here, we will derive Hölder estimates for the Newtonian potential.

Theorem 2.28. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$ and $0 < \alpha < 1$. If $u = \Gamma * f$ where Γ is the Newtonian potential (2.12), then*

$$[\partial_{ij}u]_{0,\alpha} \le C\,[f]_{0,\alpha}$$

where $[\cdot]_{0,\alpha}$ denotes the Hölder semi-norm (1.1) and C is a constant that depends only on α and n.

Proof. Let Ω be a smooth open set that contains the support of f. We write (2.25) as

$$\partial_{ij}u = Tf - fg \tag{2.30}$$

where the linear operator

$$T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is defined by

$$Tf(x) = \int_\Omega K(x-y)\,[f(y) - f(x)]\; dy, \qquad K = \partial_{ij}\Gamma,$$

and the function $g : \mathbb{R}^n \to \mathbb{R}$ is given by

$$g(x) = \int_{\partial\Omega} \partial_i\Gamma(x-y)\nu_j(y)\, dS(y). \tag{2.31}$$

If $x, x' \in \mathbb{R}^n$, then

$$\partial_{ij}u(x) - \partial_{ij}u(x') = Tf(x) - Tf(x') - [f(x)g(x) - f(x')g(x')]$$

[3]Juliusz Schauder (1899–1943) was a Polish mathematician. In addition to the Schauder theory for elliptic PDEs, he is known for the Leray-Schauder fixed point theorem, and Schauder bases of a Banach space. He was killed by the Nazi's while they occupied Lvov during the second world war.

The main part of the proof is to estimate the difference of the terms that involve Tf.

In order to do this, let

$$\bar{x} = \frac{1}{2}(x + x'), \qquad \delta = |x - x'|,$$

and choose Ω so that it contains $B_{2\delta}(\bar{x})$. We have

$$\begin{aligned} &Tf(x) - Tf(x') \\ &\quad = \int_\Omega \{K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]\}\, dy. \end{aligned} \tag{2.32}$$

We will separate the the integral over Ω in (2.32) into two parts: (a) $|y - \bar{x}| < \delta$; (b) $|y - \bar{x}| \geq \delta$. In region (a), which contains the points $y = x$, $y = x'$ where K is singular, we will use the Hölder continuity of f and the smallness of the integration region to estimate the integral. In region (b), we will use the Hölder continuity of f and the smoothness of K to estimate the integral.

(a) Suppose that $|y - \bar{x}| < \delta$, meaning that $y \in B_\delta(\bar{x})$. Then

$$|x - y| \leq |x - \bar{x}| + |\bar{x} - y| \leq \frac{3}{2}\delta,$$

so $y \in B_{3\delta/2}(x)$, and similarly for x'. Using the Hölder continuity of f and the fact that K is homogeneous of degree $-n$, we have

$$\begin{aligned} &|K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]| \\ &\qquad\qquad \leq C\,[f]_{0,\alpha}\left\{|x-y|^{\alpha-n} + |x'-y|^{\alpha-n}\right\}. \end{aligned}$$

Thus, using C to denote a generic constant depending on α and n, we get

$$\begin{aligned} &\int_{B_\delta(\bar{x})} |K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')]|\, dy \\ &\qquad \leq C\,[f]_{0,\alpha}\int_{B_\delta(\bar{x})}\left[|x-y|^{\alpha-n} + |x'-y|^{\alpha-n}\right] dy \\ &\qquad \leq C\,[f]_{0,\alpha}\int_{B_{3\delta/2}(0)} |y|^{\alpha-n} dy \\ &\qquad \leq C\,[f]_{0,\alpha}\,\delta^\alpha. \end{aligned}$$

(b) Suppose that $|y - \bar{x}| \geq \delta$. We write

$$\begin{aligned} &K(x-y)[f(y) - f(x)] - K(x'-y)[f(y) - f(x')] \\ &\quad = [K(x-y) - K(x'-y)][f(y) - f(x)] - K(x'-y)[f(x) - f(x')] \end{aligned} \tag{2.33}$$

and estimate the two terms on the right hand side separately. For the first term, we use the the Hölder continuity of f and the smoothness of K; for the second term we use the Hölder continuity of f and the divergence theorem to estimate the integral of K.

(b1) Since DK is homogeneous of degree $-(n+1)$, the mean value theorem implies that

$$|K(x-y) - K(x'-y)| \leq C\frac{|x - x'|}{|\xi - y|^{n+1}}$$

for $\xi = \theta x + (1-\theta)x'$ with $0 < \theta < 1$. Using this estimate and the Hölder continuity of f, we get

$$\left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right| \le C\,[f]_{0,\alpha}\,\delta \frac{|y-x|^{\alpha}}{|\xi - y|^{n+1}}.$$

We have

$$\begin{aligned}
|y-x| &\le |y-\bar{x}| + |\bar{x}-x| = |y-\bar{x}| + \frac{1}{2}\delta \le \frac{3}{2}|y-\bar{x}|,\\
|\xi-y| &\ge |y-\bar{x}| - |\bar{x}-\xi| \ge |y-\bar{x}| - \frac{1}{2}\delta \ge \frac{1}{2}|y-\bar{x}|.
\end{aligned}$$

It follows that

$$\left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right| \le C\,[f]_{0,\alpha}\,\delta |y-\bar{x}|^{\alpha-n-1}.$$

Thus,

$$\begin{aligned}
&\int_{\Omega\setminus B_\delta(\bar{x})} \left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right|\,dy\\
&\qquad\le \int_{\mathbb{R}^n\setminus B_\delta(\bar{x})} \left|\left[K(x-y) - K(x'-y)\right]\left[f(y) - f(x)\right]\right|\,dy\\
&\qquad\le C\,[f]_{0,\alpha}\,\delta \int_{|y|\ge\delta} |y|^{\alpha-n-1}\,dy\\
&\qquad\le C\,[f]_{0,\alpha}\,\delta^{\alpha}.
\end{aligned}$$

Note that the integral does not converge at infinity if $\alpha = 1$; this is where we require $\alpha < 1$.

(b2) To estimate the second term in (2.33), we suppose that $\Omega = B_R(\bar{x})$ where $B_R(\bar{x})$ contains the support of f and $R \ge 2\delta$. (All of the estimates above apply for this choice of Ω.) Writing $K = \partial_{ij}\Gamma$ and using the divergence theorem we get

$$\begin{aligned}
&\int_{B_R(\bar{x})\setminus B_\delta(\bar{x})} K(x-y)\,dy\\
&\qquad= \int_{\partial B_R(\bar{x})} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y) - \int_{\partial B_\delta(\bar{x})} \partial_i\Gamma(x-y)\nu_j(y)\,dS(y).
\end{aligned}$$

If $y \in \partial B_R(\bar{x})$, then

$$|x-y| \ge |y-\bar{x}| - |\bar{x}-x| \ge R - \frac{1}{2}\delta \ge \frac{3}{4}R;$$

and If $y \in \partial B_\delta(\bar{x})$, then

$$|x-y| \ge |y-\bar{x}| - |\bar{x}-x| \ge \delta - \frac{1}{2}\delta \ge \frac{1}{2}\delta.$$

Thus, using the fact that $D\Gamma$ is homogeneous of degree $-n+1$, we compute that

$$\int_{\partial B_R(\bar{x})} \left|\partial_i\Gamma(x-y)\nu_j(y)\right|\,dS(y) \le CR^{n-1}\frac{1}{R^{n-1}} \le C \tag{2.34}$$

and

$$\int_{\partial B_\delta(\bar{x})} \left|\partial_i\Gamma(x-y)\nu_j(y)\,dS(y)\right| C\delta^{n-1}\frac{1}{\delta^{n-1}} \le C$$

Thus, using the Hölder continuity of f, we get

$$\left|[f(x) - f(x')] \int_{\Omega \setminus B_\delta(\bar{x})} K(x' - y)\, dy\right| \leq C\, [f]_{0,\alpha}\, \delta^\alpha.$$

Putting these estimates together, we conclude that

$$|Tf(x) - Tf(x')| \leq C\, [f]_{0,\alpha}\, |x - x'|^\alpha$$

where C is a constant that depends only on α and n.

(c) Finally, to estimate the Hölder norm of the remaining term fg in (2.30), we continue to assume that $\Omega = B_R(\bar{x})$. From (2.31),

$$g(\bar{x} + h) = \int_{\partial B_R(0)} \partial_i \Gamma(h - y) \nu_j(y)\, dS(y).$$

Changing $y \mapsto -y$ in the integral, we find that $g(\bar{x} + h) = g(\bar{x} - h)$. Hence $g(x) = g(x')$. Moreover, from (2.34), we have $|g(x)| \leq C$. It therefore follows that

$$|f(x)g(x) - f(x')g(x')| \leq C\, |f(x) - f(x')| \leq C\, [f]_{0,\alpha}\, |x - x'|^\alpha,$$

which completes the proof. □

These Hölder estimates, and their generalizations, are fundamental to theory of elliptic PDEs. Their derivation by direct estimation of the Newtonian potential is only one of many methods to obtain them (although it was the original method). For example, they can also be obtained by the use of Campanato spaces, which provide Hölder estimates in terms of suitable integral norms [**23**], or by the use of Littlewood-Payley theory, which provides Hölder estimates in terms of dyadic decompositions of the Fourier transform [**5**].

2.8. Singular integral operators

Using (2.29), we may define a linear operator

$$T_{ij} : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

that gives the second derivatives of a function in terms of its Laplacian,

$$\partial_{ij} u = T_{ij} \Delta u.$$

Explicitly,

$$T_{ij} f(x) = \int_{B_R(x)} K_{ij}(x - y) \left[f(y) - f(x)\right]\, dy + \frac{1}{n} f(x) \delta_{ij} \tag{2.35}$$

where $B_R(x) \supset \operatorname{supp} f$ and $K_{ij} = -\partial_{ij}\Gamma$ is given by

$$K_{ij}(x) = \frac{1}{\alpha_n |x|^n} \left(\frac{1}{n}\delta_{ij} - \frac{x_i x_j}{|x|^2}\right). \tag{2.36}$$

This function is homogeneous of degree $-n$, the borderline power for integrability, so it is not locally integrable. Thus, Young's inequality does not imply that convolution with K_{ij} is a bounded operator on L^∞_{loc}, which explains why we cannot bound the maximum norm of $D^2 u$ in terms of the maximum norm of f.

The kernel K_{ij} in (2.36) has zero integral over any sphere, meaning that

$$\int_{B_R(0)} K_{ij}(y)\, dS(y) = 0.$$

Thus, we may alternatively write T_{ij} as

$$\begin{aligned} T_{ij}f(x) - \frac{1}{n}f(x)\delta_{ij} &= \lim_{\epsilon\to 0^+}\int_{B_R(x)\setminus B_\epsilon(x)} K_{ij}(x-y)\left[f(y)-f(x)\right]\,dy \\ &= \lim_{\epsilon\to 0^+}\int_{B_R(x)\setminus B_\epsilon(x)} K_{ij}(x-y)f(y)\,dy \\ &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} K_{ij}(x-y)f(y)\,dy. \end{aligned}$$

This is an example of a singular integral operator.

The operator T_{ij} can also be expressed in terms of the Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n}\int f(x)e^{-i\cdot\xi}\,dx$$

as

$$\widehat{(T_{ij}f)}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}\hat{f}(\xi).$$

Since the multiplier $m_{ij} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$m_{ij}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}$$

belongs to $L^\infty(\mathbb{R}^n)$, it follows from Plancherel's theorem that T_{ij} extends to a bounded linear operator on $L^2(\mathbb{R}^n)$.

In more generality, consider a function $K : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable in $\mathbb{R}^n \setminus 0$ and satisfies the following conditions:

$$\begin{aligned} K(\lambda x) &= \frac{1}{\lambda^n}K(x) \qquad \text{for } \lambda > 0; \\ \int_{\partial B_R(0)} K\,dS &= 0 \qquad \text{for } R > 0. \end{aligned} \tag{2.37}$$

That is, K is homogeneous of degree $-n$, and its integral over any sphere centered at zero is zero. We may then write

$$K(x) = \frac{\Omega\left(\hat{x}\right)}{|x|^n}, \qquad \hat{x} = \frac{x}{|x|}$$

where $\Omega : \mathbb{S}^{n-1} \to \mathbb{R}$ is a C^1-function such that

$$\int_{\mathbb{S}^{n-1}} \Omega\,dS = 0.$$

We define a singular integral operator $T : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ of convolution type with smooth, homogeneous kernel K by

$$Tf(x) = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\epsilon(x)} K(x-y)f(y)\,dy. \tag{2.38}$$

This operator is well-defined, since if $B_R(x) \supset \operatorname{supp} f$, we may write

$$\begin{aligned} Tf(x) &= \lim_{\epsilon\to 0^+} \int_{B_R(x)\setminus B_\epsilon(x)} K(x-y)f(y)\,dy. \\ &= \lim_{\epsilon\to 0^+} \left\{ \int_{B_R(x)\setminus B_\epsilon(x)} K(x-y)\left[f(y)-f(x)\right]\,dy \right. \\ &\qquad\qquad \left. + f(x)\int_{B_R(x)\setminus B_\epsilon(x)} K(x-y)\,dy \right\} \\ &= \int_{B_R(x)} K(x-y)\left[f(y)-f(x)\right]\,dy. \end{aligned}$$

Here, we use the dominated convergence theorem and the fact that

$$\int_{B_R(0)\setminus B_\epsilon(0)} K(y)\,dy = 0$$

since K has zero mean over spheres centered at the origin. Thus, the cancelation due to the fact that K has zero mean over spheres compensates for the non-integrability of K at the origin to give a finite limit.

Calderón and Zygmund (1952) proved that such operators, and generalizations of them, extend to bounded linear operators on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$ (see *e.g.* [**7**]). As a result, we also 'gain' two derivatives in inverting the Laplacian when derivatives are measured in L^p for $1 < p < \infty$.

CHAPTER 3

Sobolev spaces

> These spaces, at least in the particular case $p = 2$, were known since the very beginning of this century, to the Italian mathematicians Beppo Levi and Guido Fubini who investigated the Dirichlet minimum principle for elliptic equations. Later on many mathematicians have used these spaces in their work. Some French mathematicians, at the beginning of the fifties, decided to invent a name for such spaces as, very often, French mathematicians like to do. They proposed the name Beppo Levi spaces. Although this name is not very exciting in the Italian language and it sounds because of the name "Beppo", somewhat peasant, the outcome in French must be gorgeous since the special French pronunciation of the names makes it to sound very impressive. Unfortunately this choice was deeply disliked by Beppo Levi, who at that time was still alive, and — as many elderly people — was strongly against the modern way of viewing mathematics. In a review of a paper of an Italian mathematician, who, imitating the Frenchman, had written something on "Beppo Levi spaces", he practically said that he did not want to leave his name mixed up with this kind of things. Thus the name had to be changed. A good choice was to name the spaces after S. L. Sobolev. Sobolev did not object and the name Sobolev spaces is nowadays universally accepted.[1]

We will give only the most basic results here. For more information, see Shkoller [**37**], Evans [**9**] (Chapter 5), and Leoni [**26**]. A standard reference is [**1**].

3.1. Weak derivatives

Suppose, as usual, that Ω is an open set in $\mathbb{R}^n$.

Definition 3.1. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable with respect to x_i if there exists a function $g_i \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega f \partial_i \phi \, dx = -\int_\Omega g_i \phi \, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

The function g_i is called the weak ith partial derivative of f, and is denoted by $\partial_i f$.

Thus, for weak derivatives, the integration by parts formula

$$\int_\Omega f \, \partial_i \phi \, dx = -\int_\Omega \partial_i f \, \phi \, dx$$

[1]Fichera, 1977, quoted by Naumann [**31**].

holds by definition for all $\phi \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^1_{\mathrm{loc}}(\Omega)$, the weak derivative of a function, if it exists, is unique up to pointwise almost everywhere equivalence. Moreover, the weak derivative of a continuously differentiable function agrees with the pointwise derivative. The existence of a weak derivative is, however, *not* equivalent to the existence of a pointwise derivative almost everywhere; see Examples 3.4 and 3.5.

Unless stated otherwise, we will always interpret derivatives as weak derivatives, and we use the same notation for weak derivatives and continuous pointwise derivatives. Higher-order weak derivatives are defined in a similar way.

DEFINITION 3.2. Suppose that $\alpha \in \mathbb{N}_0^n$ is a multi-index. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$ if

$$\int_\Omega (\partial^\alpha f)\,\phi\,dx = (-1)^{|\alpha|}\int_\Omega f\,(\partial^\alpha \phi)\,dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

3.2. Examples

Let us consider some examples of weak derivatives that illustrate the definition. We denote the weak derivative of a function of a single variable by a prime.

EXAMPLE 3.3. Define $f \in C(\mathbb{R})$ by

$$f(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

We also write $f(x) = x_+$. Then f is weakly differentiable, with

$$f' = \chi_{[0,\infty)}, \tag{3.1}$$

where $\chi_{[0,\infty)}$ is the step function

$$\chi_{[0,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The choice of the value of $f'(x)$ at $x = 0$ is irrelevant, since the weak derivative is only defined up to pointwise almost everwhere equivalence. To prove (3.1), note that for any $\phi \in C_c^\infty(\mathbb{R})$, an integration by parts gives

$$\int f\phi'\,dx = \int_0^\infty x\phi'\,dx = -\int_0^\infty \phi\,dx = -\int \chi_{[0,\infty)}\phi\,dx.$$

EXAMPLE 3.4. The discontinuous function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

is not weakly differentiable. To prove this, note that for any $\phi \in C_c^\infty(\mathbb{R})$,

$$\int f\phi'\,dx = \int_0^\infty \phi'\,dx = -\phi(0).$$

Thus, the weak derivative $g = f'$ would have to satisfy

$$\int g\phi\,dx = \phi(0) \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}). \tag{3.2}$$

Assume for contradiction that $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ satisfies (3.2). By considering test functions with $\phi(0) = 0$, we see that g is equal to zero pointwise almost everywhere, and then (3.2) does not hold for test functions with $\phi(0) \neq 0$.

The pointwise derivative of the discontinuous function f in the previous example exists and is zero except at 0, where the function is discontinuous, but the function is not weakly differentiable. The next example shows that even a continuous function that is pointwise differentiable almost everywhere need not have a weak derivative.

EXAMPLE 3.5. Let $f \in C(\mathbb{R})$ be the Cantor function, which may be constructed as a uniform limit of piecewise constant functions defined on the standard 'middle-thirds' Cantor set C. For example, $f(x) = 1/2$ for $1/3 \le x \le 2/3$, $f(x) = 1/4$ for $1/9 \le x \le 2/9$, $f(x) = 3/4$ for $7/9 \le x \le 8/9$, and so on.[2] Then f is not weakly differentiable. To see this, suppose that $f' = g$ where

$$\int g\phi\, dx = -\int f\phi'\, dx$$

for all test functions ϕ. The complement of the Cantor set in $[0, 1]$ is a union of open intervals,

$$[0,1] \setminus C = \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \dots,$$

whose measure is equal to one. Taking test functions ϕ whose supports are compactly contained in one of these intervals, call it I, and using the fact that $f = c_I$ is constant on I, we find that

$$\int g\phi\, dx = -\int_I f\phi'\, dx = -c_I \int_I \phi'\, dx = 0.$$

It follows that $g = 0$ pointwise a.e. on $[0, 1] \setminus C$, and hence if f is weakly differentiable, then $f' = 0$. From the following proposition, however, the only functions with zero weak derivative are the ones that are equivalent to a constant function. This is a contradiction, so the Cantor function is not weakly differentiable.

PROPOSITION 3.6. *If $f : (a, b) \to \mathbb{R}$ is weakly differentiable and $f' = 0$, then f is a constant function.*

PROOF. The condition that the weak derivative f' is zero means that

$$\int f\phi'\, dx = 0 \qquad \text{for all } \phi \in C_c^\infty(a, b). \tag{3.3}$$

Choose a fixed test function $\eta \in C_c^\infty(a, b)$ whose integral is equal to one. We may represent an arbitrary test function $\phi \in C_c^\infty(a, b)$ as

$$\phi = A\eta + \psi'$$

[2]The Cantor function is given explicitly by: $f(x) = 0$ if $x \le 0$; $f(x) = 1$ if $x \ge 1$;

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{c_n}{2^n}$$

if $x = \sum_{n=1}^{\infty} c_n/3^n$ with $c_n \in \{0, 2\}$ for all $n \in \mathbb{N}$; and

$$f(x) = \frac{1}{2}\sum_{n=1}^{N} \frac{c_n}{2^n} + \frac{1}{2^{N+1}}$$

if $x = \sum_{n=1}^{\infty} c_n/3^n$, with $c_n \in \{0, 2\}$ for $1 \le n < k$ and $c_k = 1$.

where $A \in \mathbb{R}$ and $\psi \in C_c^\infty(a,b)$ are given by

$$A = \int_a^b \phi\, dx, \qquad \psi(x) = \int_a^x [\phi(t) - A\eta(t)]\, dt.$$

Then (3.3) implies that

$$\int f\phi\, dx = A\int f\eta\, dx = c\int \phi\, dx, \qquad c = \int f\eta\, dx.$$

It follows that

$$\int (f-c)\,\phi\, dx = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b),$$

which implies that $f = c$ pointwise almost everywhere, so f is equivalent to a constant function. □

As this discussion illustrates, in defining 'strong' solutions of a differential equation that satisfy the equation pointwise *a.e.*, but which are not necessarily continuously differentiable 'classical' solutions, it is important to include the condition that the solutions are weakly differentiable. For example, up to pointwise *a.e.* equivalence, the only weakly differentiable functions $u : \mathbb{R} \to \mathbb{R}$ that satisfy the ODE

$$u' = 0 \qquad \text{pointwise } a.e.$$

are the constant functions. There are, however, many non-constant functions that are differentiable pointwise *a.e.* and satisfy the ODE pointwise *a.e.*, but these solutions are not weakly differentiable; the step function and the Cantor function are examples.

Example 3.7. For $a \in \mathbb{R}$, define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{|x|^a}. \tag{3.4}$$

Then f is weakly differentiable if $a + 1 < n$ with weak derivative

$$\partial_i f(x) = -\frac{a}{|x|^{a+1}}\frac{x_i}{|x|}.$$

That is, f is weakly differentiable provided that the pointwise derivative, which is defined almost everywhere, is locally integrable.

To prove this claim, suppose that $\epsilon > 0$, and let $\phi^\epsilon \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function that is equal to one in $B_\epsilon(0)$ and zero outside $B_{2\epsilon}(0)$. Then

$$f^\epsilon(x) = \frac{1 - \phi^\epsilon(x)}{|x|^a}$$

belongs to $C^\infty(\mathbb{R}^n)$ and $f^\epsilon = f$ in $|x| \geq 2\epsilon$. Integrating by parts, we get

$$\int (\partial_i f^\epsilon)\, \phi\, dx = -\int f^\epsilon\, (\partial_i \phi)\, dx. \tag{3.5}$$

We have

$$\partial_i f^\epsilon(x) = -\frac{a}{|x|^{a+1}}\frac{x_i}{|x|}\left[1 - \phi^\epsilon(x)\right] - \frac{1}{|x|^a}\partial_i \phi^\epsilon(x).$$

Since $|\partial_i \phi^\epsilon| \leq C/\epsilon$ and $|\partial_i \phi^\epsilon| = 0$ when $|x| \leq \epsilon$ or $|x| \geq 2\epsilon$, we have

$$|\partial_i \phi^\epsilon(x)| \leq \frac{C}{|x|}.$$

It follows that
$$|\partial_i f^\epsilon(x)| \leq \frac{C'}{|x|^{a+1}}$$
where C' is a constant independent of ϵ. Moreover
$$\partial_i f^\epsilon(x) \to -\frac{a}{|x|^{a+1}}\frac{x_i}{|x|} \qquad \text{pointwise a.e. as } \epsilon \to 0^+.$$
If $|x|^{-(a+1)}$ is locally integrable, then by taking the limit of (3.5) as $\epsilon \to 0^+$ and using the Lebesgue dominated convergence theorem, we get
$$\int \left(-\frac{a}{|x|^{a+1}}\frac{x_i}{|x|}\right)\phi\, dx = -\int f\,(\partial_i\phi)\; dx,$$
which proves the claim.

Alternatively, instead of mollifying f, we can use the truncated function
$$f^\epsilon(x) = \frac{\chi_{B_\epsilon(0)}(x)}{|x|^a}.$$

3.3. Distributions

Although we will not make extensive use of the theory of distributions, it is useful to understand the interpretation of a weak derivative as a distributional derivative. Let Ω be an open set in $\mathbb{R}^n$.

DEFINITION 3.8. A sequence $\{\phi_n : n \in \mathbb{N}\}$ of functions $\phi_n \in C_c^\infty(\Omega)$ converges to $\phi \in C_c^\infty(\Omega)$ in the sense of test functions if:

(a) there exists $\Omega' \Subset \Omega$ such that $\operatorname{supp}\phi_n \subset \Omega'$ for every $n \in \mathbb{N}$;

(b) $\partial^\alpha\phi_n \to \partial^\alpha\phi$ as $n \to \infty$ uniformly on Ω for every $\alpha \in \mathbb{N}_0^n$.

The topological vector space $\mathcal{D}(\Omega)$ consists of $C_c^\infty(\Omega)$ equipped with the topology that corresponds to convergence in the sense of test functions.

Note that since the supports of the ϕ_n are contained in the same compactly contained subset, the limit has compact support; and since the derivatives of all orders converge uniformly, the limit is smooth.

The space $\mathcal{D}(\Omega)$ is not metrizable, but it can be shown that the sequential convergence of test functions is sufficient to determine its topology [**19**].

A linear functional on $\mathcal{D}(\Omega)$ is a linear map $T : \mathcal{D}(\Omega) \to \mathbb{R}$. We denote the value of T acting on a test function ϕ by $\langle T, \phi\rangle$; thus, T is linear if
$$\langle T, \lambda\phi + \mu\psi\rangle = \lambda\langle T, \phi\rangle + \mu\langle T, \psi\rangle \qquad \text{for all } \lambda, \mu \in \mathbb{R} \text{ and } \phi, \psi \in \mathcal{D}(\Omega).$$
A functional T is continuous if $\phi_n \to \phi$ in the sense of test functions implies that $\langle T, \phi_n\rangle \to \langle T, \phi\rangle$ in $\mathbb{R}$

DEFINITION 3.9. A distribution on Ω is a continuous linear functional
$$T : \mathcal{D}(\Omega) \to \mathbb{R}.$$
A sequence $\{T_n : n \in \mathbb{N}\}$ of distributions converges to a distribution T, written $T_n \rightharpoonup T$, if $\langle T_n, \phi\rangle \to \langle T, \phi\rangle$ for every $\phi \in \mathcal{D}(\Omega)$. The topological vector space $\mathcal{D}'(\Omega)$ consists of the distributions on Ω equipped with the topology corresponding to this notion of convergence.

Thus, the space of distributions is the topological dual of the space of test functions.

EXAMPLE 3.10. The delta-function supported at $a \in \Omega$ is the distribution

$$\delta_a : \mathcal{D}(\Omega) \to \mathbb{R}$$

defined by evaluation of a test function at a:

$$\langle \delta_a, \phi \rangle = \phi(a).$$

This functional is continuous since $\phi_n \to \phi$ in the sense of test functions implies, in particular, that $\phi_n(a) \to \phi(a)$

EXAMPLE 3.11. Any function $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$\langle T_f, \phi \rangle = \int_\Omega f\phi \, dx.$$

The linear functional T_f is continuous since if $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$, then

$$\sup_{\Omega'} |\phi_n - \phi| \to 0$$

on a set $\Omega' \Subset \Omega$ that contains the supports of the ϕ_n, so

$$|\langle T, \phi_n \rangle - \langle T, \phi \rangle| = \left| \int_{\Omega'} f\,(\phi_n - \phi)\, dx \right| \leq \left(\int_{\Omega'} |f|\, dx \right) \sup_{\Omega'} |\phi_n - \phi| \to 0.$$

Any distribution associated with a locally integrable function in this way is called a regular distribution. We typically regard the function f and the distribution T_f as equivalent.

EXAMPLE 3.12. If μ is a Radon measure on Ω, then

$$\langle I_\mu, \phi \rangle = \int_\Omega \phi \, d\mu$$

defines a distribution $I_\mu \in \mathcal{D}'(\Omega)$. This distribution is regular if and only if μ is locally absolutely continuous with respect to Lebesgue measure λ, in which case the Radon-Nikodym derivative

$$f = \frac{d\mu}{d\lambda} \in L^1_{\mathrm{loc}}(\Omega)$$

is locally integrable, and

$$\langle I_\mu, \phi \rangle = \int_\Omega f\phi \, dx$$

so $I_\mu = T_f$. On the other hand, if μ is singular with respect to Lebesgue measure (for example, if $\mu = \delta_a$ is the unit point measure supported at $a \in \Omega$), then I_μ is not a regular distribution.

One of the main advantages of distributions is that, in contrast to functions, every distribution is differentiable. The space of distributions may be thought of as the smallest extension of the space of continuous functions that is closed under differentiation.

DEFINITION 3.13. For $1 \leq i \leq n$, the partial derivative of a distribution $T \in \mathcal{D}'(\Omega)$ with respect to x_i is the distribution $\partial_i T \in \mathcal{D}'(\Omega)$ defined by

$$\langle \partial_i T, \phi \rangle = -\langle T, \partial_i \phi \rangle \qquad \text{for all } \phi \in \mathcal{D}(\Omega).$$

For $\alpha \in \mathbb{N}_0^n$, the derivative $\partial^\alpha T \in \mathcal{D}'(\Omega)$ of order $|\alpha|$ is defined by

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle \qquad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Note that if $T \in \mathcal{D}'(\Omega)$, then it follows from the linearity and continuity of the derivative $\partial^\alpha : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ on the space of test functions that $\partial^\alpha T$ is a continuous linear functional on $\mathcal{D}(\Omega)$. Thus, $\partial^\alpha T \in \mathcal{D}'(\Omega)$ for any $T \in \mathcal{D}'(\Omega)$. It also follows that the distributional derivative $\partial^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is linear and continuous on the space of distributions; in particular if $T_n \rightharpoonup T$, then $\partial^\alpha T_n \rightharpoonup \partial^\alpha T$.

Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be a locally integrable function and $T_f \in \mathcal{D}'(\Omega)$ the associated regular distribution defined in Example 3.11. Suppose that the distributional derivative of T_f is a regular distribution

$$\partial_i T_f = T_{g_i} \qquad g_i \in L^1_{\mathrm{loc}}(\Omega).$$

Then it follows from the definitions that

$$\int_\Omega f \partial_i \phi \, dx = -\int_\Omega g_i \phi \, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

Thus, Definition 3.1 of the weak derivative may be restated as follows: A locally integrable function is weakly differentiable if its distributional derivative is regular, and its weak derivative is the locally integrable function corresponding to the distributional derivative.

The distributional derivative of a function exists even if the function is not weakly differentiable.

Example 3.14. If f is a function of bounded variation, then the distributional derivative of f is a finite Radon measure, which need not be regular. For example, the distributional derivative of the step function is the delta-function, and the distributional derivative of the Cantor function is the corresponding Lebesgue-Stieltjes measure supported on the Cantor set.

Example 3.15. The derivative of the delta-function δ_a supported at a, defined in Example 3.10, is the distribution $\partial_i \delta_a$ defined by

$$\langle \partial_i \delta_a, \phi \rangle = -\partial_i \phi(a).$$

This distribution is neither regular nor a Radon measure.

Differential equations are typically thought of as equations that relate functions. The use of weak derivatives and distribution theory leads to an alternative point of view of linear differential equations as linear functionals acting on test functions. Using this perspective, given suitable estimates, one can obtain simple and general existence results for weak solutions of linear PDEs by the use of the Hahn-Banach, Riesz representation, or other duality theorems for the existence of bounded linear functionals.

While distribution theory provides an effective general framework for the analysis of linear PDEs, it is less useful for nonlinear PDEs because one cannot define a product of distributions that extends the usual product of smooth functions in an unambiguous way. For example, what is $T_f \delta_a$ if f is a locally integrable function that is discontinuous at a? There are difficulties even for regular distributions. For example, $f : x \mapsto |x|^{-n/2}$ is locally integrable on $\mathbb{R}^n$ but f^2 is not, so how should one define the distribution $(T_f)^2$?

3.4. Properties of weak derivatives

We collect here some properties of weak derivatives. The first result is a product rule.

PROPOSITION 3.16. *If $f \in L^1_{\mathrm{loc}}(\Omega)$ has weak partial derivative $\partial_i f \in L^1_{\mathrm{loc}}(\Omega)$ and $\psi \in C^\infty(\Omega)$, then ψf is weakly differentiable with respect to x_i and*

$$\partial_i(\psi f) = (\partial_i \psi) f + \psi(\partial_i f). \tag{3.6}$$

PROOF. Let $\phi \in C_c^\infty(\Omega)$ be any test function. Then $\psi\phi \in C_c^\infty(\Omega)$ and the weak differentiability of f implies that

$$\int_\Omega f \partial_i(\psi\phi)\, dx = -\int_\Omega (\partial_i f)\psi\phi\, dx.$$

Expanding $\partial_i(\psi\phi) = \psi(\partial_i\phi) + (\partial_i\psi)\phi$ in this equation and rearranging the result, we get

$$\int_\Omega \psi f(\partial_i\phi)\, dx = -\int_\Omega \left[(\partial_i\psi) f + \psi(\partial_i f)\right]\phi\, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

Thus, ψf is weakly differentiable and its weak derivative is given by (3.6). □

The commutativity of weak derivatives follows immediately from the commutativity of derivatives applied to smooth functions.

PROPOSITION 3.17. *Suppose that $f \in L^1_{\mathrm{loc}}(\Omega)$ and that the weak derivatives $\partial^\alpha f$, $\partial^\beta f$ exist for multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. Then if any one of the weak derivatives $\partial^{\alpha+\beta} f$, $\partial^\alpha\partial^\beta f$, $\partial^\beta\partial^\alpha f$ exists, all three derivatives exist and are equal.*

PROOF. Using the existence of $\partial^\alpha u$, and the fact that $\partial^\beta\phi \in C_c^\infty(\Omega)$ for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \partial^\alpha u\, \partial^\beta\phi\, dx = (-1)^{|\alpha|}\int_\Omega u\, \partial^{\alpha+\beta}\phi\, dx.$$

This equation shows that $\partial^{\alpha+\beta} u$ exists if and only if $\partial^\beta\partial^\alpha u$ exists, and in that case the weak derivatives are equal. Using the same argument with α and β exchanged, we get the result. □

EXAMPLE 3.18. Consider functions of the form

$$u(x, y) = f(x) + g(y).$$

Then $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ if and only if $f, g \in L^1_{\mathrm{loc}}(\mathbb{R})$. The weak derivative $\partial_x u$ exists if and only if the weak derivative f' exists, and then $\partial_x u(x, y) = f'(x)$. To see this, we use Fubini's theorem to get for any $\phi \in C_c^\infty(\mathbb{R}^2)$ that

$$\begin{aligned}\int & u(x,y)\partial_x\phi(x,y)\, dxdy \\ &= \int f(x)\partial_x\left[\int \phi(x,y)\, dy\right] dx + \int g(y)\left[\int \partial_x\phi(x,y)\, dx\right] dy.\end{aligned}$$

Since ϕ has compact support,

$$\int \partial_x\phi(x,y)\, dx = 0.$$

Also,

$$\int \phi(x,y)\, dy = \xi(x)$$

is a test function $\xi \in C_c^\infty(\mathbb{R})$. Moreover, by taking $\phi(x,y) = \xi(x)\eta(y)$, where $\eta \in C_c^\infty(\mathbb{R})$ is an arbitrary test function with integral equal to one, we can get every $\xi \in C_c^\infty(\mathbb{R})$. Since

$$\int u(x,y)\partial_x\phi(x,y)\,dxdy = \int f(x)\xi'(x)\,dx,$$

it follows that $\partial_x u$ exists if and only if f' exists, and then $\partial_x u = f'$.

In that case, the mixed derivative $\partial_y\partial_x u$ also exists, and is zero, since using Fubini's theorem as before

$$\int f'(x)\partial_y\phi(x,y)\,dxdy = \int f'(x)\left[\int \partial_y\phi(x,y)\,dy\right]\,dx = 0.$$

Similarly $\partial_y u$ exists if and only if g' exists, and then $\partial_y u = g'$ and $\partial_x\partial_y u = 0$.

The second-order weak derivative $\partial_{xy}u$ exists without any differentiability assumptions on $f, g \in L^1_{\mathrm{loc}}(\mathbb{R})$ and is equal to zero. For any $\phi \in C_c^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned}
&\int u(x,y)\partial_{xy}\phi(x,y)\,dxdy \\
&\qquad = \int f(x)\partial_x\left(\int \partial_y\phi(x,y)\,dy\right)dx + \int g(y)\partial_y\left(\int \partial_x\phi(x,y)\,dx\right)dy \\
&\qquad = 0.
\end{aligned}$$

Thus, the mixed derivatives $\partial_x\partial_y u$ and $\partial_y\partial_x u$ are equal, and are equal to the second-order derivative $\partial_{xy}u$, whenever both are defined.

Weak derivatives combine well with mollifiers. If Ω is an open set in $\mathbb{R}^n$ and $\epsilon > 0$, we define Ω^ϵ as in (1.7) and let η^ϵ be the standard mollifier (1.6).

THEOREM 3.19. *Suppose that $f \in L^1_{\mathrm{loc}}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$. Then $\eta^\epsilon * f \in C^\infty(\Omega^\epsilon)$ and*

$$\partial^\alpha\left(\eta^\epsilon * f\right) = \eta^\epsilon * \left(\partial^\alpha f\right).$$

Moreover,

$$\partial^\alpha\left(\eta^\epsilon * f\right) \to \partial^\alpha f \qquad \text{in } L^1_{\mathrm{loc}}(\Omega) \text{ as } \epsilon \to 0^+.$$

PROOF. From Theorem 1.28, we have $\eta^\epsilon * f \in C^\infty(\Omega^\epsilon)$ and

$$\partial^\alpha\left(\eta^\epsilon * f\right) = \left(\partial^\alpha\eta^\epsilon\right) * f.$$

Using the fact that $y \mapsto \eta^\epsilon(x-y)$ defines a test function in $C_c^\infty(\Omega)$ for any fixed $x \in \Omega^\epsilon$ and the definition of the weak derivative, we have

$$\begin{aligned}
\left(\partial^\alpha\eta^\epsilon\right) * f(x) &= \int \partial_x^\alpha\eta^\epsilon(x-y)f(y)\,dy \\
&= (-1)^{|\alpha|}\int \partial_y^\alpha\eta^\epsilon(x-y)f(y) \\
&= \int \eta^\epsilon(x-y)\partial^\alpha f(y)\,dy \\
&= \eta^\epsilon * \left(\partial^\alpha f\right)(x)
\end{aligned}$$

Thus $\left(\partial^\alpha\eta^\epsilon\right) * f = \eta^\epsilon * \left(\partial^\alpha f\right)$. Since $\partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$, Theorem 1.28 implies that

$$\eta^\epsilon * \left(\partial^\alpha f\right) \to \partial^\alpha f$$

in $L^1_{\mathrm{loc}}(\Omega)$, which proves the result. $\square$

The next result gives an alternative way to characterize weak derivatives as limits of derivatives of smooth functions.

THEOREM 3.20. *A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable in Ω if and only if there is a sequence $\{f_n\}$ of functions $f_n \in C^\infty(\Omega)$ such that $f_n \to f$ and $\partial^\alpha f_n \to g$ in $L^1_{\mathrm{loc}}(\Omega)$. In that case the weak derivative of f is given by $g = \partial^\alpha f \in L^1_{\mathrm{loc}}(\Omega)$.*

PROOF. If f is weakly differentiable, we may construct an appropriate sequence by mollification as in Theorem 3.19. Conversely, suppose that such a sequence exists. Note that if $f_n \to f$ in $L^1_{\mathrm{loc}}(\Omega)$ and $\phi \in C_c(\Omega)$, then

$$\int_\Omega f_n \phi \, dx \to \int_\Omega f \phi \, dx \qquad \text{as } n \to \infty,$$

since if $K = \operatorname{supp} \phi \Subset \Omega$

$$\left| \int_\Omega f_n \phi \, dx - \int_\Omega f \phi \, dx \right| = \left| \int_K (f_n - f)\, \phi \, dx \right| \le \sup_K |\phi| \int_K |f_n - f| \, dx \to 0.$$

Thus, for any $\phi \in C_c^\infty(\Omega)$, the L^1_{loc}-convergence of f_n and $\partial^\alpha f_n$ implies that

$$\begin{aligned} \int_\Omega f \partial^\alpha \phi \, dx &= \lim_{n\to\infty} \int_\Omega f_n \partial^\alpha \phi \, dx \\ &= (-1)^{|\alpha|} \lim_{n\to\infty} \int_\Omega \partial^\alpha f_n \phi \, dx \\ &= (-1)^{|\alpha|} \int_\Omega g \phi \, dx. \end{aligned}$$

So f is weakly differentiable and $\partial^\alpha f = g$. □

We can use this approximation result to derive properties of the weak derivative as a limit of corresponding properties of smooth functions. The following weak versions of the product and chain rule, which are not stated in maximum generality, may be derived in this way.

PROPOSITION 3.21. *Let Ω be an open set in $\mathbb{R}^n$.*

(1) *Suppose that $a \in C^1(\Omega)$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable. Then au is weakly differentiable and*

$$\partial_i(au) = a\,(\partial_i u) + (\partial_i a)\, u.$$

(2) *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with $f' \in L^\infty(\mathbb{R})$ bounded, and $u \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable. Then $v = f \circ u$ is weakly differentiable and*

$$\partial_i v = f'(u) \partial_i u.$$

(3) *Suppose that $\phi : \Omega \to \widetilde{\Omega}$ is a C^1-diffeomorphism of Ω onto $\widetilde{\Omega} = \phi(\Omega) \subset \mathbb{R}^n$. For $u \in L^1_{\mathrm{loc}}(\Omega)$, define $v \in L^1_{\mathrm{loc}}(\widetilde{\Omega})$ by $v = u \circ \phi^{-1}$. Then v is weakly differentiable in $\widetilde{\Omega}$ if and only if u is weakly differentiable in Ω, and*

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n \frac{\partial \phi_j}{\partial x_i} \frac{\partial v}{\partial y_j} \circ \phi.$$

PROOF. We prove (2) as an example. Since $f' \in L^\infty$, f is globally Lipschitz and there exists a constant M such that

$$|f(s) - f(t)| \le M|s-t| \qquad \text{for all } s,t \in \mathbb{R}.$$

Choose $u_n \in C^\infty(\Omega)$ such that $u_n \to u$ and $\partial_i u_n \to \partial_i u$ in $L^1_{\text{loc}}(\Omega)$, where $u_n \to u$ pointwise almost everywhere in Ω. Let $v = f \circ u$ and $v_n = f \circ u_n \in C^1(\Omega)$, with

$$\partial_i v_n = f'(u_n)\partial_i u_n \in C(\Omega).$$

If $\Omega' \Subset \Omega$, then

$$\int_{\Omega'} |v_n - v| \; dx = \int_{\Omega'} |f(u_n) - f(u)| \; dx \le M \int_{\Omega'} |u_n - u| \; dx \to 0$$

as $n \to \infty$. Also, we have

$$\begin{aligned} \int_{\Omega'} |\partial_i v_n - f'(u)\partial_i u| \; dx &= \int_{\Omega'} |f'(u_n)\partial_i u_n - f'(u)\partial_i u| \; dx \\ &\le \int_{\Omega'} |f'(u_n)| \; |\partial_i u_n - \partial_i u| \; dx \\ &\quad + \int_{\Omega'} |f'(u_n) - f'(u)| \; |\partial_i u| \; dx. \end{aligned}$$

Then

$$\int_{\Omega'} |f'(u_n)| \; |\partial_i u_n - \partial_i u| \; dx \le M \int_{\Omega'} |\partial_i u_n - \partial_i u| \; dx \to 0.$$

Moreover, since $f'(u_n) \to f'(u)$ pointwise *a.e.* and

$$|f'(u_n) - f'(u)| \; |\partial_i u| \le 2M \, |\partial_i u| \, ,$$

the dominated convergence theorem implies that

$$\int_{\Omega'} |f'(u_n) - f'(u)| \; |\partial_i u| \; dx \to 0 \qquad \text{as } n \to \infty.$$

It follows that $v_n \to f \circ u$ and $\partial_i v_n \to f'(u)\partial_i u$ in L^1_{loc}. Then Theorem 3.20, which still applies if the approximating functions are C^1, implies that $f \circ u$ is weakly differentiable with the weak derivative stated. □

In fact, (2) remains valid if $f \in W^{1,\infty}(\mathbb{R})$ is globally Lipschitz but not necessarily C^1. We will prove this is the useful special case that $f(u) = |u|$.

PROPOSITION 3.22. *If $u \in L^1_{\text{loc}}(\Omega)$ has the weak derivative $\partial_i u \in L^1_{\text{loc}}(\Omega)$, then $|u| \in L^1_{\text{loc}}(\Omega)$ is weakly differentiable and*

$$\partial_i |u| = \begin{cases} \partial_i u & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -\partial_i u & \text{if } u < 0. \end{cases} \tag{3.7}$$

PROOF. Let

$$f^\epsilon(t) = \sqrt{t^2 + \epsilon^2}.$$

Since f^ϵ is C^1 and globally Lipschitz, Proposition 3.21 implies that $f^\epsilon(u)$ is weakly differentiable, and for every $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega f^\epsilon(u)\partial_i \phi \, dx = -\int_\Omega \frac{u\partial_i u}{\sqrt{u^2 + \epsilon^2}} \phi \, dx.$$

Taking the limit of this equation as $\epsilon \to 0$ and using the dominated convergence theorem, we conclude that

$$\int_\Omega |u|\partial_i\phi\, dx = -\int_\Omega (\partial_i|u|)\phi\, dx$$

where $\partial_i|u|$ is given by (3.7). □

It follows immediately from this result that the positive and negative parts of $u = u^+ - u^-$, given by

$$u^+ = \frac{1}{2}\left(|u| + u\right), \qquad u^- = \frac{1}{2}\left(|u| - u\right),$$

are weakly differentiable if u is weakly differentiable, with

$$\partial_i u^+ = \begin{cases} \partial_i u & \text{if } u > 0, \\ 0 & \text{if } u \le 0, \end{cases} \qquad \partial_i u^- = \begin{cases} 0 & \text{if } u \ge 0, \\ -\partial_i u & \text{if } u < 0, \end{cases}$$

3.5. Sobolev spaces

Sobolev spaces consist of functions whose weak derivatives belong to L^p. These spaces provide one of the most useful settings for the analysis of PDEs.

Definition 3.23. Suppose that Ω is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that

$$\partial^\alpha f \in L^p(\Omega) \qquad \text{for } 0 \le |\alpha| \le k.$$

We write $W^{k,2}(\Omega) = H^k(\Omega)$.

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha f|^p\, dx\right)^{1/p}$$

for $1 \le p < \infty$ and

$$\|f\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|\le k} \sup_\Omega |\partial^\alpha f|.$$

As usual, we identify functions that are equal almost everywhere. We will use these norms as the standard ones on $W^{k,p}(\Omega)$, but there are other equivalent norms *e.g.*

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \left(\int_\Omega |\partial^\alpha f|^p\, dx\right)^{1/p},$$

$$\|f\|_{W^{k,p}(\Omega)} = \max_{|\alpha|\le k} \left(\int_\Omega |\partial^\alpha f|^p\, dx\right)^{1/p}.$$

The space $H^k(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{|\alpha|\le k} \int_\Omega (\partial^\alpha f)(\partial^\alpha g)\, dx.$$

We will consider the following properties of Sobolev spaces in the simplest settings.

(1) Approximation of Sobolev functions by smooth functions;
(2) Embedding theorems;
(3) Boundary values of Sobolev functions and trace theorems;

(4) Compactness results.

3.6. Approximation of Sobolev functions

To begin with, we consider Sobolev functions defined on all of $\mathbb{R}^n$. They may be approximated in the Sobolev norm by by test functions.

THEOREM 3.24. *For $k \in \mathbb{N}$ and $1 \leq p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$*

PROOF. Let $\eta^\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the standard mollifier and $f \in W^{k,p}(\mathbb{R}^n)$. Then Theorem 1.28 and Theorem 3.19 imply that $\eta^\epsilon * f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and for $|\alpha| \leq k$

$$\partial^\alpha (\eta^\epsilon * f) = \eta^\epsilon * (\partial^\alpha f) \to \partial^\alpha f \qquad \text{in } L^p(\mathbb{R}^n) \text{ as } \epsilon \to 0^+.$$

It follows that $\eta^\epsilon * f \to f$ in $W^{k,p}(\mathbb{R}^n)$ as $\epsilon \to 0$. Therefore $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Now suppose that $f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$, and let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Define $\phi^R(x) = \phi(x/R)$ and $f^R = \phi^R f \in C_c^\infty(\mathbb{R}^n)$. Then, by the Leibnitz rule,

$$\partial^\alpha f^R = \phi^R \partial^\alpha f + \frac{1}{R} h^R$$

where h^R is bounded in L^p uniformly in R. Hence, by the dominated convergence theorem

$$\partial^\alpha f^R \to \partial^\alpha f \qquad \text{in } L^p \text{ as } R \to \infty,$$

so $f^R \to f$ in $W^{k,p}(\mathbb{R}^n)$ as $R \to \infty$. It follows that $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\mathbb{R}^n)$. □

If Ω is a proper open subset of $\mathbb{R}^n$, then $C_c^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$. Instead, its closure is the space of functions $W_0^{k,p}(\Omega)$ that 'vanish on the boundary $\partial\Omega$.' We discuss this further below. The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for any open set Ω (Meyers and Serrin, 1964), so that $W^{k,p}(\Omega)$ may alternatively be defined as the completion of the space of smooth functions in Ω whose derivatives of order less than or equal to k belong to $L^p(\Omega)$. Such functions need not extend to continuous functions on $\overline{\Omega}$ or be bounded on Ω.

3.7. Sobolev embedding: $p < n$

> G. H. Hardy reported Harald Bohr as saying 'all analysts spend half their time hunting through the literature for inequalities which they want to use but cannot prove.'[3]

Let us first consider the following basic question: Can we estimate the $L^q(\mathbb{R}^n)$-norm of a smooth, compactly supported function in terms of the $L^p(\mathbb{R}^n)$-norm of its derivative? As we will show, given $1 \leq p < n$, this is possible for a unique value of q, called the Sobolev conjugate of p.

We may motivate the answer by means of a scaling argument. We are looking for an estimate of the form

$$\|f\|_{L^q} \leq C \|Df\|_{L^p} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n) \tag{3.8}$$

[3]From the Introduction of [**16**].

for some constant $C = C(p, q, n)$. For $\lambda > 0$, let f_λ denote the rescaled function

$$f_\lambda(x) = f\left(\frac{x}{\lambda}\right).$$

Then, changing variables $x \mapsto \lambda x$ in the integrals that define the L^p, L^q norms, with $1 \leq p, q < \infty$, and using the fact that

$$Df_\lambda = \frac{1}{\lambda}(Df)_\lambda$$

we find that

$$\left(\int_{\mathbb{R}^n} |Df_\lambda|^p\, dx\right)^{1/p} = \lambda^{n/p-1}\left(\int_{\mathbb{R}^n} |Df|^p\, dx\right)^{1/p},$$
$$\left(\int_{\mathbb{R}^n} |f_\lambda|^q\, dx\right)^{1/q} = \lambda^{n/q}\left(\int_{\mathbb{R}^n} |f|^q\, dx\right)^{1/q}.$$

These norms must scale according to the same exponent if we are to have an inequality of the desired form, otherwise we can violate the inequality by taking $\lambda \to 0$ or $\lambda \to \infty$. The equality of exponents implies that $q = p^*$ where p^* satifies

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{3.9}$$

Note that we need $1 \leq p < n$ to ensure that $p^* > 0$, in which case $p < p^* < \infty$. We assume that $n \geq 2$. Writing the solution of (3.9) for p^* explicitly, we make the following definition.

Definition 3.25. If $1 \leq p < n$, then the Sobolev conjugate p^* of p is

$$p^* = \frac{np}{n-p}.$$

Thus, an estimate of the form (3.8) is possible only if $q = p^*$; we will show that (3.8) is, in fact, true when $q = p^*$. This result was obtained by Sobolev (1938), who used potential-theoretic methods (*c.f.* Section 5.D). The proof we give is due to Nirenberg (1959). The inequality is usually called the Gagliardo-Nirenberg inequality or Sobolev inequality (or Gagliardo-Nirenberg-Sobolev inequality ...).

Before describing the proof, we introduce some notation, explain the main idea, and establish a preliminary inequality.

For $1 \leq i \leq n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let

$$x_i' = (x_1, \dots, \hat{x}_i, \dots x_n) \in \mathbb{R}^{n-1},$$

where the 'hat' means that the ith coordinate is omitted. We write $x = (x_i, x_i')$ and denote the value of a function $f : \mathbb{R}^n \to \mathbb{R}$ at x by

$$f(x) = f(x_i, x_i').$$

We denote the partial derivative with respect to x_i by ∂_i.

If f is smooth with compact support, then the fundamental theorem of calculus implies that

$$f(x) = \int_{-\infty}^{x_i} \partial_i f(t, x_i')\, dt.$$

Taking absolute values, we get

$$|f(x)| \leq \int_{-\infty}^{\infty} |\partial_i f(t, x_i')|\, dt.$$

We can improve the constant in this estimate by using the fact that

$$\int_{-\infty}^{\infty} \partial_i f(t, x_i')\, dt = 0.$$

LEMMA 3.26. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is an integrable function with compact support such that $\int g\, dt = 0$. If*

$$f(x) = \int_{-\infty}^{x} g(t)\, dt,$$

then

$$|f(x)| \leq \frac{1}{2} \int |g|\, dt.$$

PROOF. Let $g = g_+ - g_-$ where the nonnegative functions g_+, g_- are defined by $g_+ = \max(g, 0)$, $g_- = \max(-g, 0)$. Then $|g| = g_+ + g_-$ and

$$\int g_+\, dt = \int g_-\, dt = \frac{1}{2} \int |g|\, dt.$$

It follows that

$$f(x) \leq \int_{-\infty}^{x} g_+(t)\, dt \leq \int_{-\infty}^{\infty} g_+(t)\, dt = \frac{1}{2} \int |g|\, dt,$$
$$f(x) \geq -\int_{-\infty}^{x} g_-(t)\, dt \geq -\int_{-\infty}^{\infty} g_-(t)\, dt = -\frac{1}{2} \int |g|\, dt,$$

which proves the result. □

Thus, for $1 \leq i \leq n$ we have

$$|f(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |\partial_i f(t, x_i')|\, dt.$$

The idea of the proof is to average a suitable power of this inequality over the i-directions and integrate the result to estimate f in terms of Df. In order to do this, we use the following inequality, which estimates the L^1-norm of a function of $x \in \mathbb{R}^n$ in terms of the L^{n-1}-norms of n functions of $x_i' \in \mathbb{R}^{n-1}$ whose product bounds the original function pointwise.

THEOREM 3.27. *Suppose that $n \geq 2$ and*

$$\left\{ g_i \in C_c^{\infty}(\mathbb{R}^{n-1}) : 1 \leq i \leq n \right\}$$

are nonnegative functions. Define $g \in C_c^{\infty}(\mathbb{R}^n)$ by

$$g(x) = \prod_{i=1}^{n} g_i(x_i').$$

Then

$$\int g\, dx \leq \prod_{i=1}^{n} \|g_i\|_{n-1}. \tag{3.10}$$

Before proving the theorem, we consider what it says in more detail. If $n = 2$, the theorem states that

$$\int g_1(x_2) g_2(x_1)\, dx_1 dx_2 \leq \left(\int g_1(x_2)\, dx_2 \right) \left(\int g_2(x_1)\, dx_1 \right),$$

which follows immediately from Fubini's theorem. If $n = 3$, the theorem states that

$$\int g_1(x_2,x_3)g_2(x_1,x_3)g_3(x_1,x_2)\,dx_1dx_2dx_3$$
$$\leq \left(\int g_1^2(x_2,x_3)\,dx_2dx_3\right)^{1/2}\left(\int g_2^2(x_1,x_3)\,dx_1dx_3\right)^{1/2}\left(\int g_3^2(x_1,x_2)\,dx_1dx_2\right)^{1/2}.$$

To prove the inequality in this case, we fix x_1 and apply the Cauchy-Schwartz inequality to the x_2x_3-integral of $g_1 \cdot g_2g_3$. We then use the inequality for $n = 2$ to estimate the x_2x_3-integral of g_2g_3, and integrate the result over x_1. An analogous approach works for higher n.

Note that under the scaling $g_i \mapsto \lambda g_i$, both sides of (3.10) scale in the same way,

$$\int g\,dx \mapsto \left(\prod_{i=1}^n \lambda_i\right)\int g\,dx, \qquad \prod_{i=1}^n \|g_i\|_{n-1} \mapsto \left(\prod_{i=1}^n \lambda_i\right)\prod_{i=1}^n \|g_i\|_{n-1}$$

as must be true for any inequality involving norms. Also, under the spatial rescaling $x \mapsto \lambda x$, we have

$$\int g\,dx \mapsto \lambda^{-n}\int g\,dx,$$

while $\|g_i\|_p \mapsto \lambda^{-(n-1)/p}\|g_i\|_p$, so

$$\prod_{i=1}^n \|g_i\|_p \mapsto \lambda^{-n(n-1)/p}\prod_{i=1}^n \|g_i\|_p$$

Thus, if $p = n-1$ the two terms scale in the same way, which explains the appearance of the L^{n-1}-norms of the g_i's on the right hand side of (3.10).

PROOF. We use proof by induction. The result is true when $n = 2$. Suppose that it is true for $n-1$ where $n \geq 3$.

For $1 \leq i \leq n$, let $g_i : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g\ : \mathbb{R}^n \to \mathbb{R}$ be the functions given in the theorem. Fix $x_1 \in \mathbb{R}$ and define $g_{x_1} : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_{x_1}(x_1') = g(x_1, x_1').$$

For $2 \leq i \leq n$, let $x_i' = \left(x_1, x_{1,i}'\right)$ where

$$x_{1,i}' = (\hat{x}_1, \dots, \hat{x}_i, \dots x_n) \in \mathbb{R}^{n-2}.$$

Define $g_{i,x_1} : \mathbb{R}^{n-2} \to \mathbb{R}$ and $\tilde{g}_{i,x_1} : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_{i,x_1}\left(x_{1,i}'\right) = g_i\left(x_1, x_{1,i}'\right).$$

Then

$$g_{x_1}(x_1') = g_1(x_1')\prod_{i=2}^n g_{i,x_1}\left(x_{1,i}'\right).$$

Using Hölder's inequality with $q = n-1$ and $q' = (n-1)/(n-2)$, we get

$$\begin{aligned}\int g_{x_1}\,dx_1' &= \int g_1\left(\prod_{i=2}^n g_{i,x_1}\left(x_{1,i}'\right)\right)dx_1' \\ &\leq \|g_1\|_{n-1}\left[\int\left(\prod_{i=2}^n g_{i,x_1}\left(x_{1,i}'\right)\right)^{(n-1)/(n-2)} dx_1'\right]^{(n-2)/(n-1)}.\end{aligned}$$

The induction hypothesis implies that

$$\begin{aligned}\int \left(\prod_{i=2}^{n} g_{i,x_1}\left(x'_{1,i}\right)\right)^{(n-1)/(n-2)} dx'_1 &\leq \prod_{i=2}^{n} \left\| g_{i,x_1}^{(n-1)/(n-2)} \right\|_{n-2} \\ &\leq \prod_{i=2}^{n} \left\| g_{i,x_1} \right\|_{n-1}^{(n-1)/(n-2)}.\end{aligned}$$

Hence,

$$\int g_{x_1}\, dx'_1 \leq \left\| g_1 \right\|_{n-1} \prod_{i=2}^{n} \left\| g_{i,x_1} \right\|_{n-1}.$$

Integrating this equation over x_1 and using the generalized Hölder inequality with $p_2 = p_3 = \cdots = p_n = n-1$, we get

$$\begin{aligned}\int g\, dx &\leq \left\| g_1 \right\|_{n-1} \int \left(\prod_{i=2}^{n} \left\| g_{i,x_1} \right\|_{n-1}\right) dx_1 \\ &\leq \left\| g_1 \right\|_{n-1} \left(\prod_{i=2}^{n} \int \left\| g_{i,x_1} \right\|_{n-1}^{n-1} dx_1\right)^{1/(n-1)}.\end{aligned}$$

Thus, since

$$\begin{aligned}\int \left\| g_{i,x_1} \right\|_{n-1}^{n-1} dx_1 &= \int \left(\int \left| g_{i,x_1}(x'_{1,i}) \right|^{n-1} dx'_{1,i}\right) dx_1 \\ &= \int \left| g_i(x'_i) \right|^{n-1} dx'_i \\ &= \left\| g_i \right\|_{n-1}^{n-1},\end{aligned}$$

we find that

$$\int g\, dx \leq \prod_{i=1}^{n} \left\| g_i \right\|_{n-1}.$$

The result follows by induction. □

We now prove the main result.

THEOREM 3.28. *Let $1 \leq p < n$, where $n \geq 2$, and let p^* be the Sobolev conjugate of p given in Definition 3.25. Then*

$$\|f\|_{p^*} \leq C \left\| Df \right\|_p, \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n)$$

where

$$C(n,p) = \frac{p}{2n}\left(\frac{n-1}{n-p}\right). \tag{3.11}$$

PROOF. First, we prove the result for $p = 1$. For $1 \leq i \leq n$, we have

$$|f(x)| \leq \frac{1}{2} \int \left| \partial_i f(t, x'_i) \right| dt.$$

Multiplying these inequalities and taking the $(n-1)$th root, we get

$$|f|^{n/(n-1)} \leq \frac{1}{2^{n/(n-1)}}\, g, \qquad g = \prod_{i=1}^{n} \tilde{g}_i$$

where $\tilde{g}_i(x) = g_i(x_i')$ with

$$g_i(x_i') = \left(\int |\partial_i f(t, x_i')| \, dt\right)^{1/(n-1)}.$$

Theorem 3.27 implies that

$$\int g \, dx \leq \prod_{i=1}^{n} \|g_i\|_{n-1}.$$

Since

$$\|g_i\|_{n-1} = \left(\int |\partial_i f| \, dx\right)^{1/(n-1)}$$

it follows that

$$\int |f|^{n/(n-1)} \, dx \leq \frac{1}{2^{n/(n-1)}} \left(\prod_{i=1}^{n} \int |\partial_i f| \, dx\right)^{1/(n-1)}.$$

Note that $n/(n-1) = 1^*$ is the Sobolev conjugate of 1.

Using the arithmetic-geometric mean inequality,

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i,$$

we get

$$\int |f|^{n/(n-1)} \, dx \leq \left(\frac{1}{2n} \sum_{i=1}^{n} \int |\partial_i f| \, dx\right)^{n/(n-1)},$$

or

$$\|f\|_{1^*} \leq \frac{1}{2n} \|Df\|_1,$$

which proves the result when $p = 1$.

Next suppose that $1 < p < n$. For any $s > 1$, we have

$$\frac{d}{dx}|x|^s = s \operatorname{sgn} x |x|^{s-1}.$$

Thus,

$$\begin{aligned} |f(x)|^s &= \int_{-\infty}^{x_i} \partial_i |f(t, x_i')|^s \, dt \\ &= s \int_{-\infty}^{x_i} |f(t, x_i')|^{s-1} \operatorname{sgn}[f(t, x_i')] \, \partial_i f(t, x_i') \, dt. \end{aligned}$$

Using Lemma 3.26, it follows that

$$|f(x)|^s \leq \frac{s}{2} \int_{-\infty}^{\infty} \left| f^{s-1}(t, x_i') \partial_i f(t, x_i') \right| \, dt,$$

and multiplication of these inequalities gives

$$|f(x)|^{sn} \leq \left(\frac{s}{2}\right)^n \prod_{i=1}^{n} \int_{-\infty}^{\infty} \left| f^{s-1}(t, x_i') \partial_i f(t, x_i') \right| \, dt.$$

Applying Theorem 3.27 with the functions

$$g_i(x_i') = \left[\int_{-\infty}^{\infty} \left| f^{s-1}(t, x_i') \partial_i f(t, x_i') \right| \, dt\right]^{1/(n-1)}$$

we find that

$$\|f\|_{sn/(n-1)}^{sn} \leq \frac{s}{2}\prod_{i=1}^{n}\left\|f^{s-1}\partial_i f\right\|_1 .$$

From Hölder's inequality,

$$\left\|f^{s-1}\partial_i f\right\|_1 \leq \left\|f^{s-1}\right\|_{p'}\|\partial_i f\|_p .$$

We have

$$\left\|f^{s-1}\right\|_{p'} = \|f\|_{p'(s-1)}^{s-1}$$

We choose $s > 1$ so that

$$p'(s-1) = \frac{sn}{n-1},$$

which holds if

$$s = p\left(\frac{n-1}{n-p}\right), \qquad \frac{sn}{n-1} = p^*.$$

Then

$$\|f\|_{p^*} \leq \frac{s}{2}\left(\prod_{i=1}^{n}\|\partial_i f\|_p\right)^{1/n} .$$

Using the arithmetic-geometric mean inequality, we get

$$\|f\|_{p^*} \leq \frac{s}{2n}\left(\sum_{i=1}^{n}\|\partial_i f\|_p^p\right)^{1/p},$$

which proves the result. □

We can interpret this result roughly as follows: Differentiation of a function increases the strength of its local singularities and improves its decay at infinity. Thus, if $Df \in L^p$, it is reasonable to expect that $f \in L^{p^*}$ for some $p^* > p$ since L^{p^*}-functions have weaker singularities and decay more slowly at infinity than L^p-functions.

Example 3.29. For $a > 0$, let $f_a : \mathbb{R}^n \to \mathbb{R}$ be the function

$$f_a(x) = \frac{1}{|x|^a}$$

considered in Example 3.7. This function does not belong to $L^q(\mathbb{R}^n)$ for any a since the integral at infinity diverges whenever the integral at zero converges. Let ϕ be a smooth cut-off function that is equal to one for $|x| \leq 1$ and zero for $|x| \geq 2$. Then $g_a = \phi f_a$ is an unbounded function with compact support. We have $g_a \in L^q(\mathbb{R}^n)$ if $aq < n$, and $Dg_a \in L^p(\mathbb{R}^n)$ if $p(a+1) < n$ or $ap^* < n$. Thus if $Dg_a \in L^p(\mathbb{R}^n)$, then $g_a \in L^q(\mathbb{R}^n)$ for $1 \leq q \leq p^*$. On the other hand, the function $h_a = (1-\phi)f_a$ is smooth and decays like $|x|^{-a}$ as $x \to \infty$. We have $h_a \in L^q(\mathbb{R}^n)$ if $qa > n$ and $Dh_a \in L^p(\mathbb{R}^n)$ if $p(a+1) > n$ or $p^*a > n$. Thus, if $Dh_a \in L^p(\mathbb{R}^n)$, then $f \in L^q(\mathbb{R}^n)$ for $p^* \leq q < \infty$. The function $f_{ab} = g_a + h_b$ belongs to $L^{p^*}(\mathbb{R}^n)$ for any choice of $a, b > 0$ such that $Df_{ab} \in L^p(\mathbb{R}^n)$. On the other hand, for any $1 \leq q \leq \infty$ such that $q \neq p^*$, there is a choice of $a, b > 0$ such that $Df_{ab} \in L^p(\mathbb{R}^n)$ but $f_{ab} \notin L^q(\mathbb{R}^n)$.

The constant in Theorem 3.28 is not optimal. For $p = 1$, the best constant is

$$C(n,1) = \frac{1}{n\alpha_n^{1/n}}$$

where α_n is the volume of the unit ball, or

$$C(n,1) = \frac{1}{n\sqrt{\pi}}\left[\Gamma\left(1+\frac{n}{2}\right)\right]^{1/n}$$

where Γ is the Γ-function. Equality is obtained in the limit of functions that approach the characteristic function of a ball. This result for the best Sobolev constant is equivalent to the isoperimetric inequality that a sphere has minimal area among all surfaces enclosing a given volume.

For $1 < p < n$, the best constant is (Talenti, 1976)

$$C(n,p) = \frac{1}{n^{1/p}\sqrt{\pi}}\left(\frac{p-1}{n-p}\right)^{1-1/p}\left[\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right]^{1/n}.$$

Equality holds for functions of the form

$$f(x) = \left(a + b|x|^{p/(p-1)}\right)^{1-n/p}$$

where a, b are positive constants.

The Sobolev inequality in Theorem 3.28 does not hold in the limiting case $p \to n$, $p^* \to \infty$.

Example 3.30. If $\phi(x)$ is a smooth cut-off function that is equal to one for $|x| \le 1$ and zero for $|x| \ge 2$, and

$$f(x) = \phi(x)\log\log\left(1 + \frac{1}{|x|}\right),$$

then $Df \in L^n(\mathbb{R}^n)$, and $f \in W^{1,n}(\mathbb{R})$, but $f \notin L^\infty(\mathbb{R}^n)$.

We can use the Sobolev inequality to prove various embedding theorems. In general, we say that a Banach space X is continuously embedded, or embedded for short, in a Banach space Y if there is a one-to-one, bounded linear map $\imath : X \to Y$. We often think of $\imath$ as identifying elements of the smaller space X with elements of the larger space Y; if X is a subset of Y, then $\imath$ is the inclusion map. The boundedness of $\imath$ means that there is a constant C such that $\|\imath x\|_Y \le C\|x\|_X$ for all $x \in X$, so the weaker Y-norm of $\imath x$ is controlled by the stronger X-norm of x.

We write an embedding as $X \hookrightarrow Y$, or as $X \subset Y$ when the boundedness is understood.

Theorem 3.31. *Suppose that $1 \le p < n$ and $p \le q \le p^*$ where p^* is the Sobolev conjugate of p. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and*

$$\|f\|_q \le C\|f\|_{W^{1,p}} \qquad \text{for all } f \in W^{1,p}(\mathbb{R}^n)$$

for some constant $C = C(n,p,q)$.

Proof. If $f \in W^{1,p}(\mathbb{R}^n)$, then by Theorem 3.24 there is a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^n)$ that converges to f in $W^{1,p}(\mathbb{R}^n)$. Theorem 3.28 implies that $f_n \to f$ in $L^{p^*}(\mathbb{R}^n)$. In detail: $\{Df_n\}$ converges to Df in L^p so it is Cauchy in L^p; since

$$\|f_n - f_m\|_{p^*} \le C\|Df_n - Df_m\|_p$$

$\{f_n\}$ is Cauchy in L^{p^*}; therefore $f_n \to \tilde{f}$ for some $\tilde{f} \in L^{p^*}$ since L^{p^*} is complete; and $\tilde{f}$ is equivalent to f since a subsequence of $\{f_n\}$ converges pointwise a.e. to $\tilde{f}$, from the L^{p^*} convergence, and to f, from the L^p-convergence.

Thus, $f \in L^{p^*}(\mathbb{R}^n)$ and

$$\|f\|_{p^*} \leq C\|Df\|_p.$$

Since $f \in L^p(\mathbb{R}^n)$, Lemma 1.11 implies that for $p < q < p^*$

$$\|f\|_q \leq \|f\|_p^\theta \|f\|_{p^*}^{1-\theta}$$

where $0 < \theta < 1$ is defined by

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}.$$

Therefore, using Theorem 3.28 and the inequality

$$a^\theta b^{1-\theta} \leq \left[\theta^\theta (1-\theta)^{1-\theta}\right]^{1/p} \left(a^p + b^p\right)^{1/p},$$

we get

$$\begin{aligned}
\|f\|_q &\leq C^{1-\theta}\|f\|_p^\theta \|Df\|_p^{1-\theta} \\
&\leq C^{1-\theta}\left[\theta^\theta (1-\theta)^{1-\theta}\right]^{1/p} \left(\|f\|_p^p + \|Df\|_p^p\right)^{1/p} \\
&\leq C^{1-\theta}\left[\theta^\theta (1-\theta)^{1-\theta}\right]^{1/p} \|f\|_{W^{1,p}}.
\end{aligned}$$

□

Sobolev embedding gives a stronger conclusion for sets Ω with finite measure. In that case, $L^{p^*}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q \leq p^*$, so $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p^*$, not just $p \leq q \leq p^*$.

Theorem 3.28 does not, of course, imply that $f \in L^{p^*}(\mathbb{R}^n)$ whenever $Df \in L^p(\mathbb{R}^n)$, since constant functions have zero derivative. To ensure that $f \in L^{p^*}(\mathbb{R}^n)$, we also need to impose a decay condition on f that eliminates the constant functions. In Theorem 3.31, this is provided by the assumption that $f \in L^p(\mathbb{R}^n)$ in addition to $Df \in L^p(\mathbb{R}^n)$. We can instead impose the following weaker decay condition.

Definition 3.32. A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ vanishes at infinity if for every $\epsilon > 0$ the set $\{x \in \mathbb{R}^n : |f(x)| > \epsilon\}$ has finite Lebesgue measure.

If $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, then f vanishes at infinity. Note that this does not imply that $\lim_{|x|\to\infty} f(x) = 0$.

Example 3.33. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f = \sum_{n\in\mathbb{N}} \chi_{I_n}, \qquad I_n = \left[n, n + \frac{1}{n^2}\right]$$

where χ_I is the characteristic function of the interval I. Then

$$\int f\, dx = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty,$$

so $f \in L^1(\mathbb{R})$. The limit of $f(x)$ as $|x| \to \infty$ does not exist since $f(x)$ takes on the values 0 and 1 for arbitrarily large values of x. Nevertheless, f vanishes at infinity since for any $\epsilon < 1$,

$$|\{x \in \mathbb{R} : |f(x)| > \epsilon\}| = \sum_{n\in\mathbb{N}} \frac{1}{n^2},$$

which is finite.

EXAMPLE 3.34. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/\log x & \text{if } x \geq 2 \\ 0 & \text{if } x < 2 \end{cases}$$

vanishes at infinity, but $f \notin L^p(\mathbb{R})$ for any $1 \leq p < \infty$.

The Sobolev embedding theorem remains true for functions that vanish at infinity.

THEOREM 3.35. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is weakly differentiable with $Df \in L^p(\mathbb{R}^n)$ where $1 \leq p < n$ and f vanishes at infinity. Then $f \in L^{p^*}(\mathbb{R}^n)$ and*

$$\|f\|_{p^*} \leq C\|Df\|_p$$

where C is given in (3.11).

As before, we prove this by approximating f with smooth compactly supported functions. We omit the details.

3.8. Sobolev embedding: $p > n$

> Friedrichs was a great lover of inequalities, and that affected me very much. The point of view was that the inequalities are more interesting than the equalities, the identities.[4]

In the previous section, we saw that if the weak derivative of a function that vanishes at infinity belongs to $L^p(\mathbb{R}^n)$ with $p < n$, then the function has improved integrability properties and belongs to $L^{p^*}(\mathbb{R}^n)$. Even though the function is weakly differentiable, it need not be continuous. In this section, we show that if the derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$ then the function (or a pointwise a.e. equivalent version of it) is continuous, and in fact Hölder continuous. The following result is due to Morrey (1940). The main idea is to estimate the difference $|f(x) - f(y)|$ in terms of Df by the mean value theorem, average the result over a ball $B_r(x)$ and estimate the result in terms of $\|Df\|_p$ by Hölder's inequality.

THEOREM 3.36. *Let $n < p < \infty$ and*

$$\alpha = 1 - \frac{n}{p},$$

with $\alpha = 1$ if $p = \infty$. Then there are constants $C = C(n,p)$ such that

$$[f]_\alpha \leq C\,\|Df\|_p \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{3.12}$$

$$\sup_{\mathbb{R}^n} |f| \leq C\,\|f\|_{W^{1,p}} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{3.13}$$

where $[\cdot]_\alpha$ denotes the Hölder seminorm $[\cdot]_{\alpha,\mathbb{R}^n}$ defined in (1.1).

PROOF. First we prove that there exists a constant C depending only on n such that for any ball $B_r(x)$

$$\fint_{B_r(x)} |f(x) - f(y)|\,dy \leq C \int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\,dy \tag{3.14}$$

Let $w \in \partial B_1(0)$ be a unit vector. For $s > 0$

$$f(x+sw) - f(x) = \int_0^s \frac{d}{dt} f(x+tw)\,dt = \int_0^s Df(x+tw)\cdot w\,dt,$$

[4]Louis Nirenberg on K. O. Friedrichs, from *Notices of the AMS*, April 2002.

and therefore since $|w| = 1$

$$|f(x+sw) - f(x)| \leq \int_0^s |Df(x+tw)|\, dt.$$

Integrating this inequality with respect to w over the unit sphere, we get

$$\int_{\partial B_1(0)} |f(x) - f(x+sw)|\, dS(w) \leq \int_{\partial B_1(0)} \left(\int_0^s |Df(x+tw)|\, dt \right) dS(w).$$

From Proposition 1.45,

$$\begin{aligned}\int_{\partial B_1(0)} \left(\int_0^s |Df(x+tw)|\, dt \right) dS(w) &= \int_{\partial B_1(0)} \int_0^s \frac{|Df(x+tw)|}{t^{n-1}}\, t^{n-1}\, dt dS(w) \\ &= \int_{B_s(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy,\end{aligned}$$

Thus,

$$\int_{\partial B_1(0)} |f(x) - f(x+sw)|\, dS(w) \leq \int_{B_s(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy.$$

Using Proposition 1.45 together with this inequality, and estimating the integral over $B_s(x)$ by the integral over $B_r(x)$ for $s \leq r$, we find that

$$\begin{aligned}\int_{B_r(x)} |f(x) - f(y)|\, dy &= \int_0^r \left(\int_{\partial B_1(0)} |f(x) - f(x+sw)|\, dS(w) \right) s^{n-1}\, ds \\ &\leq \int_0^r \left(\int_{B_s(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy \right) s^{n-1}\, ds \\ &\leq \left(\int_0^r s^{n-1}\, ds \right) \left(\int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy \right) \\ &\leq \frac{r^n}{n} \int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy\end{aligned}$$

This gives (3.14) with $C = (n\alpha_n)^{-1}$.

Next, we prove (3.12). Suppose that $x, y \in \mathbb{R}^n$. Let $r = |x-y|$ and $\Omega = B_r(x) \cap B_r(y)$. Then averaging the inequality

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(y) - f(z)|$$

with respect to z over Ω, we get

$$|f(x) - f(y)| \leq \fint_\Omega |f(x) - f(z)|\, dz + \fint_\Omega |f(y) - f(z)|\, dz. \tag{3.15}$$

From (3.14) and Hölder's inequality,

$$\begin{aligned}\fint_\Omega |f(x) - f(z)|\, dz &\leq \fint_{B_r(x)} |f(x) - f(z)|\, dz \\ &\leq C \int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy \\ &\leq C \left(\int_{B_r(x)} |Df|^p\, dz \right)^{1/p} \left(\int_{B_r(x)} \frac{dz}{|x-z|^{p'(n-1)}} \right)^{1/p'}.\end{aligned}$$

We have

$$\left(\int_{B_r(x)} \frac{dz}{|x-z|^{p'(n-1)}}\right)^{1/p'} = C\left(\int_0^r \frac{r^{n-1}dr}{r^{p'(n-1)}}\right)^{1/p'} = Cr^{1-n/p}$$

where C denotes a generic constant depending on n and p. Thus,

$$⨍_\Omega |f(x) - f(z)|\, dz \leq Cr^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)},$$

with a similar estimate for the integral in which x is replaced by y. Using these estimates in (3.15) and setting $r = |x-y|$, we get

$$|f(x) - f(y)| \leq C|x-y|^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)}, \tag{3.16}$$

which proves (3.12).

Finally, we prove (3.13). For any $x \in \mathbb{R}^n$, using (3.16), we find that

$$\begin{aligned} |f(x)| &\leq ⨍_{B_1(x)} |f(x) - f(y)|\, dy + ⨍_{B_1(x)} |f(y)|\, dy \\ &\leq C\|Df\|_{L^p(\mathbb{R}^n)} + C\|f\|_{L^p(B_1(x))} \\ &\leq C\|f\|_{W^{1,p}(\mathbb{R}^n)}, \end{aligned}$$

and taking the supremum with respect to x, we get (3.13). □

Combining these estimates for

$$\|f\|_{C^{0,\alpha}} = \sup|f| + [f]_\alpha$$

and using a density argument, we get the following theorem. We denote by $C_0^{0,\alpha}(\mathbb{R}^n)$ the space of Hölder continuous functions f whose limit as $x \to \infty$ is zero, meaning that for every $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|f(x)| < \epsilon$ if $x \in \mathbb{R}^n \setminus K$.

THEOREM 3.37. *Let $n < p < \infty$ and $\alpha = 1 - n/p$. Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$$

and there is a constant $C = C(n,p)$ such that

$$\|f\|_{C^{0,\alpha}} \leq C\|f\|_{W^{1,p}} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

PROOF. From Theorem 3.24, the mollified functions $\eta^\epsilon * f^\epsilon \to f$ in $W^{1,p}(\mathbb{R}^n)$ as $\epsilon \to 0^+$, and by Theorem 3.36

$$|f^\epsilon(x) - f^\epsilon(y)| \leq C|x-y|^{1-n/p} \|Df^\epsilon\|_{L^p}.$$

Letting $\epsilon \to 0^+$, we find that

$$|f(x) - f(y)| \leq C|x-y|^{1-n/p} \|Df\|_{L^p}$$

for all Lebesgue points $x, y \in \mathbb{R}^n$ of f. Since these form a set of measure zero, f extends by uniform continuity to a uniformly continuous function on $\mathbb{R}^n$.

Also from Theorem 3.24, the function $f \in W^{1,p}(\mathbb{R}^n)$ is a limit of compactly supported functions, and from (3.13), f is the uniform limit of compactly supported functions, which implies that its limit as $x \to \infty$ is zero. □

We state two related results without proof (see §5.8 of [**9**]).

For $p = \infty$, the same proof as the proof of (3.12), using Hölder's inequality with $p = \infty$ and $p' = 1$, shows that $f \in W^{1,\infty}(\mathbb{R}^n)$ is globally Lipschitz continuous, with

$$[f]_1 \leq C \|Df\|_{L^\infty}.$$

A function in $W^{1,\infty}(\mathbb{R}^n)$ need not approach zero at infinity. We have in this case the following characterization of Lipschitz functions.

THEOREM 3.38. *A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is globally Lipschitz continuous if and only if it is weakly differentiable and $Df \in L^\infty(\mathbb{R}^n)$.*

When $n < p \leq \infty$, the above estimates can be used to prove that the pointwise derivative of a Sobolev function exists almost everywhere and agrees with the weak derivative.

THEOREM 3.39. *If $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $n < p \leq \infty$, then f is differentiable pointwise a.e. and the pointwise derivative coincides with the weak derivative.*

3.9. Boundary values of Sobolev functions

If $f \in C(\overline{\Omega})$ is a continuous function on the closure of a smooth domain Ω, then we can define the boundary values of f pointwise as a continuous function on the boundary $\partial\Omega$. We can also do this when Sobolev embedding implies that a function is Hölder continuous. In general, however, a Sobolev function is not equivalent pointwise a.e. to a continuous function and the boundary of a smooth open set has measure zero, so the boundary values cannot be defined pointwise. For example, we cannot make sense of the boundary values of an L^p-function as an L^p-function on the boundary.

EXAMPLE 3.40. Suppose $T : C^\infty([0,1]) \to \mathbb{R}$ is the map defined by $T : \phi \mapsto \phi(0)$. If $\phi^\epsilon(x) = e^{-x^2/\epsilon}$, then $\|\phi^\epsilon\|_{L^1} \to 0$ as $\epsilon \to 0^+$, but $\phi^\epsilon(0) = 1$ for every $\epsilon > 0$. Thus, T is not bounded (or even closed) in L^1 and we cannot extend it by continuity to $L^1(0,1)$.

Nevertheless, we can define the boundary values of suitable Sobolev functions at the expense of a loss of smoothness in restricting the functions to the boundary. To do this, we show that the linear map on smooth functions that gives their boundary values is bounded with respect to appropriate Sobolev norms. We then extend the map by continuity to Sobolev functions, and the resulting trace map defines their boundary values.

We consider the basic case of a half-space $\mathbb{R}^n_+$. We write $x = (x', x_n) \in \mathbb{R}^n_+$ where $x_n > 0$ and $(x', 0) \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$.

The Sobolev space $W^{1,p}(\mathbb{R}^n_+)$ consists of functions $f \in L^p(\mathbb{R}^n_+)$ that are weakly differentiable in $\mathbb{R}^n_+$ with $Df \in L^p(\mathbb{R}^n_+)$. We begin with a result which states that we can extend functions $f \in W^{1,p}(\mathbb{R}^n_+)$ to functions in $W^{1,p}(\mathbb{R}^n)$ without increasing their norm. An extension may be constructed by reflecting a function across the boundary $\partial\mathbb{R}^n_+$ in a way that preserves its differentiability. Such an extension map E is not, of course, unique.

THEOREM 3.41. *There is a bounded linear map*

$$E : W^{1,p}(\mathbb{R}^n_+) \to W^{1,p}(\mathbb{R}^n)$$

such that $Ef = f$ pointwise a.e. in $\mathbb{R}^n_+$ and for some constant $C = C(n,p)$

$$\|Ef\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|f\|_{W^{1,p}(\mathbb{R}^n_+)} .$$

The following approximation result may be proved by extending a Sobolev function from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, mollifying the extension, and restricting the result to the half-space.

THEOREM 3.42. *The space $C_c^\infty(\overline{\mathbb{R}}^n_+)$ of smooth functions is dense in $W^{k,p}(\mathbb{R}^n_+)$.*

Functions $f : \overline{\mathbb{R}}^n_+ \to \mathbb{R}$ in $C_c^\infty(\overline{\mathbb{R}}^n_+)$ need not vanish on the boundary $\partial\mathbb{R}^n_+$. On the other hand, functions in the space $C_c^\infty(\mathbb{R}^n_+)$ of smooth functions whose support is contained in the open half space $\mathbb{R}^n_+$ do vanish on the boundary, and it is not true that this space is dense in $W^{k,p}(\mathbb{R}^n_+)$. Roughly speaking, we can only approximate Sobolev functions that 'vanish on the boundary' by functions in $C_c^\infty(\mathbb{R}^n_+)$. We make the following definition.

DEFINITION 3.43. The space $W_0^{k,p}(\mathbb{R}^n_+)$ is the closure of $C_c^\infty(\mathbb{R}^n_+)$ in $W^{k,p}(\mathbb{R}^n_+)$.

The interpretation of $W_0^{1,p}(\mathbb{R}^n_+)$ as the space of Sobolev functions that vanish on the boundary is made more precise in the following theorem, which shows the existence of a trace map T that maps a Sobolev function to its boundary values, and states that functions in $W_0^{1,p}(\mathbb{R}^n_+)$ are the ones whose trace is equal to zero.

THEOREM 3.44. *For $1 \leq p < \infty$, there is a bounded linear operator*

$$T : W^{1,p}(\mathbb{R}^n_+) \to L^p(\partial\mathbb{R}^n_+)$$

such that for any $f \in C_c^\infty(\overline{\mathbb{R}}^n_+)$

$$(Tf)(x') = f(x', 0)$$

and

$$\|Tf\|_{L^p(\mathbb{R}^{n-1})} \leq C \|f\|_{W^{1,p}(\mathbb{R}^n_+)}$$

for some constant C depending only on p. Furthermore, $f \in W_0^{k,p}(\mathbb{R}^n_+)$ if and only if $Tf = 0$.

PROOF. First, we consider $f \in C_c^\infty(\overline{\mathbb{R}}^n_+)$. For $x' \in \mathbb{R}^{n-1}$ and $p \geq 1$, we have

$$|f(x',0)|^p \leq p \int_0^\infty |f(x',t)|^{p-1} |\partial_n f(x',t)| \, dt.$$

Hence, using Hölder's inequality and the identity $p'(p-1) = p$, we get

$$\begin{aligned}
\int |f(x',0)|^p \, dx' &\leq p \int_0^\infty |f(x',t)|^{p-1} |\partial_n f(x',t)| \, dx' dt \\
&\leq p \left(\int_0^\infty |f(x',t)|^{p'(p-1)} \, dx' dt \right)^{1/p'} \left(\int_0^\infty |\partial_n f(x',t)|^p \, dx' dt \right)^{1/p} \\
&\leq p \|f\|_p^{p-1} \|\partial_n f\|_p \\
&\leq p \|f\|_{W^{k,p}}^p .
\end{aligned}$$

The trace map

$$T : C_c^\infty(\overline{\mathbb{R}}^n_+) \to C_c^\infty(\mathbb{R}^{n-1})$$

is therefore bounded with respect to the $W^{1,p}(\mathbb{R}^n_+)$ and $L^p(\partial\mathbb{R}^n_+)$ norms, and extends by density and continuity to a map between these spaces. It follows immediately that $Tf = 0$ if $f \in W_0^{k,p}(\mathbb{R}^n_+)$.

We omit the proof that $Tf = 0$ implies that $f \in W_0^{k,p}(\mathbb{R}^n_+)$. (The idea is to extend f by 0, translate the extension into the domain, and mollify the translated extension to get a smooth compactly supported approximation; see e.g., [**9**]). □

If $p = 1$, the trace $T : W^{1,1}(\mathbb{R}^n_+) \to L^1(\mathbb{R}^{n-1})$ is onto, but if $1 < p < \infty$ the range of T is not all of L^p. In that case, $T : W^{1,p}(\mathbb{R}^n_+) \to B^{1-1/p,p}(\mathbb{R}^{n-1})$ maps $W^{1,p}$ onto a Besov space $B^{1-1/p,p}$; roughly speaking, this is a Sobolev space of functions with fractional derivatives, and there is a loss of $1/p$ derivatives in restricting a function to the boundary [**26**].

An alternative, and more concrete, way to define the trace map is to show that if $f \in W^{1,1}(\mathbb{R}^n_+)$, then $f = \tilde{f}$ pointwise a.e. in $\mathbb{R}^n_+$ where $\tilde{f}(x', x_n)$ is an absolutely continuous function of $0 \le x_n < \infty$ for x' pointwise a.e. in $\mathbb{R}^{n-1}$. In that case, $(Tf)(x') = \tilde{f}(x', 0)$ is defined pointwise a.e. on the boundary by continuity [**3**].[5]

Note that if $f \in W_0^{2,p}(\mathbb{R}^n_+)$, then $\partial_i f \in W_0^{1,p}(\mathbb{R}^n_+)$, so $T(\partial_i f) = 0$. Thus, both f and Df vanish on the boundary. The correct way to formulate the condition that f has weak derivatives of order less than or equal to two and satisfies the Dirichlet condition $f = 0$ on the boundary is that $f \in W^{2,p}(\mathbb{R}^n_+) \cap W_0^{1,p}(\mathbb{R}^n_+)$.

3.10. Compactness results

A Banach space X is compactly embedded in a Banach space Y, written $X \Subset Y$, if the embedding $\imath : X \to Y$ is compact. That is, $\imath$ maps bounded sets in X to precompact sets in Y; or, equivalently, if $\{x_n\}$ is a bounded sequence in X, then $\{\imath x_n\}$ has a convergent subsequence in Y.

An important property of the Sobolev embeddings is that they are compact on domains with finite measure. This corresponds to the rough principle that uniform bounds on higher derivatives imply compactness with respect to lower derivatives. The compactness of the Sobolev embeddings, due to Rellich and Kondrachov, depend on the Arzelà-Ascoli theorem. We will prove a version for $W_0^{1,p}(\Omega)$ by use of the L^p-compactness criterion in Theorem 1.15.

THEOREM 3.45. *Let Ω be a bounded open set in $\mathbb{R}^n$, $1 \le p < n$, and $1 \le q < p^*$. If $\mathcal{F}$ is a bounded set in $W_0^{1,p}(\Omega)$, then $\mathcal{F}$ is precompact in $L^q(\mathbb{R}^n)$.*

PROOF. By a density argument, we may assume that the functions in $\mathcal{F}$ are smooth and $\operatorname{supp} f \Subset \Omega$. We may then extend the functions and their derivatives by zero to obtain smooth functions on $\mathbb{R}^n$, and prove that $\mathcal{F}$ is precompact in $L^q(\mathbb{R}^n)$.

Condition (1) in Theorem 1.15 follows immediately from the boundedness of Ω and the Sobolev embeddeding theorem: for all $f \in \mathcal{F}$,

$$\|f\|_{L^q(\mathbb{R}^n)} = \|f\|_{L^q(\Omega)} \le C\|f\|_{L^{p^*}(\Omega)} \le C\|Df\|_{L^p(\mathbb{R}^n)} \le C$$

where C denotes a generic constant that does not depend on f. Condition (2) is satisfied automatically since the supports of all functions in $\mathcal{F}$ are contained in the same bounded set.

[5]The definition of weakly differentiable functions as absolutely continuous functions on lines $x_i =$ constant, pointwise a.e. in the remaining coordinates x_i', goes back to the Italian mathematician Levi (1906) before the introduction of Sobolev spaces.

To verify (3), we first note that since Df is supported inside the bounded open set Ω,

$$\|Df\|_{L^1(\mathbb{R}^n)} \leq C \|Df\|_{L^p(\mathbb{R}^n)}.$$

Fix $h \in \mathbb{R}^n$ and let $f_h(x) = f(x+h)$ denote the translation of f by h. Then

$$|f_h(x) - f(x)| = \left| \int_0^1 h \cdot Df(x+th)\, dt \right| \leq |h| \int_0^1 |Df(x+th)|\, dt.$$

Integrating this inequality with respect to x and using Fubini's theorem to exchange the order of integration on the right-hand side, together with the fact that the inner x-integral is independent of t, we get

$$\int_{\mathbb{R}^n} |f_h(x) - f(x)|\, dx \leq |h| \|Df\|_{L^1(\mathbb{R}^n)} \leq C|h| \|Df\|_{L^p(\mathbb{R}^n)}.$$

Thus,

$$\|f_h - f\|_{L^1(\mathbb{R}^n)} \leq C|h| \|Df\|_{L^p(\mathbb{R}^n)}. \tag{3.17}$$

Using the interpolation inequality in Lemma 1.11, we get for any $1 \leq q < p^*$ that

$$\|f_h - f\|_{L^q(\mathbb{R}^n)} \leq \|f_h - f\|_{L^1(\mathbb{R}^n)}^{\theta} \|f_h - f\|_{L^{p^*}(\mathbb{R}^n)}^{1-\theta} \tag{3.18}$$

where $0 < \theta \leq 1$ is given by

$$\frac{1}{q} = \theta + \frac{1-\theta}{p^*}.$$

The Sobolev embedding theorem implies that

$$\|f_h - f\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Df\|_{L^p(\mathbb{R}^n)}.$$

Using this inequality and (3.17) in (3.18), we get

$$\|f_h - f\|_{L^q(\mathbb{R}^n)} \leq C|h|^{\theta} \|Df\|_{L^p(\mathbb{R}^n)}.$$

It follows that $\mathcal{F}$ is L^q-equicontinuous if the derivatives of functions in $\mathcal{F}$ are uniformly bounded in L^p, and the result follows. □

Equivalently, this theorem states that if $\{f_k : k \in \mathbb{N}\}$ is a sequence of functions in $W_0^{1,p}(\Omega)$ such that

$$\|f_k\|_{W^{1,p}} \leq C \qquad \text{for all } k \in \mathbb{N},$$

for some constant C, then there exists a subsequence f_{k_i} and a function $f \in L^q(\Omega)$ such that

$$f_{k_i} \to f \qquad \text{as } i \to \infty \text{ in } L^q(\Omega).$$

The assumptions that the domain Ω satisfies a boundedness condition and that $q < p^*$ are necessary.

Example 3.46. If $\phi \in W^{1,p}(\mathbb{R}^n)$ and $f_m(x) = \phi(x - c_m)$, where $c_m \to \infty$ as $m \to \infty$, then $\|f_m\|_{W^{1,p}} = \|\phi\|_{W^{1,p}}$ is constant, but $\{f_m\}$ has no convergent subsequence in L^q since the functions 'escape' to infinity. Thus, compactness does not hold without some limitation on the decay of the functions.

Example 3.47. For $1 \leq p < n$, define $f_k : \mathbb{R}^n \to \mathbb{R}$ by

$$f_k(x) = \begin{cases} k^{n/p^*}(1 - k|x|) & \text{if } |x| < 1/k, \\ 0 & \text{if } |x| \geq 1/k. \end{cases}$$

Then $\operatorname{supp} f_k \subset \overline{B}_1(0)$ for every $k \in \mathbb{N}$ and $\{f_k\}$ is bounded in $W^{1,p}(\mathbb{R}^n)$, but no subsequence converges strongly in $L^{p^*}(\mathbb{R}^n)$.

The loss of compactness in the critical case $q = p^*$ has received a great deal of study (for example, in the concentration compactness principle of P.L. Lions).

If Ω is a smooth and bounded domain, the use of an extension map implies that $W^{1,p}(\Omega) \Subset L^q(\Omega)$. For an example of the loss of this compactness in a bounded domain with an irregular boundary, see [**26**].

THEOREM 3.48. *Let Ω be a bounded open set in $\mathbb{R}^n$, and $n < p < \infty$. Suppose that $\mathcal{F}$ is a set of functions whose weak derivative belongs to $L^p(\mathbb{R}^n)$ such that: (a)* $\operatorname{supp} f \Subset \Omega$*; (b) there exists a constant C such that*

$$\|Df\|_{L^p} \le C \qquad \text{for all } f \in \mathcal{F}.$$

Then $\mathcal{F}$ is precompact in $C_0(\mathbb{R}^n)$.

PROOF. Theorem 3.36 implies that the set $\mathcal{F}$ is bounded and equicontinuous, so the result follows immediately from the Arzelà-Ascoli theorem. □

In other words, if $\{f_m : m \in \mathbb{N}\}$ is a sequence of functions in $W^{1,p}(\mathbb{R}^n)$ such that $\operatorname{supp} f_m \subset \Omega$, where $\Omega \Subset \mathbb{R}^n$, and

$$\|f_m\|_{W^{1,p}} \le C \qquad \text{for all } m \in \mathbb{N}$$

for some constant C, then there exists a subsequence f_{m_k} such that $f_{n_k} \to f$ uniformly, in which case $f \in C_c(\mathbb{R}^n)$.

3.11. Sobolev functions on $\Omega \subset \mathbb{R}^n$

Here, we briefly outline how ones transfers the results above to Sobolev spaces on domains other than $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Suppose that Ω is a smooth, bounded domain in $\mathbb{R}^n$. We may cover the closure $\overline{\Omega}$ by a collection of open balls contained in Ω and open balls with center $x \in \partial\Omega$. Since $\overline{\Omega}$ is compact, there is a finite collection $\{B_i : 1 \le i \le N\}$ of such open balls that covers $\overline{\Omega}$. There is a partition of unity $\{\psi_i : 1 \le i \le N\}$ subordinate to this cover consisting of functions $\psi_i \in C_c^\infty(B_i)$ such that $0 \le \psi_i \le 1$ and $\sum_i \psi_i = 1$ on $\overline{\Omega}$.

Given any function $f \in L^1_{\text{loc}}(\Omega)$, we may write $f = \sum_i f_i$ where $f_i = \psi_i f$ has compact support in B_i for balls whose center belongs to Ω, and in $B_i \cap \overline{\Omega}$ for balls whose center belongs to $\partial\Omega$. In these latter balls, we may 'straighten out the boundary' by a smooth map. After this change of variables, we get a function f_i that is compactly supported in $\overline{\mathbb{R}}^n_+$. We may then apply the previous results to the functions $\{f_i : 1 \le i \le N\}$.

Typically, results about $W^{k,p}_0(\Omega)$ do not require assumptions on the smoothness of $\partial\Omega$; but results about $W^{k,p}(\Omega)$ — for example, the existence of a bounded extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ — only hold if $\partial\Omega$ satisfies an appropriate smoothness or regularity condition *e.g.* a C^k, Lipschitz, segment, or cone condition [**1**].

The statement of the embedding theorem for higher order derivatives extends in a straightforward way from the one for first order derivatives. For example,

$$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \qquad \text{if} \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

The result for smooth bounded domains is summarized in the following theorem. As before, $X \subset Y$ denotes a continuous embedding of X into Y, and $X \Subset Y$ denotes a compact embedding.

THEOREM 3.49. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^1 boundary, $k, m \in \mathbb{N}$ with $k \geq m$, and $1 \leq p < \infty$.*

(1) *If $kp < n$, then*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) \qquad \text{for } 1 \leq q < np/(n-kp);$$
$$W^{k,p}(\Omega) \subset L^q(\Omega) \qquad \text{for } q = np/(n-kp).$$

More generally, if $(k-m)p < n$, then

$$W^{k,p}(\Omega) \Subset W^{m,q}(\Omega) \qquad \text{for } 1 \leq q < np/\left(n-(k-m)p\right);$$
$$W^{k,p}(\Omega) \subset W^{m,q}(\Omega) \qquad \text{for } q = np/\left(n-(k-m)p\right).$$

(2) *If $kp = n$, then*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) \qquad \text{for } 1 \leq q < \infty.$$

(3) *If $kp > n$, then*

$$W^{k,p}(\Omega) \Subset C^{0,\mu}\left(\overline{\Omega}\right)$$

for $0 < \mu < k - n/p$ if $k - n/p < 1$, for $0 < \mu < 1$ if $k - n/p = 1$, and for $\mu = 1$ if $k - n/p > 1$; and

$$W^{k,p}(\Omega) \subset C^{0,\mu}\left(\overline{\Omega}\right)$$

for $\mu = k - n/p$ if $k - n/p < 1$. More generally, if $(k-m)p > n$, then

$$W^{k,p}(\Omega) \Subset C^{m,\mu}\left(\overline{\Omega}\right)$$

for $0 < \mu < k-m-n/p$ if $k-m-n/p < 1$, for $0 < \mu < 1$ if $k-m-n/p = 1$, and for $\mu = 1$ if $k - m - n/p > 1$; and

$$W^{k,p}(\Omega) \subset C^{m,\mu}\left(\overline{\Omega}\right)$$

for $\mu = k - m - n/p$ if $k - m - n/p = 0$.

These results hold for arbitrary bounded open sets Ω if $W^{k,p}(\Omega)$ is replaced by $W_0^{k,p}(\Omega)$.

EXAMPLE 3.50. If $u \in W^{n,1}(\mathbb{R}^n)$, then $u \in C_0(\mathbb{R}^n)$. This can be seen from the equality

$$u(x) = \int_0^{x_1} \dots \int_0^{x_n} \partial_1 \cdots \partial_n u(x') dx'_1 \dots dx'_n,$$

which holds for all $u \in C_c^\infty(\mathbb{R}^n)$ and a density argument. In general, however, it is not true that $u \in L^\infty$ in the critical case $kp = n$ *c.f.* Example 3.30.

Appendix

In this appendix, we describe without proof some results from real analysis which help to understand weak and distributional derivatives in the simplest context of functions of a single variable. Proofs are given in [**11**] or [**15**], for example. These results are, in fact, easier to understand from the perspective of weak and distributional derivatives of functions, rather than pointwise derivatives.

3.A. Functions

For definiteness, we consider functions $f : [a, b] \to \mathbb{R}$ defined on a compact interval $[a, b]$. When we say that a property holds almost everywhere (*a.e.*), we mean *a.e.* with respect to Lebesgue measure unless we specify otherwise.

3.A.1. Lipschitz functions. Lipschitz continuity is a weaker condition than continuous differentiability. A Lipschitz continuous function is pointwise differentiable almost everwhere and weakly differentiable. The derivative is essentially bounded, but not necessarily continuous.

Definition 3.51. A function $f : [a, b] \to \mathbb{R}$ is uniformly Lipschitz continuous on $[a, b]$ (or Lipschitz, for short) if there is a constant C such that

$$|f(x) - f(y)| \leq C\,|x - y| \qquad \text{for all } x, y \in [a, b].$$

The Lipschitz constant of f is the infimum of constants C with this property.

We denote the space of Lipschitz functions on $[a, b]$ by $\mathrm{Lip}[a, b]$. We also define the space of locally Lipschitz functions on $\mathbb{R}$ by

$$\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \in \mathrm{Lip}[a, b] \text{ for all } a < b\}.$$

By the mean-value theorem, any function that is continuous on $[a, b]$ and pointwise differentiable in (a, b) with bounded derivative is Lipschitz. In particular, every function $f \in C^1([a, b])$ is Lipschitz, and every function $f \in C^1(\mathbb{R})$ is locally Lipschitz. On the other hand, the function $x \mapsto |x|$ is Lipschitz but not C^1 on $[-1, 1]$. The following result, called Rademacher's theorem, is true for functions of several variables, but we state it here only for the one-dimensional case.

Theorem 3.52. *If $f \in \mathrm{Lip}[a, b]$, then the pointwise derivative f' exists almost everywhere in (a, b) and is essentially bounded.*

It follows from the discussion in the next section that the pointwise derivative of a Lipschitz function is also its weak derivative (since a Lipschitz function is absolutely continuous). In fact, we have the following characterization of Lipschitz functions.

Theorem 3.53. *Suppose that $f \in L^1_{\mathrm{loc}}(a, b)$. Then $f \in \mathrm{Lip}[a, b]$ if and only if f is weakly differentiable in (a, b) and $f' \in L^\infty(a, b)$. Moreover, the Lipschitz constant of f is equal to the sup-norm of f'.*

Here, we say that $f \in L^1_{\mathrm{loc}}(a, b)$ is Lipschitz on $[a, b]$ if is equal almost everywhere to a (uniformly) Lipschitz function on (a, b), in which case f extends by uniform continuity to a Lipschitz function on $[a, b]$.

EXAMPLE 3.54. The function $f(x) = x_+$ in Example 3.3 is Lipschitz continuous on $[-1, 1]$ with Lipschitz constant 1. The pointwise derivative of f exists everywhere except at $x = 0$, and is equal to the weak derivative. The sup-norm of the weak derivative $f' = \chi_{[0,1]}$ is equal to 1.

EXAMPLE 3.55. Consider the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = x^2 \sin\left(\frac{1}{x}\right).$$

Since f is C^1 on compactly contained intervals in $(0, 1)$, an integration by parts implies that

$$\int_0^1 f\phi'\,dx = -\int_0^1 f'\phi\,dx \qquad \text{for all } \phi \in C_c^\infty(0,1).$$

Thus, the weak derivative of f in $(0, 1)$ is

$$f'(x) = -\cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right).$$

Since $f' \in L^\infty(0, 1)$, f is Lipschitz on $[0, 1]$,

Similarly, if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, if and only if f is weakly differentiable in $\mathbb{R}$ and $f' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

3.A.2. Absolutely continuous functions. Absolute continuity is a strengthening of uniform continuity that provides a necessary and sufficient condition for the fundamental theorem of calculus to hold. A function is absolutely continuous if and only if its weak derivative is integrable.

DEFINITION 3.56. A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \epsilon$$

for any finite collection $\{[a_i, b_i] : 1 \le i \le N\}$ of non-overlapping subintervals $[a_i, b_i]$ of $[a, b]$ with

$$\sum_{i=1}^{N} |b_i - a_i| < \delta$$

Here, we say that intervals are non-overlapping if their interiors are disjoint. We denote the space of absolutely continuous functions on $[a, b]$ by $\mathrm{AC}[a, b]$. We also define the space of locally absolutely continuous functions on $\mathbb{R}$ by

$$\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \in \mathrm{AC}[a, b] \text{ for all } a < b\}.$$

Restricting attention to the case $N = 1$ in Definition 3.56, we see that an absolutely continuous function is uniformly continuous, but the converse is not true (see Example 3.58).

EXAMPLE 3.57. A Lipschitz function is absolutely continuous. If the function has Lipschitz constant C, we may take $\delta = \epsilon/C$ in the definition of absolute continuity.

EXAMPLE 3.58. The Cantor function f in Example 3.5 is uniformly continuous on $[0,1]$, as is any continuous function on a compact interval, but it is not absolutely continuous. We may enclose the Cantor set in a union of disjoint intervals the sum of whose lengths is as small as we please, but the jumps in f across those intervals add up to 1. Thus for any $0 < \epsilon \leq 1$, there is no $\delta > 0$ with the property required in the definition of absolute continuity. In fact, absolutely continuous functions map sets of measure zero to sets of measure zero; by contrast, the Cantor function maps the Cantor set with measure zero onto the interval $[0,1]$ with measure one.

EXAMPLE 3.59. If $g \in L^1(a,b)$ and

$$f(x) = \int_a^x g(t)\, dt$$

then $f \in \mathrm{AC}[a,b]$ and $f' = g$ pointwise *a.e.* (at every Lebesgue point of g). This is one direction of the fundamental theorem of calculus.

According to the following result, the absolutely continuous functions are precisely the ones for which the fundamental theorem of calculus holds. This result may be regarded as giving an explicit characterization of weakly differentiable functions of a single variable.

THEOREM 3.60. *A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if: (a) the pointwise derivative f' exists almost everywhere in (a,b); (b) the derivative $f' \in L^1(a,b)$ is integrable; and (c) for every $x \in [a,b]$,*

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

To prove this result, one shows from the definition of absolute continuity that if $f \in \mathrm{AC}[a,b]$, then f' exists pointwise *a.e.* and is integrable, and if $f' = 0$, then f is constant. Then the function

$$f(x) - \int_a^x f'(t)\, dt$$

is absolutely continuous with pointwise *a.e.* derivative equal to zero, so the result follows.

EXAMPLE 3.61. We recover the function $f(x) = x_+$ in Example 3.3 by integrating its derivative $\chi_{[0,\infty)}$. On the other hand, the pointwise *a.e.* derivative of the Cantor function in Example 3.5 is zero, so integration of its pointwise derivative (which exists *a.e.* and is integrable) gives zero instead of the original function.

Integration by parts holds for absolutely continuous functions.

THEOREM 3.62. *If $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous, then*

$$\int_a^b f g'\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f' g\, dx \tag{3.19}$$

where f', g' denote the pointwise a.e. derivatives of f, g.

This result is not true under the assumption that f, g that are continuous and differentiable pointwise *a.e.*, as can be seen by taking f, g to be Cantor functions on $[0,1]$.

In particular, taking $g \in C_c^\infty(a,b)$ in (3.19), we see that an absolutely continuous function f is weakly differentiable on (a,b) with integrable derivative, and the weak derivative is equal to the pointwise *a.e.* derivative. Thus, we have the following characterization of absolutely continuous functions in terms of weak derivatives.

THEOREM 3.63. *Suppose that $f \in L^1_{\mathrm{loc}}(a,b)$. Then $f \in \mathrm{AC}[a,b]$ if and only if f is weakly differentiable in (a,b) and $f' \in L^1(a,b)$.*

It follows that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is weakly differentiable if and only if $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$, in which case $f' \in L^1_{\mathrm{loc}}(\mathbb{R})$.

3.A.3. Functions of bounded variation. Functions of bounded variation are functions with finite oscillation or variation. A function of bounded variation need not be weakly differentiable, but its distributional derivative is a Radon measure.

DEFINITION 3.64. The total variation $\mathrm{V}_f([a,b])$ of a function $f : [a,b] \to \mathbb{R}$ on the interval $[a,b]$ is

$$\mathrm{V}_f([a,b]) = \sup \left\{ \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \right\}$$

where the supremum is taken over all partitions

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

of the interval $[a,b]$. A function f has bounded variation on $[a,b]$ if $\mathrm{V}_f([a,b])$ is finite.

We denote the space of functions of bounded variation on $[a,b]$ by $\mathrm{BV}[a,b]$, and refer to a function of bounded variation as a BV-function. We also define the space of locally BV-functions on $\mathbb{R}$ by

$$\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : f \in \mathrm{BV}[a,b] \text{ for all } a < b \}.$$

EXAMPLE 3.65. Every Lipschitz continuous function $f : [a,b] \to \mathbb{R}$ has bounded variation, and

$$\mathrm{V}_f([a,b]) \le C(b-a)$$

where C is the Lipschitz constant of f.

A BV-function is bounded, and an absolutely continuous function is BV; but a BV-function need not be continuous, and a continuous function need not be BV.

EXAMPLE 3.66. The discontinuous step function in Example 3.4 has bounded variation on the interval $[-1,1]$, and the continuous Cantor function in Example 3.5 has bounded variation on $[0,1]$. The total variation of both functions is equal to one. More generally, any monotone function $f : [a,b] \to \mathbb{R}$ has bounded variation, and its total variation on $[a,b]$ is equal to $|f(b) - f(a)|$.

EXAMPLE 3.67. The function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is bounded $[0,1]$, but it is not of bounded variation on $[0,1]$.

EXAMPLE 3.68. The function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous on $[0, 1]$, but it is not of bounded variation on $[0, 1]$ since its total variation is proportional to the divergent harmonic series $\sum 1/n$.

The following result states that any BV-functions is a difference of monotone increasing functions. We say that a function f is monotone increasing if $f(x) \le f(y)$ for $x \le y$; we do not require that the function is strictly increasing.

THEOREM 3.69. *A function $f : [a, b] \to \mathbb{R}$ has bounded variation on $[a, b]$ if and only if $f = f_+ - f_-$, where $f_+, f_- : [a, b] \to \mathbb{R}$ are bounded monotone increasing functions.*

To prove the theorem, we define an increasing variation function $v : [a, b] \to \mathbb{R}$ by $v(a) = 0$ and

$$v(x) = \mathrm{V}_f([a, x]) \qquad \text{for } x > a.$$

We then choose f_+, f_- so that

$$f = f_+ - f_-, \qquad v = f_+ + f_-, \tag{3.20}$$

and show that f_+, f_- are increasing functions.

The decomposition in Theorem 3.69 is not unique, since we may add an arbitrary increasing function to both f_+ and f_-, but it is unique if we add the condition that $f_+ + f_- = \mathrm{V}_f$.

A monotone function is differentiable pointwise *a.e.*, and thus so is a BV-function. In general, a BV-function contains a singular component that is not weakly differentiable in addition to an absolutely continuous component that is weakly differentiable

DEFINITION 3.70. A function $f \in \mathrm{BV}[a, b]$ is singular on $[a, b]$ if the pointwise derivative f' is equal to zero *a.e.* in $[a, b]$.

The step function and the Cantor function are examples of non-constant singular functions.[6]

THEOREM 3.71. *If $f \in \mathrm{BV}[a, b]$, then $f = f_{ac} + f_s$ where $f_{ac} \in \mathrm{AC}[a, b]$ and f_s is singular. The functions f_{ac}, f_s are unique up to an additive constant.*

The absolutely continuous part f_{ac} of f is given by

$$f_{ac}(x) = \int_a^x f'(x)\, dx$$

and the remainder $f_s = f - f_{ac}$ is the singular part. We may further decompose the singular part into a jump-function (such as the step function) and a singular continuous part (such as the Cantor function).

For $f \in \mathrm{BV}[a, b]$, let $D \subset [a, b]$ denote the set of points of discontinuity of f. Since f is the difference of monotone functions, it can only contain jump discontinuities at which its left and right limits exist (excluding the left limit at a and the right limit at b), and D is necessarily countable.

[6]Sometimes a singular function is required to be continuous, but our definition allows jump discontinuities.

If $c \in D$, let

$$[f](c) = f(c^+) - f(c^-)$$

denote the jump of f at c (with $f(a^-) = f(a)$, $f(b^+) = f(b)$ if $a, b \in D$). Define

$$f_p(x) = \sum_{c \in D \cap [a,x]} [f](c) \qquad \text{if } x \notin D.$$

Then f_p has the same jump discontinuities as f and, with an appropriate choice of $f_p(c)$ for $c \in D$, the function $f - f_p$ is continuous on $[a, b]$. Decomposing this continuous part into and absolutely continuous and a singular continuous part, we get the following result.

THEOREM 3.72. *If $f \in \mathrm{BV}[a, b]$, then $f = f_{ac} + f_p + f_{sc}$ where $f_{ac} \in \mathrm{AC}[a, b]$, f_p is a jump function, and f_{sc} is a singular continuous function. The functions f_{ac}, f_p, f_{sc} are unique up to an additive constant.*

EXAMPLE 3.73. Let $Q = \{q_n : n \in \mathbb{N}\}$ be an enumeration of the rational numbers in $[0, 1]$ and $\{p_n : n \in \mathbb{N}\}$ any sequence of real numbers such that $\sum p_n$ is absolutely convergent. Define $f : [a, b] \to \mathbb{R}$ by $f(0) = 0$ and

$$f(x) = \sum_{a \leq q_n \leq x} p_n \qquad \text{for } x > 0.$$

Then $f \in \mathrm{BV}[a, b]$, with

$$\mathrm{V}_f[a, b] = \sum_{n \in \mathbb{N}} |p_n|.$$

This function is a singular jump function with zero pointwise derivative at every irrational number in $[0, 1]$.

3.B. Measures

We denote the extended real numbers by $\overline{\mathbb{R}} = [-\infty, \infty]$ and the extended nonnegative real numbers by $\overline{\mathbb{R}}_+ = [0, \infty]$. We make the natural conventions for algebraic operations and limits that involve extended real numbers.

3.B.1. Borel measures. The Borel σ-algebra of a topological space X is the smallest collection of subsets of X that contains the open and closed sets, and is closed under complements, countable unions, and countable intersections. Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}$, and $\overline{\mathcal{B}}$ the Borel σ-algebra of $\overline{\mathbb{R}}$.

DEFINITION 3.74. A Borel measure on $\mathbb{R}$ is a function $\mu : \mathcal{B} \to \overline{\mathbb{R}}_+$, such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n)$$

for any countable collection of disjoint sets $\{E_n \in \mathcal{B} : n \in \mathbb{N}\}$.

The measure μ is finite if $\mu(\mathbb{R}) < \infty$, in which case $\mu : \mathcal{B} \to [0, \infty)$. The measure is σ-finite if $\mathbb{R}$ is a countable union of Borel sets with finite measure.

EXAMPLE 3.75. Lebesgue measure $\lambda : \mathcal{B} \to \overline{\mathbb{R}}_+$ is a Borel measure that assigns to each interval its length. Lebesgue measure on $\mathcal{B}$ may be extended to a complete measure on a larger σ-algebra of Lebesgue measurable sets by the inclusion of all subsets of sets with Lebesgue measure zero. Here we consider it as a Borel measure.

EXAMPLE 3.76. For $c \in \mathbb{R}$, the unit point measure $\delta_c : \mathcal{B} \to [0, \infty)$ supported on c is defined by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

This measure is a finite Borel measure. More generally, if $\{c_n : n \in \mathbb{N}\}$ is a countable set of points in $\mathbb{R}$ and $\{p_n \geq 0 : n \in \mathbb{N}\}$, we define a point measure

$$\mu = \sum_{n \in \mathbb{N}} p_n \delta_{c_n}, \qquad \mu(E) = \sum_{c_n \in E} p_n.$$

This measure is σ-finite, and finite if $\sum p_n < \infty$.

EXAMPLE 3.77. Counting measure $\nu : \mathcal{B} \to \overline{\mathbb{R}}_+$ is defined by $\nu(E) = \#E$ where $\#E$ denotes the number of points in E. Thus, $\nu(\emptyset) = 0$ and $\nu(E) = \infty$ if E contains infinitely many points. This measure is not σ-finite.

In order to describe the decomposition of measures, we introduce the idea of singular measures that 'live' on different sets.

DEFINITION 3.78. Two measures $\mu, \nu : \mathcal{B} \to \overline{\mathbb{R}}_+$ are mutually singular, written $\mu \perp \nu$, if there is a set $E \in \mathcal{B}$ such that $\mu(E) = 0$ and $\nu(E^c) = 0$.

We also say that μ is singular with respect to ν, or ν is singular with respect to μ. In particular, a measure is singular with respect to Lebesgue measure if it assigns full measure to a set of Lebesgue measure zero.

EXAMPLE 3.79. The point measures in Example 3.76 are singular with respect to Lebesgue measure.

Next we consider signed measures which can take negative as well as positive values.

DEFINITION 3.80. A signed Borel measure is a map $\mu : \mathcal{B} \to \overline{\mathbb{R}}$ of the form

$$\mu = \mu_+ - \mu_-$$

where $\mu_+, \mu_- : \mathcal{B} \to \overline{\mathbb{R}}_+$ are Borel measures, at least one of which is finite.

The condition that at least one of μ_+, μ_- is finite is needed to avoid meaningless expressions such as $\mu(\mathbb{R}) = \infty - \infty$. Thus, μ takes at most one of the values ∞, $-\infty$.

According to the Jordan decomposition theorem, we may choose μ_+, μ_- in Definition 3.80 so that $\mu_+ \perp \mu_-$, in which case the decomposition is unique. The total variation of μ is then measure $|\mu| : \mathcal{B} \to \overline{\mathbb{R}}_+$ defined by

$$|\mu| = \mu_+ + \mu_-.$$

DEFINITION 3.81. Let $\mu : \mathcal{B} \to \overline{\mathbb{R}}_+$ be a measure. A signed measure $\nu : \mathcal{B} \to \overline{\mathbb{R}}$ is absolutely continuous with respect to μ, written $\nu \ll \mu$, if $\mu(E) = 0$ implies that $\nu(E) = 0$ for any $E \in \mathcal{B}$.

The condition $\nu \ll \mu$ is equivalent to $|\nu| \ll \mu$. In that case ν 'lives' on the same sets as μ; thus absolute continuity is at the opposite extreme to singularity. In particular, a signed measure ν is absolutely continuous with respect to Lebesgue measure if it assigns zero measure to any set with zero Lebesgue measure, If $g \in L^1(\mathbb{R})$, then

$$\nu(E) = \int_E g\, dx \tag{3.21}$$

defines a finite signed Borel measure $\nu : \mathcal{B} \to \mathbb{R}$. This measure is absolutely continuous with respect to Lebesgue measure, since $\int_E g\, dx = 0$ for any set E with Lebesgue measure zero.

If $g \geq 0$, then ν is a measure. If the set $\{x : g(x) = 0\}$ has non-zero Lebesgue measure, then Lebesgue measure is not absolutely continuous with respect to ν. Thus $\nu \ll \mu$ does not imply that $\mu \ll \nu$.

The Radon-Nikodym theorem (which holds in greater generality) implies that every absolutely continuous measure is given by the above example.

Theorem 3.82. *If ν is a Borel measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure λ then there exists a function $g \in L^1(\mathbb{R})$ such that ν is given by (3.21).*

The function g in this theorem is called the Radon-Nikodym derivative of ν with respect to λ, and is denoted by

$$g = \frac{d\nu}{d\lambda}.$$

The following result gives an alternative characterization of absolute continuity of measures, which has a direct connection with the absolute continuity of functions.

Theorem 3.83. *A signed measure $\nu : \mathcal{B} \to \overline{\mathbb{R}}$ is absolutely continuous with respect to a measure $\mu : \mathcal{B} \to \overline{\mathbb{R}}_+$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies that $|\nu(E)| \leq \epsilon$ for all $E \in \mathcal{B}$.*

3.B.2. Radon measures. The most important Borel measures for distribution theory are the Radon measures. The essential property of a Radon measure μ is that integration against μ defines a positive linear functional on the space of continuous functions ϕ with compact support,

$$\phi \mapsto \int \phi\, d\mu.$$

(See Theorem 3.96 below.) This link is the fundamental connection between measures and distributions. The condition in the following definition characterizes all such measures on $\mathbb{R}$ (and $\mathbb{R}^n$).

Definition 3.84. A Radon measure on $\mathbb{R}$ is a Borel measure that is finite on compact sets.

We note in passing that a Radon measure μ has the following regularity property: For any $E \in \mathcal{B}$,

$$\mu(E) = \inf \{\mu(G) : G \supset E \text{ open}\}, \quad \mu(E) = \sup \{\mu(K) : K \subset E \text{ compact}\}.$$

Thus, any Borel set may be approximated in a measure-theoretic sense by open sets from the outside and compact sets from the inside.

Example 3.85. Lebesgue measure λ in Example 3.75 and the point measure δ_c in Example 3.76 are Radon measures on $\mathbb{R}$.

Example 3.86. The counting measure ν in Example 3.77 is not a Radon measure since, for example, $\nu[0,1] = \infty$. This measure is not outer regular: If $\{c\}$ is a singleton set, then $\nu(\{c\}) = 1$ but

$$\inf\{\nu(G) : c \in G,\ G \text{ open}\} = \infty.$$

The following is the Lebesgue decomposition of a Radon measure.

Theorem 3.87. *Let μ, ν be Radon measures on $\mathbb{R}$. There are unique measures ν_{ac}, ν_s such that*

$$\nu = \nu_{ac} + \nu_s, \qquad \textit{where } \nu_{ac} \ll \mu \textit{ and } \nu_s \perp \mu.$$

3.B.3. Lebesgue-Stieltjes measures. Given a Radon measure μ on $\mathbb{R}$, we may define a monotone increasing, right-continuous distribution function $f : \mathbb{R} \to \mathbb{R}$, which is unique up to an arbitrary additive constant, such that

$$\mu(a,b] = f(b) - f(a).$$

The function f is right-continuous since

$$\lim_{x\to b^+} f(b) - f(a) = \lim_{x\to b^+} \mu(a,x] = \mu(a,b] = f(b) - f(a).$$

Conversely, every such function f defines a Radon measure μ_f, called the Lebesgue-Stieltjes measure associated with f. Thus, Radon measures on $\mathbb{R}$ may be characterized explicitly as Lebesgue-Stieltjes measures.

Theorem 3.88. *If $f : \mathbb{R} \to \mathbb{R}$ is a monotone increasing, right-continuous function, there is a unique Radon measure μ_f such that*

$$\mu_f(a,b] = f(b) - f(a)$$

for any half-open interval $(a,b] \subset \mathbb{R}$.

The standard proof is due to Carathéodory. One uses f to define a countably sub-additive outer measure μ_f^* on all subsets of $\mathbb{R}$, then restricts μ_f^* to a measure on the σ-algebra of μ_f^*-measurable sets, which includes all of the Borel sets [**11**].

The Lebesgue-Stieltjes measure of a compact interval $[a,b]$ is given by

$$\mu_f[a,b] = \lim_{x\to a^-} \mu_f(x,b] = f(b) - \lim_{x\to a^-} f(a).$$

Thus, the measure of the set consisting of a single point is equal to the jump in f at the point,

$$\mu_f\{a\} = f(a) - \lim_{x\to a^-} f(a),$$

and $\mu_f\{a\} = 0$ if and only if f is continuous at a.

Example 3.89. If $f(x) = x$, then μ_f is Lebesgue measure (restricted to the Borel sets) in $\mathbb{R}$.

Example 3.90. If $c \in \mathbb{R}$ and

$$f(x) = \begin{cases} 1 & \text{if } x \geq c, \\ 0 & \text{if } x < c, \end{cases}$$

then μ_f is the point measure δ_c in Example 3.76.

Example 3.91. If f is the Cantor function defined in Example 3.5, then μ_f assigns measure one to the Cantor set C and measure zero to $\mathbb{R} \setminus C$. Thus, μ_f is singular with respect to Lebesgue measure. Nevertheless, since f is continuous, the measure of any set consisting of a single point, and therefore any countable set, is zero.

If $f : \mathbb{R} \to \mathbb{R}$ is the difference $f = f_+ - f_-$ of two right-continuous monotone increasing functions $f_+, f_- : \mathbb{R} \to \mathbb{R}$, at least one of which is bounded, we may define a signed Radon measure $\mu_f : \mathcal{B} \to \overline{\mathbb{R}}$ by

$$\mu_f = \mu_{f_+} - \mu_{f_-}.$$

If we add the condition that $\mu_{f_+} \perp \mu_{f_-}$, then this decomposition is unique, and corresponds to the decomposition of f in (3.20).

3.C. Integration

A function $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable if $\phi^{-1}(E) \in \mathcal{B}$ for every $E \in \overline{\mathcal{B}}$. In particular, every continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

Given a Borel measure μ, and a non-negative, Borel measurable function ϕ, we define the integral of ϕ with respect to μ as follows. If

$$\psi = \sum_{i \in \mathbb{N}} c_i \chi_{E_i}$$

is a simple function, where $c_i \in \overline{\mathbb{R}}_+$ and χ_{E_i} is the characteristic function of a set $E_i \in \mathcal{B}$, then

$$\int \psi \, d\mu = \sum_{i \in \mathbb{N}} c_i \mu(E_i).$$

Here, we define $0 \cdot \infty = 0$ for the integral of a zero value on a set of infinite measure, or an infinite value on a set of measure zero. If $\phi : \mathbb{R} \to \overline{\mathbb{R}}_+$ is a non-negative Borel-measurable function, we define

$$\int \phi \, d\mu = \sup \left\{ \int \psi \, d\mu : 0 \leq \psi \leq \phi \right\}$$

where the supremum is taken over all non-negative simple functions ψ that are bounded from above by ϕ.

If $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ is a general Borel function, we split ϕ into its positive and negative parts,

$$\phi = \phi_+ - \phi_-, \qquad \phi_+ = \max(\phi, 0), \quad \phi_- = \max(-\phi, 0),$$

and define

$$\int \phi \, d\mu = \int \phi_+ \, d\mu - \int \phi_- \, d\mu$$

provided that at least one of these integrals is finite.

The continual annoyance of excluding $\infty - \infty$ as meaningless is often viewed as a defect of the Lebesgue integral, which cannot cope directly with the cancelation between infinite positive and negative components. For example, the improper integral

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$$

does not hold as a Lebesgue integral since $|\sin(x)/x|$ is not integrable. Nevertheless, other definitions of the integral — such as the Henstock-Kurzweil integral — have not proved to be as useful.

EXAMPLE 3.92. The integral of ϕ with respect to Lebesgue measure λ in Example 3.75 is the usual Lebesgue integral

$$\int \phi \, d\lambda = \int \phi \, dx.$$

EXAMPLE 3.93. The integral of ϕ with respect to the point measure δ_c in Example 3.76 is

$$\int \phi \, d\delta_c = \phi(c).$$

Note that $\phi = \psi$ pointwise *a.e.* with respect to δ_c if and only if $\phi(c) = \psi(c)$.

EXAMPLE 3.94. If f is absolutely continuous, the associated Lebesgue-Stieltjes measure μ_f is absolutely continuous with respect to Lebesgue measure, and

$$\int \phi \, d\mu_f = \int \phi f' \, dx.$$

Next, we consider linear functionals on the space $C_c(\mathbb{R})$ of linear functions with compact support.

DEFINITION 3.95. A linear functional $I : C_c(\mathbb{R}) \to \mathbb{R}$ is positive if $I(\phi) \geq 0$ whenever $\phi \geq 0$, and locally bounded if for every compact set K in $\mathbb{R}$ there is a constant C_K such that

$$|I(\phi)| \leq C_K \|\phi\|_\infty \qquad \text{for all } \phi \in C_c(\mathbb{R}) \text{ with } \operatorname{supp} \phi \subset K.$$

A positive functional is locally bounded, and a locally bounded functional I defines a distribution $I \in \mathcal{D}'(\mathbb{R})$ by restriction to $C_c^\infty(\mathbb{R})$. We also write $I(\phi) = \langle I, \phi \rangle$. If μ is a Radon measure, then

$$\langle I_\mu, \phi \rangle = \int \phi \, d\mu$$

defines a positive linear functional $I_\mu : C_c(\mathbb{R}) \to \mathbb{R}$, and if μ_+, μ_- are Radon measures, then $I_{\mu_+} - I_{\mu_-}$ is a locally bounded functional.

Conversely, according to the following Riesz representation theorem, all locally bounded linear functionals on $C_c(\mathbb{R})$ are of this form

THEOREM 3.96. *If $I : C_c(\mathbb{R}) \to \mathbb{R}_+$ is a positive linear functional on the space of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ with compact support, then there is a unique Radon measure μ such that*

$$I(\phi) = \int \phi \, d\mu.$$

If $I : C_c(\mathbb{R}) \to \overline{\mathbb{R}}_+$ is locally bounded linear functional, then there are unique Radon measures μ_+, μ_- such that

$$I(\phi) = \int \phi \, d\mu_+ - \int \phi \, d\mu_-.$$

Note that the functional $\mu = \mu_+ - \mu_-$ is not well-defined as a signed Radon measure if both μ_+ and μ_- are infinite.

Every distribution $T \in \mathcal{D}'(\mathbb{R})$ such that

$$\langle T, \phi \rangle \leq C_K \|\phi\|_\infty \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}) \text{ with } \operatorname{supp} \phi \subset K$$

may be extended by continuity to a locally bounded linear functional on $C_c(\mathbb{R})$, and therefore is given by $T = I_{\mu_+} - I_{\mu_-}$ for Radon measures μ_+, μ_- . We typically identify a Radon measure μ with the corresponding distribution I_μ. If μ is absolutely continuous with respect to Lebesgue measure, then $\mu = \mu_f$ for some $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$, meaning that

$$\mu_f(E) = \int_E f' \, dx,$$

and I_μ is the same as the regular distribution $T_{f'}$. Thus, with these identifications, and denoting the Radon measures by $\mathcal{M}$, we have the following local inclusions:

$$\mathrm{AC} \subset \mathrm{BV} \subset L^1 \subset \mathcal{M} \subset \mathcal{D}'.$$

The distributional derivative of an AC function is an integrable function, and the following integration by parts formula shows that the distributional derivative of a BV function is a Radon measure.

THEOREM 3.97. *Suppose that $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ and $g \in \mathrm{AC}_c(\mathbb{R})$ is absolutely continuous with compact support. Then*

$$\int g \, d\mu_f = - \int f g' \, dx.$$

Thus, the distributional derivative of $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ is the functional I_{μ_f} associated with the corresponding Radon measure μ_f. If

$$f = f_{ac} + f_p + f_{sc}$$

is the decomposition of f into a locally absolutely continuous part, a jump function, and a singular continuous function, then

$$\mu_f = \mu_{ac} + \mu_p + \mu_{sc},$$

where μ_{ac} is absolutely continuous with respect to Lebesgue measure with density f'_{ac}, μ_p is a point measure of the form

$$\mu_p = \sum_{n \in \mathbb{N}} p_n \delta_{c_n}$$

where the c_n are the points of discontinuity of f and the p_n are the jumps, and μ_{sc} is a measure with continuous distribution function that is singular with respect to Lebesgue measure. The function is weakly differentiable if and only if it is locally absolutely continuous.

Thus, to return to our original one-dimensional examples, the function x_+ in Example 3.3 is absolutely continuous and its weak derivative is the step function. The weak derivative is bounded since the function is Lipschitz. The step function in Example 3.4 is not weakly differentiable; its distributional derivative is the δ-measure. The Cantor function f in Example 3.5 is not weakly differentiable; its distributional derivative is the singular continuous Lebesgue-Stieltjes measure μ_f associated with f.

We summarize the above discussion in a table.

Function	*Weak Derivative*
Smooth (C^1)	Continuous (C^0)
Lipschitz	Bounded (L^∞)
Absolutely Continuous	Integrable (L^1)
Bounded Variation	Distributional derivative is Radon measure

The correspondences shown in this table continue to hold for functions of several variables, although the study of fine structure of weakly differentiable functions and functions of bounded variation is more involved than in the one-dimensional case.

CHAPTER 4

Elliptic PDEs

One of the main advantages of extending the class of solutions of a PDE from classical solutions with continuous derivatives to weak solutions with weak derivatives is that it is easier to prove the existence of weak solutions. Having established the existence of weak solutions, one may then study their properties, such as uniqueness and regularity, and perhaps prove under appropriate assumptions that the weak solutions are, in fact, classical solutions.

There is often considerable freedom in how one defines a weak solution of a PDE; for example, the function space to which a solution is required to belong is not given *a priori* by the PDE itself. Typically, we look for a weak formulation that reduces to the classical formulation under appropriate smoothness assumptions and which is amenable to a mathematical analysis; the notion of solution and the spaces to which solutions belong are dictated by the available estimates and analysis.

4.1. Weak formulation of the Dirichlet problem

Let us consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions on a bounded domain Ω in $\mathbb{R}^n$,

$$-\Delta u = f \qquad \text{in } \Omega, \tag{4.1}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4.2}$$

First, suppose that the boundary of Ω is smooth and $u, f : \overline{\Omega} \to \mathbb{R}$ are smooth functions. Multiplying (4.1) by a test function ϕ, integrating the result over Ω, and using the divergence theorem, we get

$$\int_\Omega Du \cdot D\phi \, dx = \int_\Omega f\phi \, dx \qquad \text{for all } \phi \in C_c^\infty(\Omega). \tag{4.3}$$

The boundary terms vanish because $\phi = 0$ on the boundary. Conversely, if f and Ω are smooth, then any smooth function u that satisfies (4.3) is a solution of (4.1).

Next, we formulate weaker assumptions under which (4.3) makes sense. We use the flexibility of choice to define weak solutions with L^2-derivatives that belong to a Hilbert space; this is helpful because Hilbert spaces are easier to work with than Banach spaces.[1] Furthermore, it leads to a variational form of the equation that is symmetric in the solution u and the test function ϕ. Our goal of obtaining a symmetric weak formulation also explains why we only integrate by parts once in (4.3). We briefly discuss some other ways to define weak solutions at the end of this section.

[1] We would need to use Banach spaces to study the solutions of Laplace's equation whose derivatives lie in L^p for $p \neq 2$, and we may be forced to use Banach spaces for some PDEs, especially if they are nonlinear.

By the Cauchy-Schwartz inequality, the integral on the left-hand side of (4.3) is finite if Du belongs to $L^2(\Omega)$, so we suppose that $u \in H^1(\Omega)$. We impose the boundary condition (4.2) in a weak sense by requiring that $u \in H_0^1(\Omega)$. The left hand side of (4.3) then extends by continuity to $\phi \in H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}$.

The right hand side of (4.3) is well-defined for all $\phi \in H_0^1(\Omega)$ if $f \in L^2(\Omega)$, but this is not the most general f for which it makes sense; we can define the right-hand for any f in the dual space of $H_0^1(\Omega)$.

Definition 4.1. The space of bounded linear maps $f : H_0^1(\Omega) \to \mathbb{R}$ is denoted by $H^{-1}(\Omega) = H_0^1(\Omega)^*$, and the action of $f \in H^{-1}(\Omega)$ on $\phi \in H_0^1(\Omega)$ by $\langle f, \phi\rangle$. The norm of $f \in H^{-1}(\Omega)$ is given by

$$\|f\|_{H^{-1}} = \sup\left\{\frac{|\langle f, \phi\rangle|}{\|\phi\|_{H_0^1}} : \phi \in H_0^1, \phi \neq 0\right\}.$$

A function $f \in L^2(\Omega)$ defines a linear functional $F_f \in H^{-1}(\Omega)$ by

$$\langle F_f, v\rangle = \int_\Omega fv\, dx = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega).$$

Here, $(\cdot,\cdot)_{L^2}$ denotes the standard inner product on $L^2(\Omega)$. The functional F_f is bounded on $H_0^1(\Omega)$ with $\|F_f\|_{H^{-1}} \leq \|f\|_{L^2}$ since, by the Cauchy-Schwartz inequality,

$$|\langle F_f, v\rangle| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H_0^1}.$$

We identify F_f with f, and write both simply as f.

Such linear functionals are, however, not the only elements of $H^{-1}(\Omega)$. As we will show below, $H^{-1}(\Omega)$ may be identified with the space of distributions on Ω that are sums of first-order distributional derivatives of functions in $L^2(\Omega)$.

Thus, after identifying functions with regular distributions, we have the following triple of Hilbert spaces

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega), \qquad H^{-1}(\Omega) = H_0^1(\Omega)^*.$$

Moreover, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, then

$$\langle f, u\rangle = (f, u)_{L^2},$$

so the duality pairing coincides with the L^2-inner product when both are defined.

This discussion motivates the following definition.

Definition 4.2. Let Ω be an open set in $\mathbb{R}^n$ and $f \in H^{-1}(\Omega)$. A function $u : \Omega \to \mathbb{R}$ is a weak solution of (4.1)–(4.2) if: (a) $u \in H_0^1(\Omega)$; (b)

$$\int_\Omega Du \cdot D\phi\, dx = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega). \tag{4.4}$$

Here, strictly speaking, 'function' means an equivalence class of functions with respect to pointwise a.e. equality.

We have assumed homogeneous boundary conditions to simplify the discussion. If Ω is smooth and $g : \partial\Omega \to \mathbb{R}$ is a function on the boundary that is in the range of the trace map $T : H^1(\Omega) \to L^2(\partial\Omega)$, say $g = Tw$, then we obtain a weak formulation of the nonhomogeneous Dirichet problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

by replacing (a) in Definition 4.2 with the condition that $u - w \in H_0^1(\Omega)$. The definition is otherwise the same. The range of the trace map on $H^1(\Omega)$ for a smooth domain Ω is the fractional-order Sobolev space $H^{1/2}(\partial\Omega)$; thus if the boundary data g is so rough that $g \notin H^{1/2}(\partial\Omega)$, then there is no solution $u \in H^1(\Omega)$ of the nonhomogeneous BVP.

Finally, we comment on some other ways to define weak solutions of Poisson's equation. If we integrate by parts again in (4.3), we find that every smooth solution u of (4.1) satisfies

$$-\int_\Omega u\Delta\phi\,dx = \int_\Omega f\phi\,dx \qquad \text{for all } \phi \in C_c^\infty(\Omega). \tag{4.5}$$

This condition makes sense without any differentiability assumptions on u, and we can define a locally integrable function $u \in L^1_{\mathrm{loc}}(\Omega)$ to be a weak solution of $-\Delta u = f$ for $f \in L^1_{\mathrm{loc}}(\Omega)$ if it satisfies (4.5). One problem with using this definition is that general functions $u \in L^p(\Omega)$ do not have enough regularity to make sense of their boundary values on $\partial\Omega$.[2]

More generally, we can define distributional solutions $T \in \mathcal{D}'(\Omega)$ of Poisson's equation $-\Delta T = f$ with $f \in \mathcal{D}'(\Omega)$ by

$$-\langle T, \Delta\phi\rangle = \langle f, \phi\rangle \qquad \text{for all } \phi \in C_c^\infty(\Omega). \tag{4.6}$$

While these definitions appear more general, because of elliptic regularity they turn out not to extend the class of variational solutions we consider here if $f \in H^{-1}(\Omega)$, and we will not use them below.

4.2. Variational formulation

Definition 4.2 of a weak solution in is closely connected with the variational formulation of the Dirichlet problem for Poisson's equation. To explain this connection, we first summarize some definitions of the differentiability of functionals (scalar-valued functions) acting on a Banach space.

Definition 4.3. A functional $J : X \to \mathbb{R}$ on a Banach space X is differentiable at $x \in X$ if there is a bounded linear functional $A : X \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|J(x+h) - J(x) - Ah|}{\|h\|_X} = 0.$$

If A exists, then it is unique, and it is called the derivative, or differential, of J at x, denoted $DJ(x) = A$.

This definition expresses the basic idea of a differentiable function as one which can be approximated locally by a linear map. If J is differentiable at every point of X, then $DJ : X \to X^*$ maps $x \in X$ to the linear functional $DJ(x) \in X^*$ that approximates J near x.

[2] For example, if Ω is bounded and $\partial\Omega$ is smooth, then pointwise evaluation $\phi \mapsto \phi|_{\partial\Omega}$ on $C(\overline{\Omega})$ extends to a bounded, linear trace map $T : H^s(\Omega) \to H^{s-1/2}(\Omega)$ if $s > 1/2$ but not if $s \le 1/2$. In particular, there is no sensible definition of the boundary values of a general function $u \in L^2(\Omega)$. We remark, however, that if $u \in L^2(\Omega)$ is a weak solution of $-\Delta u = f$ where $f \in L^2(\Omega)$, then elliptic regularity implies that $u \in H^2(\Omega)$, so it does have a well-defined boundary value $u|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$; on the other hand, if $f \in H^{-2}(\Omega)$, then $u \in L^2(\Omega)$ and we cannot make sense of $u|_{\partial\Omega}$.

A weaker notion of differentiability (even for functions $J:\mathbb{R}^2\to\mathbb{R}$ — see Example 4.4) is the existence of directional derivatives

$$\delta J(x;h)=\lim_{\epsilon\to 0}\left[\frac{J(x+\epsilon h)-J(x)}{\epsilon}\right]=\frac{d}{d\epsilon}J(x+\epsilon h)\bigg|_{\epsilon=0}.$$

If the directional derivative at x exists for every $h\in X$ and is a bounded linear functional on h, then $\delta J(x;h)=\delta J(x)h$ where $\delta J(x)\in X^*$. We call $\delta J(x)$ the Gâteaux derivative of J at x. The derivative DJ is then called the Fréchet derivative to distinguish it from the directional or Gâteaux derivative. If J is differentiable at x, then it is Gâteaux-differentiable at x and $DJ(x)=\delta J(x)$, but the converse is not true.

Example 4.4. Define $f:\mathbb{R}^2\to\mathbb{R}$ by $f(0,0)=0$ and

$$f(x,y)=\left(\frac{xy^2}{x^2+y^4}\right)^2\qquad\text{if }(x,y)\neq(0,0).$$

Then f is Gâteaux-differentiable at 0, with $\delta f(0)=0$, but f is not Fréchet-differentiable at 0.

If $J:X\to\mathbb{R}$ attains a local minimum at $x\in X$ and J is differentiable at x, then for every $h\in X$ the function $J_{x;h}:\mathbb{R}\to\mathbb{R}$ defined by $J_{x;h}(t)=J(x+th)$ is differentiable at $t=0$ and attains a minimum at $t=0$. It follows that

$$\frac{dJ_{x;h}}{dt}(0)=\delta J(x;h)=0\qquad\text{for every }h\in X.$$

Hence $DJ(x)=0$. Thus, just as in multivariable calculus, an extreme point of a differentiable functional is a critical point where the derivative is zero.

Given $f\in H^{-1}(\Omega)$, define a quadratic functional $J:H_0^1(\Omega)\to\mathbb{R}$ by

$$J(u)=\frac{1}{2}\int_\Omega|Du|^2\,dx-\langle f,u\rangle. \tag{4.7}$$

Clearly, J is well-defined.

Proposition 4.5. *The functional $J:H_0^1(\Omega)\to\mathbb{R}$ in (4.7) is differentiable. Its derivative $DJ(u):H_0^1(\Omega)\to\mathbb{R}$ at $u\in H_0^1(\Omega)$ is given by*

$$DJ(u)h=\int_\Omega Du\cdot Dh\,dx-\langle f,h\rangle\qquad\textit{for } h\in H_0^1(\Omega).$$

Proof. Given $u\in H_0^1(\Omega)$, define the linear map $A:H_0^1(\Omega)\to\mathbb{R}$ by

$$Ah=\int_\Omega Du\cdot Dh\,dx-\langle f,h\rangle.$$

Then A is bounded, with $\|A\|\le\|Du\|_{L^2}+\|f\|_{H^{-1}}$, since

$$|Ah|\le\|Du\|_{L^2}\|Dh\|_{L^2}+\|f\|_{H^{-1}}\|h\|_{H_0^1}\le\left(\|Du\|_{L^2}+\|f\|_{H^{-1}}\right)\|h\|_{H_0^1}.$$

For $h\in H_0^1(\Omega)$, we have

$$J(u+h)-J(u)-Ah=\frac{1}{2}\int_\Omega|Dh|^2\,dx.$$

It follows that

$$|J(u+h)-J(u)-Ah|\le\frac{1}{2}\|h\|_{H_0^1}^2,$$

and therefore

$$\lim_{h\to 0} \frac{|J(u+h) - J(u) - Ah|}{\|h\|_{H_0^1}} = 0,$$

which proves that J is differentiable on $H_0^1(\Omega)$ with $DJ(u) = A$. □

Note that $DJ(u) = 0$ if and only if u is a weak solution of Poisson's equation in the sense of Definition 4.2. Thus, we have the following result.

COROLLARY 4.6. *If $J : H_0^1(\Omega) \to \mathbb{R}$ defined in (4.7) attains a minimum at $u \in H_0^1(\Omega)$, then u is a weak solution of $-\Delta u = f$ in the sense of Definition 4.2.*

In the direct method of the calculus of variations, we prove the existence of a minimizer of J by showing that a minimizing sequence $\{u_n\}$ converges in a suitable sense to a minimizer u. This minimizer is then a weak solution of (4.1)–(4.2). We will not follow this method here, and instead establish the existence of a weak solution by use of the Riesz representation theorem. The Riesz representation theorem is, however, typically proved by a similar argument to the one used in the direct method of the calculus of variations, so in essence the proofs are equivalent.

4.3. The space $H^{-1}(\Omega)$

The negative order Sobolev space $H^{-1}(\Omega)$ can be described as a space of distributions on Ω.

THEOREM 4.7. *The space $H^{-1}(\Omega)$ consists of all distributions $f \in \mathcal{D}'(\Omega)$ of the form*

$$f = f_0 + \sum_{i=1}^{n} \partial_i f_i \qquad \textit{where } f_0, f_i \in L^2(\Omega). \tag{4.8}$$

These distributions extend uniquely by continuity from $\mathcal{D}(\Omega)$ to bounded linear functionals on $H_0^1(\Omega)$. Moreover,

$$\|f\|_{H^{-1}(\Omega)} = \inf \left\{ \left(\sum_{i=0}^{n} \int_\Omega f_i^2 \, dx \right)^{1/2} : \textit{such that } f_0,\ f_i \textit{ satisfy (4.8)} \right\}. \tag{4.9}$$

PROOF. First suppose that $f \in H^{-1}(\Omega)$. By the Riesz representation theorem there is a function $g \in H_0^1(\Omega)$ such that

$$\langle f, \phi \rangle = (g, \phi)_{H_0^1} \qquad \text{for all } \phi \in H_0^1(\Omega). \tag{4.10}$$

Here, $(\cdot,\cdot)_{H_0^1}$ denotes the standard inner product on $H_0^1(\Omega)$,

$$(u, v)_{H_0^1} = \int_\Omega (uv + Du \cdot Dv) \, dx.$$

Identifying a function $g \in L^2(\Omega)$ with its corresponding regular distribution, restricting f to $\phi \in \mathcal{D}(\Omega) \subset H_0^1(\Omega)$, and using the definition of the distributional derivative, we have

$$\begin{aligned}\langle f, \phi\rangle &= \int_\Omega g\phi \, dx + \sum_{i=1}^n \int_\Omega \partial_i g \, \partial_i \phi \, dx \\ &= \langle g, \phi\rangle + \sum_{i=1}^n \langle \partial_i g, \partial_i \phi\rangle \\ &= \left\langle g - \sum_{i=1}^n \partial_i g_i, \phi \right\rangle \qquad \text{for all } \phi \in \mathcal{D}(\Omega),\end{aligned}$$

where $g_i = \partial_i g \in L^2(\Omega)$. Thus the restriction of every $f \in H^{-1}(\Omega)$ from $H_0^1(\Omega)$ to $\mathcal{D}(\Omega)$ is a distribution

$$f = g - \sum_{i=1}^n \partial_i g_i$$

of the form (4.8). Also note that taking $\phi = g$ in (4.10), we get $\langle f, g\rangle = \|g\|_{H_0^1}^2$, which implies that

$$\|f\|_{H^{-1}} \geq \|g\|_{H_0^1} = \left(\int_\Omega g^2 \, dx + \sum_{i=1}^n \int_\Omega g_i^2 \, dx\right)^{1/2},$$

which proves inequality in one direction of (4.9).

Conversely, suppose that $f \in \mathcal{D}'(\Omega)$ is a distribution of the form (4.8). Then, using the definition of the distributional derivative, we have for any $\phi \in \mathcal{D}(\Omega)$ that

$$\langle f, \phi\rangle = \langle f_0, \phi\rangle + \sum_{i=1}^n \langle \partial_i f_i, \phi\rangle = \langle f_0, \phi\rangle - \sum_{i=1}^n \langle f_i, \partial_i \phi\rangle.$$

Use of the Cauchy-Schwartz inequality gives

$$|\langle f, \phi\rangle| \leq \left(\langle f_0, \phi\rangle^2 + \sum_{i=1}^n \langle f_i, \partial_i \phi\rangle^2\right)^{1/2}.$$

Moreover, since the f_i are regular distributions belonging to $L^2(\Omega)$

$$|\langle f_i, \partial_i \phi\rangle| = \left|\int_\Omega f_i \partial_i \phi \, dx\right| \leq \left(\int_\Omega f_i^2 \, dx\right)^{1/2} \left(\int_\Omega \partial_i \phi^2 \, dx\right)^{1/2},$$

so

$$|\langle f, \phi\rangle| \leq \left[\left(\int_\Omega f_0^2 \, dx\right)\left(\int_\Omega \phi^2 \, dx\right) + \sum_{i=1}^n \left(\int_\Omega f_i^2 \, dx\right)\left(\int_\Omega \partial_i \phi^2 \, dx\right)\right]^{1/2},$$

and

$$\begin{aligned}|\langle f, \phi\rangle| &\leq \left(\int_\Omega f_0^2 \, dx + \sum_{i=1}^n \int_\Omega f_i^2 \, dx\right)^{1/2} \left(\int_\Omega \phi^2 + \int_\Omega \partial_i \phi^2 \, dx\right)^{1/2} \\ &\leq \left(\sum_{i=0}^n \int_\Omega f_i^2 \, dx\right)^{1/2} \|\phi\|_{H_0^1}\end{aligned}$$

Thus the distribution $f : \mathcal{D}(\Omega) \to \mathbb{R}$ is bounded with respect to the $H_0^1(\Omega)$-norm on the dense subset $\mathcal{D}(\Omega)$. It therefore extends in a unique way to a bounded linear functional on $H_0^1(\Omega)$, which we still denote by f. Moreover,

$$\|f\|_{H^{-1}} \leq \left(\sum_{i=0}^{n} \int_\Omega f_i^2 \, dx \right)^{1/2},$$

which proves inequality in the other direction of (4.9). □

The dual space of $H^1(\Omega)$ cannot be identified with a space of distributions on Ω because $\mathcal{D}(\Omega)$ is not a dense subspace. Any linear functional $f \in H^1(\Omega)^*$ defines a distribution by restriction to $\mathcal{D}(\Omega)$, but the same distribution arises from different linear functionals. Conversely, any distribution $T \in \mathcal{D}'(\Omega)$ that is bounded with respect to the H^1-norm extends uniquely to a bounded linear functional on H_0^1, but the extension of the functional to the orthogonal complement $(H_0^1)^\perp$ in H^1 is arbitrary (subject to maintaining its boundedness). Roughly speaking, distributions are defined on functions whose boundary values or trace is zero, but general linear functionals on H^1 depend on the trace of the function on the boundary $\partial\Omega$.

Example 4.8. The one-dimensional Sobolev space $H^1(0,1)$ is embedded in the space $C([0,1])$ of continuous functions, since $p > n$ for $p = 2$ and $n = 1$. In fact, according to the Sobolev embedding theorem $H^1(0,1) \hookrightarrow C^{0,1/2}([0,1])$, as can be seen directly from the Cauchy-Schwartz inequality:

$$\begin{aligned} |f(x) - f(y)| &\leq \int_y^x |f'(t)| \, dt \\ &\leq \left(\int_y^x 1 \, dt \right)^{1/2} \left(\int_y^x |f'(t)|^2 \, dt \right)^{1/2} \\ &\leq \left(\int_0^1 |f'(t)|^2 \, dt \right)^{1/2} |x - y|^{1/2}. \end{aligned}$$

As usual, we identify an element of $H^1(0,1)$ with its continuous representative in $C([0,1])$. By the trace theorem,

$$H_0^1(0,1) = \left\{ u \in H^1(0,1) : u(0) = 0, \; u(1) = 0 \right\}.$$

The orthogonal complement is

$$H_0^1(0,1)^\perp = \left\{ u \in H^1(0,1) : \text{such that } (u,v)_{H^1} = 0 \text{ for every } v \in H_0^1(0,1) \right\}.$$

This condition implies that $u \in H_0^1(0,1)^\perp$ if and only if

$$\int_0^1 (uv + u'v') \, dx = 0 \qquad \text{for all } v \in H_0^1(0,1),$$

which means that u is a weak solution of the ODE

$$-u'' + u = 0.$$

It follows that $u(x) = c_1 e^x + c_2 e^{-x}$, so

$$H^1(0,1) = H_0^1(0,1) \oplus E$$

where E is the two dimensional subspace of $H^1(0,1)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$. Thus,

$$H^1(0,1)^* = H^{-1}(0,1) \oplus E^*.$$

If $f \in H^1(0,1)^*$ and $u = u_0 + c_1 e^x + c_2 e^{-x}$ where $u_0 \in H_0^1(0,1)$, then

$$\langle f, u\rangle = \langle f_0, u_0\rangle + a_1 c_1 + a_2 c_2$$

where $f_0 \in H^{-1}(0,1)$ is the restriction of f to $H_0^1(0,1)$ and

$$a_1 = \langle f, e^x\rangle, \qquad a_2 = \langle f, e^{-x}\rangle.$$

The constants a_1, a_2 determine how the functional $f \in H^1(0,1)^*$ acts on the boundary values $u(0)$, $u(1)$ of a function $u \in H^1(0,1)$.

4.4. The Poincaré inequality for $H_0^1(\Omega)$

We cannot, in general, estimate a norm of a function in terms of a norm of its derivative since constant functions have zero derivative. Such estimates are possible if we add an additional condition that eliminates non-zero constant functions. For example, we can require that the function vanishes on the boundary of a domain, or that it has zero mean. We typically also need some sort of boundedness condition on the domain of the function, since even if a function vanishes at some point we cannot expect to estimate the size of a function over arbitrarily large distances by the size of its derivative. The resulting inequalities are called Poincaré inequalities.

The inequality we prove here is a basic example of a Poincaré inequality. We say that an open set Ω in $\mathbb{R}^n$ is bounded in some direction if there is a unit vector $e \in \mathbb{R}^n$ and constants a, b such that $a < x \cdot e < b$ for all $x \in \Omega$.

THEOREM 4.9. *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded is some direction. Then there is a constant C such that*

$$\int_\Omega u^2\,dx \le C \int_\Omega |Du|^2\,dx \qquad \text{for all } u \in H_0^1(\Omega). \tag{4.11}$$

PROOF. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it is sufficient to prove the inequality for $u \in C_c^\infty(\Omega)$. The inequality is invariant under rotations and translations, so we can assume without loss of generality that the domain is bounded in the x_n-direction and lies between $0 < x_n < a$.

Writing $x = (x', x_n)$ where $x' = (x_1, \dots, , x_{n-1})$, we have

$$|u(x', x_n)| = \left|\int_0^{x_n} \partial_n u(x', t)\,dt\right| \le \int_0^a |\partial_n u(x', t)|\,dt.$$

The Cauchy-Schwartz inequality implies that

$$\int_0^a |\partial_n u(x', t)|\,dt = \int_0^a 1 \cdot |\partial_n u(x', t)|\,dt \le a^{1/2}\left(\int_0^a |\partial_n u(x', t)|^2\,dt\right)^{1/2}.$$

Hence,

$$|u(x', x_n)|^2 \le a \int_0^a |\partial_n u(x', t)|^2\,dt.$$

Integrating this inequality with respect to x_n, we get

$$\int_0^a |u(x', x_n)|^2\,dx_n \le a^2 \int_0^a |\partial_n u(x', t)|^2\,dt.$$

A further integration with respect to x' gives

$$\int_\Omega |u(x)|^2\,dx \le a^2 \int_\Omega |\partial_n u(x)|^2\,dx.$$

Since $|\partial_n u| \le |Du|$, the result follows with $C = a^2$. □

This inequality implies that we may use as an equivalent inner-product on H_0^1 an expression that involves only the derivatives of the functions and not the functions themselves.

COROLLARY 4.10. *If Ω is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product*

$$(u, v)_0 = \int_\Omega Du \cdot Dv \, dx \tag{4.12}$$

is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.

PROOF. We denote the norm associated with the inner-product (4.12) by

$$\|u\|_0 = \left(\int_\Omega |Du|^2 \, dx \right)^{1/2},$$

and the standard norm and inner product by

$$\begin{aligned} \|u\|_1 &= \left(\int_\Omega \left[u^2 + |Du|^2 \right] \, dx \right)^{1/2}, \\ (u, v)_1 &= \int_\Omega (uv + Du \cdot Dv) \, dx. \end{aligned} \tag{4.13}$$

Then, using the Poincaré inequality (4.11), we have

$$\|u\|_0 \leq \|u\|_1 \leq (C+1)^{1/2} \|u\|_0.$$

Thus, the two norms are equivalent; in particular, $(H_0^1, (\cdot, \cdot)_0)$ is complete since $(H_0^1, (\cdot, \cdot)_1)$ is complete, so it is a Hilbert space with respect to the inner product (4.12). □

4.5. Existence of weak solutions of the Dirichlet problem

With these preparations, the existence of weak solutions is an immediate consequence of the Riesz representation theorem.

THEOREM 4.11. *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded in some direction and $f \in H^{-1}(\Omega)$. Then there is a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in the sense of Definition 4.2.*

PROOF. We equip $H_0^1(\Omega)$ with the inner product (4.12). Then, since Ω is bounded in some direction, the resulting norm is equivalent to the standard norm, and f is a bounded linear functional on $\left(H_0^1(\Omega), (,)_0\right)$. By the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, \phi)_0 = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega),$$

which is equivalent to the condition that u is a weak solution. □

The same approach works for other symmetric linear elliptic PDEs. Let us give some examples.

EXAMPLE 4.12. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u + u = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned}$$

Then $u \in H^1_0(\Omega)$ is a weak solution if

$$\int_\Omega (Du \cdot D\phi + u\phi)\, dx = \langle f, \phi\rangle \qquad \text{for all } \phi \in H^1_0(\Omega).$$

This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi\rangle \qquad \text{for all } \phi \in H^1_0(\Omega).$$

where $(\cdot,\cdot)_1$ is the standard inner product on $H^1_0(\Omega)$ given in (4.13). Thus, the Riesz representation theorem implies the existence of a unique weak solution.

Note that in this example and the next, we do not use the Poincaré inequality, so the result applies to arbitrary open sets, including $\Omega = \mathbb{R}^n$. In that case, $H^1_0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, and we get a unique solution $u \in H^1(\mathbb{R}^n)$ of $-\Delta u + u = f$ for every $f \in H^{-1}(\mathbb{R}^n)$. Moreover, using the standard norms, we have $\|u\|_{H^1} = \|f\|_{H^{-1}}$. Thus the operator $-\Delta + I$ is an isometry of $H^1(\mathbb{R}^n)$ onto $H^{-1}(\mathbb{R}^n)$.

Example 4.13. As a slight generalization of the previous example, suppose that $\mu > 0$. A function $u \in H^1_0(\Omega)$ is a weak solution of

$$\begin{aligned} -\Delta u + \mu u &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.14}$$

if $(u, \phi)_\mu = \langle f, \phi\rangle$ for all $\phi \in H^1_0(\Omega)$ where

$$(u, v)_\mu = \int_\Omega (\mu uv + Du \cdot Dv)\, dx$$

The norm $\|\cdot\|_\mu$ associated with this inner product is equivalent to the standard one, since

$$\frac{1}{C}\|u\|_\mu^2 \le \|u\|_1^2 \le C\|u\|_\mu^2$$

where $C = \max\{\mu, 1/\mu\}$. We therefore again get the existence of a unique weak solution from the Riesz representation theorem.

Example 4.14. Consider the last example for $\mu < 0$. If we have a Poincaré inequality $\|u\|_{L^2} \le C\|Du\|_{L^2}$ for Ω, which is the case if Ω is bounded in some direction, then

$$(u, u)_\mu = \int_\Omega \left(\mu u^2 + |Du|^2\right)\, dx \ge (1 - C|\mu|)\int_\Omega |Du|^2\, dx.$$

Thus $\|u\|_\mu$ defines a norm on $H^1_0(\Omega)$ that is equivalent to the standard norm if $-1/C < \mu < 0$, and we get a unique weak solution in this case also, provided that $|\mu|$ is sufficiently small.

For bounded domains, the Dirichlet Laplacian has an infinite sequence of real eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ such that there exists a nonzero solution $u \in H^1_0(\Omega)$ of $-\Delta u = \lambda_n u$. The best constant in the Poincaré inequality can be shown to be the minimum eigenvalue λ_1, and this method does not work if $\mu \le -\lambda_1$. For $\mu = -\lambda_n$, a weak solution of (4.14) does not exist for every $f \in H^{-1}(\Omega)$, and if one does exist it is not unique since we can add to it an arbitrary eigenfunction. Thus, not only does the method fail, but the conclusion of Theorem 4.11 may be false.

EXAMPLE 4.15. Consider the second order PDE

$$
\begin{aligned}
-\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) = f \qquad &\text{in } \Omega, \\
u = 0 \qquad &\text{on } \partial\Omega
\end{aligned}
\tag{4.15}
$$

where the coefficient functions $a_{ij} : \Omega \to \mathbb{R}$ are symmetric ($a_{ij} = a_{ji}$), bounded, and satisfy the uniform ellipticity condition that for some $\theta > 0$

$$
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n.
$$

Also, assume that Ω is bounded in some direction. Then a weak formulation of (4.15) is that $u \in H_0^1(\Omega)$ and

$$
a(u,\phi) = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega),
$$

where the symmetric bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$
a(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}\partial_i u \partial_j v \, dx.
$$

The boundedness of a_{ij}, the uniform ellipticity condition, and the Poincaré inequality imply that a defines an inner product on H_0^1 which is equivalent to the standard one. An application of the Riesz representation theorem for the bounded linear functionals f on the Hilbert space (H_0^1, a) then implies the existence of a unique weak solution. We discuss a generalization of this example in greater detail in the next section.

4.6. General linear, second order elliptic PDEs

Consider PDEs of the form

$$
Lu = f
$$

where L is a linear differential operator of the form

$$
Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu,
\tag{4.16}
$$

acting on functions $u : \Omega \to \mathbb{R}$ where Ω is an open set in $\mathbb{R}^n$. A physical interpretation of such PDEs is described briefly in Section 4.A.

We assume that the given coefficients functions $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ satisfy

$$
a_{ij}, b_i, c \in L^\infty(\Omega), \qquad a_{ij} = a_{ji}.
\tag{4.17}
$$

The operator L is elliptic if the matrix (a_{ij}) is positive definite. We will assume the stronger condition of uniformly ellipticity given in the next definition.

DEFINITION 4.16. The operator L in (4.16) is uniformly elliptic on Ω if there exists a constant $\theta > 0$ such that

$$
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2
\tag{4.18}
$$

for x almost everywhere in Ω and every $\xi \in \mathbb{R}^n$.

This uniform ellipticity condition allows us to estimate the integral of $|Du|^2$ in terms of the integral of $\sum a_{ij}\partial_i u \partial_j u$.

EXAMPLE 4.17. The Laplacian operator $L = -\Delta$ is uniformly elliptic on any open set, with $\theta = 1$.

EXAMPLE 4.18. The Tricomi operator

$$L = y\partial_x^2 + \partial_y^2$$

is elliptic in $y > 0$ and hyperbolic in $y < 0$. For any $0 < \epsilon < 1$, L is uniformly elliptic in the strip $\{(x, y) : \epsilon < y < 1\}$, with $\theta = \epsilon$, but it is not uniformly elliptic in $\{(x, y) : 0 < y < 1\}$.

For $\mu \in \mathbb{R}$, we consider the Dirichlet problem for $L + \mu I$,

$$\begin{aligned} Lu + \mu u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.19}$$

We motivate the definition of a weak solution of (4.19) in a similar way to the motivation for the Laplacian: multiply the PDE by a test function $\phi \in C_c^\infty(\Omega)$, integrate over Ω, and use integration by parts, assuming that all functions and the domain are smooth. Note that

$$\int_\Omega \partial_i(b_i u)\phi\, dx = -\int_\Omega b_i u \partial_i \phi\, dx.$$

This leads to the condition that $u \in H_0^1(\Omega)$ is a weak solution of (4.19) with L given by (4.16) if

$$\int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j \phi - \sum_{i=1}^n b_i u \partial_i \phi + cu\phi \right\} dx + \mu \int_\Omega u\phi\, dx = \langle f, \phi\rangle$$

for all $\phi \in H_0^1(\Omega)$.

To write this condition more concisely, we define a bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v - \sum_i^n b_i u \partial_i v + cuv \right\} dx. \tag{4.20}$$

This form is well-defined and bounded on $H_0^1(\Omega)$, as we check explicitly below. We denote the L^2-inner product by

$$(u, v)_{L^2} = \int_\Omega uv\, dx.$$

DEFINITION 4.19. Suppose that Ω is an open set in $\mathbb{R}^n$, $f \in H^{-1}(\Omega)$, and L is a differential operator (4.16) whose coefficients satisfy (4.17). Then $u : \Omega \to \mathbb{R}$ is a weak solution of (4.19) if: (a) $u \in H_0^1(\Omega)$; (b)

$$a(u, \phi) + \mu(u, \phi)_{L^2} = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

The form a in (4.20) is not symmetric unless $b_i = 0$. We have

$$a(v, u) = a^*(u, v)$$

where

$$a^*(u,v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v + \sum_i^n b_i(\partial_i u)v + cuv \right\} dx \tag{4.21}$$

is the bilinear form associated with the formal adjoint L^* of L,

$$L^*u = -\sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j u\right) - \sum_{i=1}^n b_i \partial_i u + cu. \tag{4.22}$$

The proof of the existence of a weak solution of (4.19) is similar to the proof for the Dirichlet Laplacian, with one exception. If L is not symmetric, we cannot use a to define an equivalent inner product on $H_0^1(\Omega)$ and appeal to the Riesz representation theorem. Instead we use a result due to Lax and Milgram which applies to non-symmetric bilinear forms.[3]

4.7. The Lax-Milgram theorem and general elliptic PDEs

We begin by stating the Lax-Milgram theorem for a bilinear form on a Hilbert space. Afterwards, we verify its hypotheses for the bilinear form associated with a general second-order uniformly elliptic PDE and use it to prove the existence of weak solutions.

THEOREM 4.20. *Let $\mathcal{H}$ be a Hilbert space with inner-product $(\cdot,\cdot) : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$, and let $a : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a bilinear form on $\mathcal{H}$. Assume that there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|u\|^2 \le a(u,u), \quad |a(u,v)| \le C_2\|u\|\,\|v\| \qquad \text{for all } u,v \in \mathcal{H}.$$

Then for every bounded linear functional $f : \mathcal{H} \to \mathbb{R}$, there exists a unique $u \in \mathcal{H}$ such that

$$\langle f, v\rangle = a(u,v) \qquad \text{for all } v \in \mathcal{H}.$$

For the proof, see [**9**]. The verification of the hypotheses for (4.20) depends on the following energy estimates.

THEOREM 4.21. *Let a be the bilinear form on $H_0^1(\Omega)$ defined in (4.20), where the coefficients satisfy (4.17) and the uniform ellipticity condition (4.18) with constant θ. Then there exist constants $C_1, C_2 > 0$ and $\gamma \in \mathbb{R}$ such that for all $u, v \in H_0^1(\Omega)$*

$$C_1\|u\|_{H_0^1}^2 \le a(u,u) + \gamma\|u\|_{L^2}^2 \tag{4.23}$$

$$|a(u,v)| \le C_2 \|u\|_{H_0^1} \|v\|_{H_0^1}, \tag{4.24}$$

If $b = 0$, we may take $\gamma = \theta - c_0$ where $c_0 = \inf_\Omega c$, and if $b \neq 0$, we may take

$$\gamma = \frac{1}{2\theta}\sum_{i=1}^n \|b_i\|_{L^\infty}^2 + \frac{\theta}{2} - c_0.$$

[3] The story behind this result — the story might be completely true or completely false — is that Lax and Milgram attended a seminar where the speaker proved existence for a symmetric PDE by use of the Riesz representation theorem, and one of them asked the other if symmetry was required; in half an hour, they convinced themselves that is wasn't, giving birth to the Lax-Milgram "lemma."

PROOF. First, we have for any $u, v \in H_0^1(\Omega)$ that

$$
\begin{aligned}
|a(u,v)| &\leq \sum_{i,j=1}^{n} \int_\Omega |a_{ij}\partial_i u \partial_j v| \; dx + \sum_{i=1}^{n} \int_\Omega |b_i u \partial_i v| \; dx + \int_\Omega |cuv| \; dx. \\
&\leq \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty} \|\partial_i u\|_{L^2} \|\partial_j v\|_{L^2} \\
&\qquad + \sum_{i=1}^{n} \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i v\|_{L^2} + \|c\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\
&\leq C \left(\sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty} + \sum_{i=1}^{n} \|b_i\|_{L^\infty} + \|c\|_{L^\infty} \right) \|u\|_{H_0^1} \|v\|_{H_0^1} \,,
\end{aligned}
$$

which shows (4.24).

Second, using the uniform ellipticity condition (4.18), we have

$$
\begin{aligned}
\theta \|Du\|_{L^2}^2 &= \theta \int_\Omega |Du|^2 \; dx \\
&\leq \sum_{i,j=1}^{n} \int_\Omega a_{ij} \partial_i u \partial_j u \, dx \\
&\leq a(u,u) + \sum_{i=1}^{n} \int_\Omega b_i u \partial_i u \, dx - \int_\Omega c u^2 \, dx \\
&\leq a(u,u) + \sum_{i=1}^{n} \int_\Omega |b_i u \partial_i u| \; dx - c_0 \int_\Omega u^2 \, dx \\
&\leq a(u,u) + \sum_{i=1}^{n} \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i u\|_{L^2} - c_0 \|u\|_{L^2} \\
&\leq a(u,u) + \beta \|u\|_{L^2} \|Du\|_{L^2} - c_0 \|u\|_{L^2} \,,
\end{aligned}
$$

where $c(x) \geq c_0$ a.e. in Ω, and

$$
\beta = \left(\sum_{i=1}^{n} \|b_i\|_{L^\infty}^2 \right)^{1/2} .
$$

If $\beta = 0$, we get (4.23) with

$$
\gamma = \theta - c_0, \qquad C_1 = \theta.
$$

If $\beta > 0$, by Cauchy's inequality with ϵ, we have for any $\epsilon > 0$ that

$$
\|u\|_{L^2} \|Du\|_{L^2} \leq \epsilon \|Du\|_{L^2}^2 + \frac{1}{4\epsilon} \|u\|_{L^2}^2 .
$$

Hence, choosing $\epsilon = \theta/2\beta$, we get

$$
\frac{\theta}{2} \|Du\|_{L^2}^2 \leq a(u,u) + \left(\frac{\beta^2}{2\theta} - c_0 \right) \|u\|_{L^2} \,,
$$

and (4.23) follows with

$$
\gamma = \frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0, \qquad C_1 = \frac{\theta}{2}.
$$

□

Equation (4.23) is called Gårding's inequality; this estimate of the H_0^1-norm of u in terms of $a(u,u)$, using the uniform ellipticity of L, is the crucial energy estimate. Equation (4.24) states that the bilinear form a is bounded on H_0^1. The expression for γ in this Theorem is not necessarily sharp. For example, as in the case of the Laplacian, the use of Poincaré's inequality gives smaller values of γ for bounded domains.

THEOREM 4.22. *Suppose that Ω is an open set in $\mathbb{R}^n$, and $f \in H^{-1}(\Omega)$. Let L be a differential operator (4.16) with coefficients that satisfy (4.17), and let $\gamma \in \mathbb{R}$ be a constant for which Theorem 4.21 holds. Then for every $\mu \geq \gamma$ there is a unique weak solution of the Dirichlet problem*

$$Lu + \mu f = 0, \qquad u \in H_0^1(\Omega)$$

in the sense of Definition 4.19.

PROOF. For $\mu \in \mathbb{R}$, define $a_\mu : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a_\mu(u,v) = a(u,v) + \mu(u,v)_{L^2} \tag{4.25}$$

where a is defined in (4.20). Then $u \in H_0^1(\Omega)$ is a weak solution of $Lu + \mu u = f$ if and only if

$$a_\mu(u,\phi) = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

From (4.24),

$$|a_\mu(u,v)| \leq C_2 \|u\|_{H_0^1} \|v\|_{H_0^1} + |\mu| \, \|u\|_{L^2}\|v\|_{L^2} \leq (C_2 + |\mu|) \, \|u\|_{H_0^1} \|v\|_{H_0^1}$$

so a_μ is bounded on $H_0^1(\Omega)$. From (4.23),

$$C_1\|u\|_{H_0^1}^2 \leq a(u,u) + \gamma\|u\|_{L^2}^2 \leq a_\mu(u,u)$$

whenever $\mu \geq \gamma$. Thus, by the Lax-Milgram theorem, for every $f \in H^{-1}(\Omega)$ there is a unique $u \in H_0^1(\Omega)$ such that $\langle f, \phi\rangle = a_\mu(u,\phi)$ for all $v \in H_0^1(\Omega)$, which proves the result. □

Although L^* is not of exactly the same form as L, since it first derivative term is not in divergence form, the same proof of the existence of weak solutions for L applies to L^* with a in (4.20) replaced by a^* in (4.21).

4.8. Compactness of the resolvent

An elliptic operator $L + \mu I$ of the type studied above is a bounded, invertible linear map from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ for sufficiently large $\mu \in \mathbb{R}$, so we may define an inverse operator $K = (L + \mu I)^{-1}$. If Ω is a bounded open set, then the Sobolev embedding theorem implies that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, and therefore K is a compact operator on $L^2(\Omega)$.

The operator $(L - \lambda I)^{-1}$ is called the resolvent of L, so this property is sometimes expressed by saying that L has compact resolvent. As discussed in Example 4.14, $L + \mu I$ may fail to be invertible at smaller values of μ, such that $\lambda = -\mu$ belongs to the spectrum $\sigma(L)$ of L, and the resolvent is not defined as a bounded operator on $L^2(\Omega)$ for $\lambda \in \sigma(L)$.

The compactness of the resolvent of elliptic operators on bounded open sets has several important consequences for the solvability of the elliptic PDE and the spectrum of the elliptic operator. Before describing some of these, we discuss the resolvent in more detail.

From Theorem 4.22, for $\mu \geq \gamma$ we can define

$$K : L^2(\Omega) \to L^2(\Omega), \qquad K = (L + \mu I)^{-1}\big|_{L^2(\Omega)}.$$

We define the inverse K on $L^2(\Omega)$, rather than $H^{-1}(\Omega)$, in which case its range is a subspace of $H_0^1(\Omega)$. If the domain Ω is sufficiently smooth for elliptic regularity theory to apply, then $u \in H^2(\Omega)$ if $f \in L^2(\Omega)$, and the range of K is $H^2(\Omega) \cap H_0^1(\Omega)$; for non-smooth domains, the range of K is more difficult to describe.

If we consider L as an operator acting in $L^2(\Omega)$, then the domain of L is $D = \operatorname{ran} K$, and

$$L : D \subset L^2(\Omega) \to L^2(\Omega)$$

is an unbounded linear operator with dense domain D. The operator L is closed, meaning that if $\{u_n\}$ is a sequence of functions in D such that $u_n \to u$ and $Lu_n \to f$ in $L^2(\Omega)$, then $u \in D$ and $Lu = f$. By using the resolvent, we can replace an analysis of the unbounded operator L by an analysis of the bounded operator K.

If $f \in L^2(\Omega)$, then $\langle f, v\rangle = (f, v)_{L^2}$. It follows from the definition of weak solution of $Lu + \mu u = f$ that

$$Kf = u \quad \text{if and only if} \quad a_\mu(u, v) = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega) \tag{4.26}$$

where a_μ is defined in (4.25). We also define the operator

$$K^* : L^2(\Omega) \to L^2(\Omega), \qquad K^* = (L^* + \mu I)^{-1}\big|_{L^2(\Omega)},$$

meaning that

$$K^* f = u \quad \text{if and only if} \quad a^*_\mu(u, v) = (f, v)_{L^2} \qquad \text{for all } v \in H_0^1(\Omega) \tag{4.27}$$

where $a^*_\mu(u, v) = a^*(u, v) + \mu\,(u, v)_{L^2}$ and a^* is given in (4.21).

THEOREM 4.23. *If $K \in \mathcal{B}\left(L^2(\Omega)\right)$ is defined by (4.26), then the adjoint of K is K^* defined by (4.27). If Ω is a bounded open set, then K is a compact operator.*

PROOF. If $f, g \in L^2(\Omega)$ and $Kf = u$, $K^* g = v$, then using (4.26) and (4.27), we get

$$(f, K^* g)_{L^2} = (f, v)_{L^2} = a_\mu(u, v) = a^*_\mu(v, u) = (g, u)_{L^2} = (u, g)_{L^2} = (Kf, g)_{L^2}.$$

Hence, K^* is the adjoint of K.

If $Kf = u$, then (4.23) with $\mu \geq \gamma$ and (4.26) imply that

$$C_1 \|u\|^2_{H_0^1} \leq a_\mu(u, u) = (f, u)_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H_0^1}.$$

Hence $\|Kf\|_{H_0^1} \leq C\|f\|_{L^2}$ where $C = 1/C_1$. It follows that K is compact if Ω is bounded, since it maps bounded sets in $L^2(\Omega)$ into bounded sets in $H_0^1(\Omega)$, which are precompact in $L^2(\Omega)$ by the Sobolev embedding theorem. □

4.9. The Fredholm alternative

Consider the Dirichlet problem

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.28}$$

where Ω is a smooth, bounded open set, and

$$Lu = -\sum_{i,j=1}^n \partial_i \left(a_{ij} \partial_j u\right) + \sum_{i=1}^n \partial_i \left(b_i u\right) + cu.$$

If $u = v = 0$ on $\partial\Omega$, Green's formula implies that

$$\int_\Omega (Lu) v\, dx = \int_\Omega u\, (L^* v)\, dx,$$

where the formal adjoint L^* of L is defined by

$$L^* v = -\sum_{i,j=1}^n \partial_i\, (a_{ij}\partial_j v) - \sum_{i=1}^n b_i \partial_i v + cv.$$

It follows that if u is a smooth solution of (4.28) and v is a smooth solution of the homogeneous adjoint problem,

$$L^* v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

then

$$\int_\Omega f v\, dx = \int_\Omega (Lu) v\, dx = \int_\Omega u L^* v\, dx = 0.$$

Thus, a necessary condition for (4.28) to be solvable is that f is orthogonal with respect to the $L^2(\Omega)$-inner product to every solution of the homogeneous adjoint problem.

For bounded domains, we will use the compactness of the resolvent to prove that this condition is necessary and sufficient for the existence of a weak solution of (4.28) where $f \in L^2(\Omega)$. Moreover, the solution is unique if and only if a solution exists for every $f \in L^2(\Omega)$.

This result is a consequence of the fact that if K is compact, then the operator $I+\sigma K$ is a Fredholm operator with index zero on $L^2(\Omega)$ for any $\sigma \in \mathbb{R}$, and therefore satisfies the Fredholm alternative (see Section 4.B.2). Thus, if $K = (L + \mu I)^{-1}$ is compact, the inverse elliptic operator $L - \lambda I$ also satisfies the Fredholm alternative.

THEOREM 4.24. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ and L is a uniformly elliptic operator of the form (4.16) whose coefficients satisfy (4.17). Let L^* be the adjoint operator (4.22) and $\lambda \in \mathbb{R}$. Then one of the following two alternatives holds.*

(1) *The only weak solution of the equation $L^* v - \lambda v = 0$ is $v = 0$. For every $f \in L^2(\Omega)$ there is a unique weak solution $u \in H_0^1(\Omega)$ of the equation $Lu - \lambda u = f$. In particular, the only solution of $Lu - \lambda u = 0$ is $u = 0$.*
(2) *The equation $L^* v - \lambda v = 0$ has a nonzero weak solution v. The solution spaces of $Lu - \lambda u = 0$ and $L^* v - \lambda v = 0$ are finite-dimensional and have the same dimension. For $f \in L^2(\Omega)$, the equation $Lu - \lambda u = f$ has a weak solution $u \in H_0^1(\Omega)$ if and only if $(f, v) = 0$ for every $v \in H_0^1(\Omega)$ such that $L^* v - \lambda v = 0$, and if a solution exists it is not unique.*

PROOF. Since $K = (L + \mu I)^{-1}$ is a compact operator on $L^2(\Omega)$, the Fredholm alternative holds for the equation

$$u + \sigma K u = g \qquad u, g \in L^2(\Omega) \tag{4.29}$$

for any $\sigma \in \mathbb{R}$. Let us consider the two alternatives separately.

First, suppose that the only solution of $v + \sigma K^* v = 0$ is $v = 0$, which implies that the only solution of $L^* v + (\mu + \sigma) v = 0$ is $v = 0$. Then the Fredholm alterative for $I + \sigma K$ implies that (4.29) has a unique solution $u \in L^2(\Omega)$ for every $g \in L^2(\Omega)$. In particular, for any $g \in \operatorname{ran} K$, there exists a unique solution $u \in L^2(\Omega)$, and the equation implies that $u \in \operatorname{ran} K$. Hence, we may apply $L + \mu I$ to (4.29), and conclude that for every $f = (L + \mu I)g \in L^2(\Omega)$, there is a unique solution $u \in \operatorname{ran} K \subset H_0^1(\Omega)$ of the equation

$$Lu + (\mu + \sigma)u = f. \tag{4.30}$$

Taking $\sigma = -(\lambda + \mu)$, we get part (1) of the Fredholm alternative for L.

Second, suppose that $v + \sigma K^* v = 0$ has a finite-dimensional subspace of solutions $v \in L^2(\Omega)$. It follows that $v \in \operatorname{ran} K^*$ (clearly, $\sigma \neq 0$ in this case) and

$$L^* v + (\mu + \sigma)v = 0.$$

By the Fredholm alternative, the equation $u + \sigma K u = 0$ has a finite-dimensional subspace of solutions of the same dimension, and hence so does

$$Lu + (\mu + \sigma)u = 0.$$

Equation (4.29) is solvable for $u \in L^2(\Omega)$ given $g \in \operatorname{ran} K$ if and only if

$$(v, g)_{L^2} = 0 \qquad \text{for all } v \in L^2(\Omega) \text{ such that } v + \sigma K^* v = 0, \tag{4.31}$$

and then $u \in \operatorname{ran} K$. It follows that the condition (4.31) with $g = Kf$ is necessary and sufficient for the solvability of (4.30) given $f \in L^2(\Omega)$. Since

$$(v, g)_{L^2} = (v, Kf)_{L^2} = (K^* v, f)_{L^2} = -\frac{1}{\sigma}(v, f)_{L^2}$$

and $v + \sigma K^* v = 0$ if and only if $L^* v + (\mu + \sigma)v = 0$, we conclude that (4.30) is solvable for u if and only if $f \in L^2(\Omega)$ satisfies

$$(v, f)_{L^2} = 0 \qquad \text{for all } v \in \operatorname{ran} K \text{ such that } L^* v + (\mu + \sigma)v = 0.$$

Taking $\sigma = -(\lambda + \mu)$, we get alternative (2) for L. □

Elliptic operators on a Riemannian manifold may have nonzero Fredholm index. The Atiyah-Singer index theorem (1968) relates the Fredholm index of such operators with a topological index of the manifold.

4.10. The spectrum of a self-adjoint elliptic operator

Suppose that L is a symmetric, uniformly elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) + cu \tag{4.32}$$

where $a_{ij} = a_{ji}$ and $a_{ij}, c \in L^\infty(\Omega)$. The associated symmetric bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

is given by

$$a(u, v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} \partial_i u \partial_j u + cuv \right) dx.$$

The resolvent $K = (L + \mu I)^{-1}$ is a compact self-adjoint operator on $L^2(\Omega)$ for sufficiently large μ. Therefore its eigenvalues are real and its eigenfunctions provide an orthonormal basis of $L^2(\Omega)$. Since L has the same eigenfunctions as K, we get the corresponding result for L.

THEOREM 4.25. *The operator L has an increasing sequence of real eigenvalues of finite multiplicity*

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots$$

such that $\lambda_n \to \infty$. There is an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $L^2(\Omega)$ consisting of eigenfunctions functions $\phi_n \in H_0^1(\Omega)$ such that

$$L\phi_n = \lambda_n \phi_n.$$

PROOF. If $K\phi = 0$ for any $\phi \in L^2(\Omega)$, then applying $L + \mu I$ to the equation we find that $\phi = 0$, so 0 is not an eigenvalue of K. If $K\phi = \kappa\phi$, for $\phi \in L^2(\Omega)$ and $\kappa \neq 0$, then $\phi \in \operatorname{ran} K$ and

$$L\phi = \left(\frac{1}{\kappa} - \mu\right)\phi,$$

so ϕ is an eigenfunction of L with eigenvalue $\lambda = 1/\kappa - \mu$. From Gårding's inequality (4.23) with $u = \phi$, and the fact that $a(\phi, \phi) = \lambda\|\phi\|_{L^2}^2$, we get

$$C_1\|\phi\|_{H_0^1}^2 \leq (\lambda + \gamma)\|\phi\|_{L^2}^2.$$

It follows that $\lambda > -\gamma$, so the eigenvalues of L are bounded from below, and at most a finite number are negative. The spectral theorem for the compact self-adjoint operator K then implies the result. □

The boundedness of the domain Ω is essential here, otherwise K need not be compact, and the spectrum of L need not consist only of eigenvalues.

EXAMPLE 4.26. Suppose that $\Omega = \mathbb{R}^n$ and $L = -\Delta$. Let $K = (-\Delta + I)^{-1}$. Then, from Example 4.12, $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The range of K is $H^2(\mathbb{R}^n)$. This operator is bounded but not compact. For example, if $\phi \in C_c^\infty(\mathbb{R}^n)$ is any nonzero function and $\{a_j\}$ is a sequence in $\mathbb{R}^n$ such that $|a_j| \uparrow \infty$ as $j \to \infty$, then the sequence $\{\phi_j\}$ defined by $\phi_j(x) = \phi(x - a_j)$ is bounded in $L^2(\mathbb{R}^n)$ but $\{K\phi_j\}$ has no convergent subsequence. In this example, K has continuous spectrum $[0, 1]$ on $L^2(\mathbb{R}^n)$ and no eigenvalues. Correspondingly, $-\Delta$ has the purely continuous spectrum $[0, \infty)$.

Finally, let us briefly consider the Fredholm alternative for a self-adjoint elliptic equation from the perspective of this spectral theory. The equation

$$Lu - \lambda u = f \tag{4.33}$$

may be solved by expansion with respect to the eigenfunctions of L. Suppose that $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\Omega)$ such that $L\phi_n = \lambda_n\phi_n$, where the eigenvalues λ_n are increasing and repeated according to their multiplicity. We get the following alternatives, where all series converge in $L^2(\Omega)$:

(1) If $\lambda \neq \lambda_n$ for any $n \in \mathbb{N}$, then (4.33) has the unique solution

$$u = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n - \lambda}\phi_n$$

for every $f \in L^2(\Omega)$;

(2) If $\lambda = \lambda_M$ for for some $M \in \mathbb{N}$ and $\lambda_n = \lambda_M$ for $M \leq n \leq N$, then (4.33) has a solution $u \in H_0^1(\Omega)$ if and only if $f \in L^2(\Omega)$ satisfies

$$(f, \phi_n) = 0 \qquad \text{for } M \leq n \leq N.$$

In that case, the solutions are

$$u = \sum_{\lambda_n \neq \lambda} \frac{(f, \phi_n)}{\lambda_n - \lambda} \phi_n + \sum_{n=M}^{N} c_n \phi_n$$

where $\{c_M, \ldots, c_N\}$ are arbitrary real constants.

4.11. Interior regularity

Roughly speaking, solutions of elliptic PDEs are as smooth as the data allows. For boundary value problems, it is convenient to consider the regularity of the solution in the interior of the domain and near the boundary separately. We begin by studying the interior regularity of solutions. We follow closely the presentation in [**9**].

To motivate the regularity theory, consider the following simple *a priori* estimate for the Laplacian. Suppose that $u \in C_c^\infty(\mathbb{R}^n)$. Then, integrating by parts twice, we get

$$\begin{aligned}
\int (\Delta u)^2 \, dx &= \sum_{i,j=1}^{n} \int \left(\partial_{ii}^2 u\right) \left(\partial_{jj}^2 u\right) \, dx \\
&= - \sum_{i,j=1}^{n} \int \left(\partial_{iij}^3 u\right) \left(\partial_{j} u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int \left(\partial_{ij}^2 u\right) \left(\partial_{ij}^2 u\right) \, dx \\
&= \int \left|D^2 u\right|^2 \, dx.
\end{aligned}$$

Hence, if $-\Delta u = f$, then

$$\left\|D^2 u\right\|_{L^2} = \|f\|_{L^2}^2.$$

Thus, we can control the L^2-norm of all second derivatives of u by the L^2-norm of the Laplacian of u. This estimate suggests that we should have $u \in H^2_{\mathrm{loc}}$ if $f, u \in L^2$, as is in fact true. The above computation is, however, not justified for weak solutions that belong to H^1; as far as we know from the previous existence theory, such solutions may not even possess second-order weak derivatives.

We will consider a PDE

$$Lu = f \qquad \text{in } \Omega \tag{4.34}$$

where Ω is an open set in $\mathbb{R}^n$, $f \in L^2(\Omega)$, and L is a uniformly elliptic of the form

$$Lu = - \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right). \tag{4.35}$$

It is straightforward to extend the proof of the regularity theorem to uniformly elliptic operators that contain lower-order terms [**9**].

A function $u \in H^1(\Omega)$ is a weak solution of (4.34)–(4.35) if

$$a(u, v) = (f, v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{4.36}$$

where the bilinear form a is given by

$$a(u,v) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \partial_i u \partial_j v \, dx. \tag{4.37}$$

We do not impose any boundary condition on u, for example by requiring that $u \in H_0^1(\Omega)$, so the interior regularity theorem applies to any weak solution of (4.34).

Before stating the theorem, we illustrate the idea of the proof with a further *a priori* estimate. To obtain a local estimate for D^2u on a subdomain $\Omega' \Subset \Omega$, we introduce a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on Ω'. We take as a test function

$$v = -\partial_k \left(\eta^2 \partial_k u\right). \tag{4.38}$$

Note that v is given by a positive-definite, symmetric operator acting on u of a similar form to L, which leads to the positivity of the resulting estimate for $D\partial_k u$.

Multiplying (4.34) by v and integrating over Ω, we get $(Lu, v) = (f, v)$. Two integrations by parts imply that

$$\begin{aligned}
(Lu,v) &= \sum_{i,j=1}^{n} \int_{\Omega} \partial_j \left(a_{ij}\partial_i u\right) \left(\partial_k \eta^2 \partial_k u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int_{\Omega} \partial_k \left(a_{ij}\partial_i u\right) \left(\partial_j \eta^2 \partial_k u\right) \, dx \\
&= \sum_{i,j=1}^{n} \int_{\Omega} \eta^2 a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_j \partial_k u\right) \, dx + F
\end{aligned}$$

where

$$\begin{aligned}
F = \sum_{i,j=1}^{n} \int_{\Omega} \Big\{ & \eta^2 \left(\partial_k a_{ij}\right) \left(\partial_i u\right) \left(\partial_j \partial_k u\right) \\
& + 2\eta \partial_j \eta \Big[a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_k u\right) + \left(\partial_k a_{ij}\right) \left(\partial_i u\right) \left(\partial_k u\right) \Big] \Big\} \, dx.
\end{aligned}$$

The term F is linear in the second derivatives of u. We use the uniform ellipticity of L to get

$$\theta \int_{\Omega'} |D\partial_k u|^2 \, dx \le \sum_{i,j=1}^{n} \int_{\Omega} \eta^2 a_{ij} \left(\partial_i \partial_k u\right) \left(\partial_j \partial_k u\right) \, dx = (f, v) - F,$$

and a Cauchy inequality with ϵ to absorb the linear terms in second derivatives on the right-hand side into the quadratic terms on the left-hand side. This results in an estimate of the form

$$\|D\partial_k u\|_{L^2(\Omega')}^2 \le C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right).$$

The proof of regularity is entirely analogous, with the derivatives in the test function (4.38) replaced by difference quotients (see Section 4.C). We obtain an $L^2(\Omega')$-bound for the difference quotients $D\partial_k^h u$ that is uniform in h, which implies that $u \in H^2(\Omega')$.

THEOREM 4.27. *Suppose that Ω is an open set in $\mathbb{R}^n$. Assume that $a_{ij} \in C^1(\Omega)$ and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (4.34)–(4.35), then $u \in H^2(\Omega')$ for every $\Omega' \Subset \Omega$. Furthermore,*

$$\|u\|_{H^2(\Omega')} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \tag{4.39}$$

where the constant C depends only on n, Ω', Ω and a_{ij}.

PROOF. Choose a cut-off function $\eta \in C_c^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on Ω'. We use the compactly supported test function

$$v = -D_k^{-h}\left(\eta^2 D_k^h u\right) \in H_0^1(\Omega)$$

in the definition (4.36)–(4.37) for weak solutions. (As in (4.38), v is given by a positive self-adjoint operator acting on u.) This implies that

$$-\sum_{i,j=1}^n \int_\Omega a_{ij}\left(\partial_i u\right) D_k^{-h}\partial_j\left(\eta^2 D_k^h u\right)\,dx = -\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)\,dx. \tag{4.40}$$

Performing a discrete integration by parts and using the product rule, we may write the left-hand side of (4.40) as

$$\begin{aligned}
-\sum_{i,j=1}^n \int_\Omega a_{ij}\left(\partial_i u\right) D_k^{-h}\partial_j\left(\eta^2 D_k^h u\right)\,dx &= \sum_{i,j=1}^n \int_\Omega D_k^h\left(a_{ij}\partial_i u\right)\partial_j\left(\eta^2 D_k^h u\right)\,dx \\
&= \sum_{i,j=1}^n \int_\Omega \eta^2 a_{ij}^h\left(D_k^h \partial_i u\right)\left(D_k^h \partial_j u\right)\,dx + F,
\end{aligned} \tag{4.41}$$

with $a_{ij}^h(x) = a_{ij}(x + he_k)$, where the error-term F is given by

$$\begin{aligned}
F = \sum_{i,j=1}^n \int_\Omega \Big\{ &\eta^2\left(D_k^h a_{ij}\right)\left(\partial_i u\right)\left(D_k^h \partial_j u\right) \\
&+ 2\eta\partial_j\eta\Big[a_{ij}^h\left(D_k^h\partial_i u\right)\left(D_k^h u\right) + \left(D_k^h a_{ij}\right)\left(\partial_i u\right)\left(D_k^h u\right)\Big]\Big\}\,dx.
\end{aligned} \tag{4.42}$$

Using the uniform ellipticity of L in (4.18), we estimate

$$\theta \int_\Omega \eta^2\left|D_k^h D u\right|^2\,dx \leq \sum_{i,j=1}^n \int_\Omega \eta^2 a_{ij}^h\left(D_k^h\partial_i u\right)\left(D_k^h\partial_j u\right)\,dx.$$

Using (4.40)–(4.41) and this inequality, we find that

$$\theta \int_\Omega \eta^2\left|D_k^h D u\right|^2\,dx \leq -\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)\,dx - F. \tag{4.43}$$

By the Cauchy-Schwartz inequality,

$$\left|\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)\,dx\right| \leq \|f\|_{L^2(\Omega)}\left\|D_k^{-h}\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)}.$$

Since $\operatorname{supp}\eta \Subset \Omega$, Theorem 4.53 implies that for sufficiently small h,

$$\begin{aligned}
\left\|D_k^{-h}\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)} &\leq \left\|\partial_k\left(\eta^2 D_k^h u\right)\right\|_{L^2(\Omega)} \\
&\leq \left\|\eta^2\partial_k D_k^h u\right\|_{L^2(\Omega)} + \left\|2\eta\left(\partial_k\eta\right) D_k^h u\right\|_{L^2(\Omega)} \\
&\leq \left\|\eta\partial_k D_k^h u\right\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)}.
\end{aligned}$$

A similar estimate of F in (4.42) gives

$$|F| \leq C\left(\|Du\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\right).$$

Using these results in (4.43), we find that

$$\begin{aligned}\theta\left\|\eta D_k^h Du\right\|^2_{L^2(\Omega)} \leq & C\Big(\|f\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)} \\ & + \|Du\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\Big).\end{aligned} \tag{4.44}$$

By Cauchy's inequality with ϵ, we have

$$\|f\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} \leq \epsilon\left\|\eta D_k^h Du\right\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon}\|f\|^2_{L^2(\Omega)},$$
$$\|Du\|_{L^2(\Omega)}\left\|\eta D_k^h Du\right\|_{L^2(\Omega)} \leq \epsilon\left\|\eta D_k^h Du\right\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon}\|Du\|^2_{L^2(\Omega)}.$$

Hence, choosing ϵ so that $4C\epsilon = \theta$, and using the result in (4.44) we get that

$$\frac{\theta}{4}\left\|\eta D_k^h Du\right\|^2_{L^2(\Omega)} \leq C\Big(\|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\Big).$$

Thus, since $\eta = 1$ on Ω',

$$\left\|D_k^h Du\right\|^2_{L^2(\Omega')} \leq C\Big(\|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\Big) \tag{4.45}$$

where the constant C depends on Ω, Ω', a_{ij}, but is independent of h, u, f. Theorem 4.53 now implies that the weak second derivatives of u exist and belong to $L^2(\Omega)$. Furthermore, the H^2-norm of u satisfies

$$\|u\|_{H^2(\Omega')} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right).$$

Finally, we replace $\|u\|_{H^1(\Omega)}$ in this estimate by $\|u\|_{L^2(\Omega)}$. First, by the previous argument, if $\Omega' \Subset \Omega'' \Subset \Omega$, then

$$\|u\|_{H^2(\Omega')} \leq C\left(\|f\|_{L^2(\Omega'')} + \|u\|_{H^1(\Omega'')}\right). \tag{4.46}$$

Let $\eta \in C_c^\infty(\Omega)$ be a cut-off function with $0 \leq \eta \leq 1$ and $\eta = 1$ on Ω''. Using the uniform ellipticity of L and taking $v = \eta^2 u$ in (4.36)–(4.37), we get that

$$\begin{aligned}\theta\int_\Omega \eta^2|Du|^2\,dx &\leq \sum_{i,j=1}^n\int_\Omega \eta^2 a_{ij}\partial_i u\partial_j u\,dx \\ &\leq \int_\Omega \eta^2 fu\,dx - \sum_{i,j=1}^n\int_\Omega 2a_{ij}\eta u\partial_i u\partial_j\eta\,dx \\ &\leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)}\|\eta Du\|_{L^2(\Omega)}.\end{aligned}$$

Cauchy's inequality with ϵ then implies that

$$\|\eta Du\|^2_{L^2(\Omega)} \leq C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right),$$

and since $\|Du\|^2_{L^2(\Omega'')} \leq \|\eta Du\|^2_{L^2(\Omega)}$, the use of this result in (4.46) gives (4.39). □

If $u \in H^2_{\mathrm{loc}}(\Omega)$ and $f \in L^2(\Omega)$, then the equation $Lu = f$ relating the weak derivatives of u and f holds pointwise a.e.; such solutions are often called strong solutions, to distinguish them from weak solutions, which may not possess weak second order derivatives, and classical solutions, which possess continuous second order derivatives.

The repeated application of these estimates leads to higher interior regularity.

THEOREM 4.28. *Suppose that $a_{ij} \in C^{k+1}(\Omega)$ and $f \in H^k(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (4.34)–(4.35), then $u \in H^{k+2}(\Omega')$ for every $\Omega' \Subset \Omega$. Furthermore,*

$$\|u\|_{H^{k+2}(\Omega')} \leq C\left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant C depends only on n, k, Ω', Ω and a_{ij}.

See [**9**] for a detailed proof. Note that if the above conditions hold with $k > n/2$, then $f \in C(\Omega)$ and $u \in C^2(\Omega)$, so u is a classical solution of the PDE $Lu = f$. Furthermore, if f and a_{ij} are smooth then so is the solution.

COROLLARY 4.29. *If $a_{ij}, f \in C^\infty(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of (4.34)–(4.35), then $u \in C^\infty(\Omega)$*

PROOF. If $\Omega' \Subset \Omega$, then $f \in H^k(\Omega')$ for every $k \in \mathbb{N}$, so by Theorem (4.28) $u \in H^{k+2}_{\text{loc}}(\Omega')$ for every $k \in \mathbb{N}$, and by the Sobolev embedding theorem $u \in C^\infty(\Omega')$. Since this holds for every open set $\Omega' \Subset \Omega$, we have $u \in C^\infty(\Omega)$. □

4.12. Boundary regularity

To study the regularity of solutions near the boundary, we localize the problem to a neighborhood of a boundary point by use of a partition of unity: We decompose the solution into a sum of functions that are compactly supported in the sets of a suitable open cover of the domain and estimate each function in the sum separately.

Assuming, as in Section 1.10, that the boundary is at least C^1, we may 'flatten' the boundary in a neighborhood U by a diffeomorphism $\varphi : U \to V$ that maps $U \cap \Omega$ to an upper half space $V = B_1(0) \cap \{y_n > 0\}$. If $\varphi^{-1} = \psi$ and $x = \psi(y)$, then by a change of variables (*c.f.* Theorem 1.44 and Proposition 3.21) the weak formulation (4.34)–(4.35) on U becomes

$$\sum_{i,j=1}^{n} \int_V \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial y_i} \frac{\partial \tilde{v}}{\partial y_j}\, dy = \int_V \tilde{f} \tilde{v}\, dy \qquad \text{for all functions } \tilde{v} \in H_0^1(V),$$

where $\tilde{u} \in H^1(V)$. Here, $\tilde{u} = u \circ \psi$, $\tilde{v} = v \circ \psi$, and

$$\tilde{a}_{ij} = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \circ \psi \left(\frac{\partial \varphi_i}{\partial x_p} \circ \psi\right)\left(\frac{\partial \varphi_j}{\partial x_q} \circ \psi\right), \qquad \tilde{f} = |\det D\psi|\, f \circ \psi.$$

The matrix $\tilde{a}_{ij}$ satisfies the uniform ellipticity condition if a_{pq} does. To see this, we define $\zeta = (D\varphi^t)\,\xi$, or

$$\zeta_p = \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_p} \xi_i.$$

Then, since $D\varphi$ and $D\psi = D\varphi^{-1}$ are invertible and bounded away from zero, we have for some constant $C > 0$ that

$$\sum_{i,j=1}^{n} \tilde{a}_{ij} \xi_i \xi_j = |\det D\psi| \sum_{p,q=1}^{n} a_{pq} \zeta_p \zeta_q \geq |\det D\psi|\ \theta |\zeta|^2 \geq C\theta |\xi|^2.$$

Thus, we obtain a problem of the same form as before after the change of variables. Note that we must require that the boundary is C^2 to ensure that $\tilde{a}_{ij}$ is C^1.

It is important to recognize that in changing variables for weak solutions, we need to verify the change of variables for the weak formulation directly and not just for the original PDE. A transformation that is valid for smooth solutions of a

PDE is not always valid for weak solutions, which may lack sufficient smoothness to justify the transformation.

We now state a boundary regularity theorem. Unlike the interior regularity theorem, we impose a boundary condition $u \in H_0^1(\Omega)$ on the solution, and we require that the boundary of the domain is smooth. A solution of an elliptic PDE with smooth coefficients and smooth right-hand side is smooth in the interior of its domain of definition, whatever its behavior near the boundary; but we cannot expect to obtain smoothness up to the boundary without imposing a smooth boundary condition on the solution and requiring that the boundary is smooth.

THEOREM 4.30. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^2-boundary. Assume that $a_{ij} \in C^1(\overline{\Omega})$ and $f \in L^2(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution of (4.34)–(4.35), then $u \in H^2(\Omega)$, and*

$$\|u\|_{H^2(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant C depends only on n, Ω and a_{ij}.

PROOF. By use of a partition of unity and a flattening of the boundary, it is sufficient to prove the result for an upper half space $\Omega = \{(x_1, \dots, x_n) : x_n > 0\}$ space and functions $u, f : \Omega \to \mathbb{R}$ that are compactly supported in $B_1(0) \cap \overline{\Omega}$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_1(0)$. We will estimate the tangential and normal difference quotients of Du separately.

First consider a test function that depends on tangential differences,

$$v = -D_k^{-h}\eta^2 D_k^h u \qquad \text{for } k = 1, 2, \dots, n-1.$$

Since the trace of u is zero on $\partial\Omega$, the trace of v on $\partial\Omega$ is zero and, by Theorem 3.44, $v \in H_0^1(\Omega)$. Thus we may use v in the definition of weak solution to get (4.40). Exactly the same argument as the one in the proof of Theorem 4.27 gives (4.45). It follows from Theorem 4.53 that the weak derivatives $\partial_k \partial_i u$ exist and satisfy

$$\|\partial_k Du\|_{L^2(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \qquad \text{for } k = 1, 2, \dots, n-1. \tag{4.47}$$

The only derivative that remains is the second-order normal derivative $\partial_n^2 u$, which we can estimate from the equation. Using (4.34)–(4.35), we have for $\phi \in C_c^\infty(\Omega)$ that

$$\int_\Omega a_{nn}\left(\partial_n u\right)\left(\partial_n \phi\right)\, dx = -\sum{}' \int_\Omega a_{ij}\left(\partial_i u\right)\left(\partial_j \phi\right)\, dx + \int_\Omega f\phi\, dx$$

where $\sum'$ denotes the sum over $1 \leq i, j \leq n$ with the term $i = j = n$ omitted. Since $a_{ij} \in C^1(\Omega)$ and $\partial_i u$ is weakly differentiable with respect to x_j unless $i = j = n$ we get, using Proposition 3.21, that

$$\int_\Omega a_{nn}\left(\partial_n u\right)\left(\partial_n \phi\right)\, dx = \sum{}' \int_\Omega \left\{\partial_j\left[a_{ij}\left(\partial_i u\right)\right] + f\right\}\phi\, dx \qquad \text{for every } \phi \in C_c^\infty(\Omega).$$

It follows that $a_{nn}\left(\partial_n u\right)$ is weakly differentiable with respect to x_n, and

$$\partial_n\left[a_{nn}\left(\partial_n u\right)\right] = -\left\{\sum{}' \partial_j\left[a_{ij}\left(\partial_i u\right)\right] + f\right\} \in L^2(\Omega).$$

From the uniform ellipticity condition (4.18) with $\xi = e_n$, we have $a_{nn} \geq \theta$. Hence, by Proposition 3.21,

$$\partial_n u = \frac{1}{a_{nn}}\, a_{nn}\partial_n u$$

is weakly differentiable with respect to x_n with derivative

$$\partial^2_{nn} u = \frac{1}{a_{nn}} \partial_n \left[a_{nn} \partial_n u\right] + \partial_n \left(\frac{1}{a_{nn}}\right) a_{nn} \partial_n u \in L^2(\Omega).$$

Furthermore, using (4.47) we get an estimate of the same form for $\|\partial^2_{nn} u\|^2_{L^2(\Omega)}$, so that

$$\left\|D^2 u\right\|_{L^2(\Omega)} \leq C \left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right)$$

□

The repeated application of these estimates leads to higher-order regularity.

THEOREM 4.31. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^{k+2}-boundary. Assume that $a_{ij} \in C^{k+1}(\overline{\Omega})$ and $f \in H^k(\Omega)$. If $u \in H^1_0(\Omega)$ is a weak solution of (4.34)–(4.35), then $u \in H^{k+2}(\Omega)$ and*

$$\|u\|_{H^{k+2}(\Omega)} \leq C \left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

where the constant C depends only on n, k, Ω, and a_{ij}.

Sobolev embedding then yields the following result.

COROLLARY 4.32. *Suppose that Ω is a bounded open set in $\mathbb{R}^n$ with C^∞ boundary. If $a_{ij}, f \in C^\infty(\overline{\Omega})$ and $u \in H^1_0(\Omega)$ is a weak solution of (4.34)–(4.35), then $u \in C^\infty(\overline{\Omega})$*

4.13. Some further perspectives

> This book is to a large extent self-contained, with the restriction that the linear theory — Schauder estimates and Campanato theory — is not presented. The reader is expected to be familiar with functional-analytic tools, like the theory of monotone operators.[4]

The above results give an existence and L^2-regularity theory for second-order, uniformly elliptic PDEs in divergence form. This theory is based on the simple *a priori* energy estimate for $\|Du\|_{L^2}$ that we obtain by multiplying the equation $Lu = f$ by u, or some derivative of u, and integrating the result by parts.

This theory is a fundamental one, but there is a bewildering variety of approaches to the existence and regularity of solutions of elliptic PDEs. In an attempt to put the above analysis in a broader context, we briefly list some of these approaches and other important results, without any claim to completeness. Many of these topics are discussed further in the references [**9, 17, 23**].

L^p-theory: If $1 < p < \infty$, there is a similar regularity result that solutions of $Lu = f$ satisfy $u \in W^{2,p}$ if $f \in L^p$. The derivation is not as simple when $p \neq 2$, however, and requires the use of more sophisticated tools from real analysis (such as the L^p-theory of Calderón-Zygmund operators).

Schauder theory: The Schauder theory provides Hölder-estimates similar to those derived in Section 2.7.2 for Laplace's equation, and a corresponding existence theory of solutions $u \in C^{2,\alpha}$ of $Lu = f$ if $f \in C^{0,\alpha}$ and L has Hölder continuous coefficients. General linear elliptic PDEs are treated by regarding them as perturbations of constant coefficient PDEs, an approach that works because there is no 'loss of derivatives' in the estimates

[4]From the introduction to [**2**].

of the solution. The Hölder estimates were originally obtained by the use of potential theory, but other ways to obtain them are now known; for example, by the use of Campanato spaces, which provide Hölder norms in terms of suitable integral norms that are easier to estimate directly.

Perron's method: Perron (1923) showed that solutions of the Dirichlet problem for Laplace's equation can be obtained as the infimum of superharmonic functions or the supremum of subharmonic functions, together with the use of barrier functions to prove that, under suitable assumptions on the boundary, the solution attains the prescribed boundary values. This method is based on maximum principle estimates.

Boundary integral methods: By the use of Green's functions, one can often reduce a linear elliptic BVP to an integral equation on the boundary, and then use the theory of integral equations to study the existence and regularity of solutions. These methods also provide efficient numerical schemes because of the lower dimensionality of the boundary.

Pseudo-differential operators: The Fourier transform provides an effective method for solving linear PDEs with constant coefficients. The theory of pseudo-differential and Fourier-integral operators is a powerful extension of this method that applies to general linear PDEs with variable coefficients, and elliptic PDEs in particular. It is, however, less well-suited to the analysis of nonlinear PDEs (although there are nonlinear generlizations, such as the theory of para-differential operators).

Variational methods: Many elliptic PDEs — especially those in divergence form — arise as Euler-Lagrange equations for variational principles. Direct methods in the calculus of variations provide a powerful and general way to analyze such PDEs, both linear and nonlinear.

Di Giorgi-Nash-Moser: Di Giorgi (1957), Nash (1958), and Moser (1960) showed that weak solutions of a second order elliptic PDE in divergence form with bounded (L^∞) coefficients are Hölder continuous ($C^{0,\alpha}$). This was the key step in developing a regularity theory for minimizers of nonlinear variational principles with elliptic Euler-Lagrange equations. Moser also obtained a Harnack inequality for weak solutions which is a crucial ingredient of the regularity theory.

Fully nonlinear equations: Krylov and Safonov (1979) obtained a Harnack inequality for second order elliptic equations in nondivergence form. This allowed the development of a regularity theory for fully nonlinear elliptic equations (*e.g.* second-order equations for u that depend nonlinearly on D^2u). Crandall and Lions (1983) introduced the notion of viscosity solutions which — despite the name — uses the maximum principle and is based on a comparison with appropriate sub and super solutions This theory applies to fully nonlinear elliptic PDEs, although it is mainly restricted to scalar equations.

Degree theory: Topological methods based on the Leray-Schauder degree of a mapping on a Banach space can be used to prove existence of solutions of various nonlinear elliptic problems [**32**]. These methods can provide global existence results for large solutions, but often do not give much detailed analytical information about the solutions.

Heat flow methods: Parabolic PDEs, such as $u_t + Lu = f$, are closely connected with the associated elliptic PDEs for stationary solutions, such as $Lu = f$. One may use this connection to obtain solutions of an elliptic PDE as the limit as $t \to \infty$ of solutions of the associated parabolic PDE. For example, Hamilton (1981) introduced the Ricci flow on a manifold, in which the metric approaches a Ricci-flat metric as $t \to \infty$, as a means to understand the topological classification of smooth manifolds, and Perelman (2003) used this approach to prove the Poincaré conjecture (that every simply connected, three-dimensional, compact manifold without boundary is homeomorphic to a three-dimensional sphere) and, more generally, the geometrization conjecture of Thurston.

Appendix

4.A. Heat flow

As a simple physical application that leads to second order PDEs, we consider the problem of finding the temperature distribution inside a body. Similar equations describe the diffusion of a solute. Steady temperature distributions satisfy an elliptic PDE, such as Laplace's equation, while unsteady distributions satisfy a parabolic PDE, such as the heat equation.

4.A.1. Steady heat flow. Suppose that the body occupies an open set Ω in $\mathbb{R}^n$. Let $u : \Omega \to \mathbb{R}$ denote the temperature, $g : \Omega \to \mathbb{R}$ the rate per unit volume at which heat sources create energy inside the body, and $\vec{q} : \Omega \to \mathbb{R}^n$ the heat flux. That is, the rate per unit area at which heat energy diffuses across a surface with normal $\vec{\nu}$ is equal to $\vec{q} \cdot \vec{\nu}$.

If the temperature distribution is steady, then conservation of energy implies that for any smooth open set $\Omega' \Subset \Omega$ the heat flux out of Ω' is equal to the rate at which heat energy is generated inside Ω'; that is,

$$\int_{\partial\Omega'} \vec{q} \cdot \vec{\nu}\, dS = \int_{\Omega'} g\, dV.$$

Here, we use dS and dV to denote integration with respect to surface area and volume, respectively.

We assume that $\vec{q}$ and g are smooth. Then, by the divergence theorem,

$$\int_{\Omega'} \operatorname{div} \vec{q}\, dV = \int_{\Omega'} g\, dV.$$

Since this equality holds for all subdomains Ω' of Ω, it follows that

$$\operatorname{div} \vec{q} = g \qquad \text{in } \Omega. \tag{4.48}$$

Equation (4.48) expresses the fundamental physical principle of conservation of energy, but this principle alone is not enough to determine the temperature distribution inside the body. We must supplement it with a constitutive relation that describes how the heat flux is related to the temperature distribution.

Fourier's law states that the heat flux at some point of the body depends linearly on the temperature gradient at the same point and is in a direction of decreasing temperature. This law is an excellent and well-confirmed approximation in a wide variety of circumstances. Thus,

$$\vec{q} = -A\nabla u \tag{4.49}$$

for a suitable conductivity tensor $A : \Omega \to L(\mathbb{R}^n, \mathbb{R}^n)$, which is required to be symmetric and positive definite. Explicitly, if $\vec{x} \in \Omega$, then $A(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map that takes the negative temperature gradient at $\vec{x}$ to the heat flux at $\vec{x}$. In a uniform, isotropic medium $A = \kappa I$ where the constant $\kappa > 0$ is the thermal conductivity. In an anisotropic medium, such as a crystal or a composite medium, A is not proportional to the identity I and the heat flux need not be in the same direction as the temperature gradient.

Using (4.49) in (4.48), we find that the temperature u satisfies

$$-\operatorname{div}(A\nabla u) = g.$$

If we denote the matrix of A with respect to the standard basis in $\mathbb{R}^n$ by (a_{ij}), then the component form of this equation is

$$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) = g.$$

This equation is in divergence or conservation form. For smooth functions $a_{ij} : \Omega \to \mathbb{R}$, we can write it in nondivergence form as

$$-\sum_{i,j=1}^{n} a_{ij} \partial_{ij} u - \sum_{j=1}^{n} b_j \partial_j u = g, \qquad b_j = \sum_{i}^{n} \partial_i a_{ij}.$$

These forms need not be equivalent if the coefficients a_{ij} are not smooth. For example, in a composite medium made up of different materials, a_{ij} may be discontinuous across boundaries that separate the materials. Such problems can be rewritten as smooth PDEs within domains occupied by a given material, together with appropriate jump conditions across the boundaries. The weak formulation incorporates both the PDEs and the jump conditions.

Next, suppose that the body is occupied by a fluid which, in addition to conducting heat, is in motion with velocity $\vec{v} : \Omega \to \mathbb{R}^n$. Let $e : \Omega \to \mathbb{R}$ denote the internal thermal energy per unit volume of the body, which we assume is a function of the location $\vec{x} \in \Omega$ of a point in the body. Then, in addition to the diffusive flux $\vec{q}$, there is a convective thermal energy flux equal to $e\vec{v}$, and conservation of energy gives

$$\int_{\partial \Omega'} (\vec{q} + e\vec{v}) \cdot \vec{\nu} \, dS = \int_{\Omega'} g \, dV.$$

Using the divergence theorem as before, we find that

$$\operatorname{div} (\vec{q} + e\vec{v}) = g,$$

If we assume that $e = c_p u$ is proportional to the temperature, where c_p is the heat capacity per unit volume of the material in the body, and Fourier's law, we get the PDE

$$-\operatorname{div} (A \nabla u) + \operatorname{div}(\vec{b} u) = g.$$

where $\vec{b} = c_p \vec{v}$.

Suppose that $g = f - cu$ where $f : \Omega \to \mathbb{R}$ is a given energy source and cu represents a linear growth or decay term with coefficient $c : \Omega \to \mathbb{R}$. For example, lateral heat loss at a rate proportional the temperature would give decay $(c > 0)$, while the effects of an exothermic temperature-dependent chemical reaction might be approximated by a linear growth term $(c < 0)$. We then get the linear PDE

$$-\operatorname{div} (A \nabla u) + \operatorname{div}(\vec{b} u) + cu = f,$$

or in component form with $\vec{b} = (b_1, \dots, b_n)$

$$-\sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu = f.$$

This PDE describes a thermal equilibrium due to the combined effects of diffusion with diffusion matrix a_{ij}, advection with normalized velocity b_i, growth or decay with coefficient c, and external sources with density f.

In the simplest case where, after nondimensionalization, $A = I$, $\vec{b} = 0$, $c = 0$, and $f = 0$, we get Laplace's equation $\Delta u = 0$.

4.A.2. Unsteady heat flow. Consider a time-dependent heat flow in a region Ω with temperature $u(\vec{x}, t)$, energy density per unit volume $e(\vec{x}, t)$, heat flux $\vec{q}(\vec{x}, t)$, advection velocity $\vec{v}(\vec{x}, t)$, and heat source density $g(\vec{x}, t)$. Conservation of energy implies that for any subregion $\Omega' \Subset \Omega$

$$\frac{d}{dt}\int_{\Omega'} e\, dV = -\int_{\partial\Omega'} (\vec{q} + e\vec{v}) \cdot \vec{\nu}\, dS + \int_{\Omega'} g\, dV.$$

Since

$$\frac{d}{dt}\int_{\Omega'} e\, dV = \int_{\Omega'} e_t\, dV,$$

the use of the divergence theorem and the same constitutive assumptions as in the steady case lead to the parabolic PDE

$$(c_p u)_t - \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} \partial_i \left(b_i u\right) + cu = f.$$

In the simplest case where, after nondimensionalization, $c_p = 1$, $A = I$, $\vec{b} = 0$, $c = 0$, and $f = 0$, we get the heat equation $u_t = \Delta u$.

4.B. Operators on Hilbert spaces

Suppose that $\mathcal{H}$ is a Hilbert space with inner product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$. We denote the space of bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. This is a Banach space with respect to the operator norm, defined by

$$\|T\| = \sup_{\substack{x \in \mathcal{H} \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

The adjoint $T^* \in \mathcal{L}(\mathcal{H})$ of $T \in \mathcal{L}(\mathcal{H})$ is the linear operator such that

$$(Tx, y) = (x, T^*y) \qquad \text{for all } x, y \in \mathcal{H}.$$

An operator T is self-adjoint if $T = T^*$. The kernel and range of $T \in \mathcal{L}(\mathcal{H})$ are the subspaces

$$\ker T = \{x \in \mathcal{H} : Tx = 0\}, \quad \operatorname{ran} T = \{y \in \mathcal{H} : y = Tx \text{ for some } x \in \mathcal{H}\}.$$

We denote by $\ell^2(\mathbb{N})$, or ℓ^2 for short, the Hilbert space of square summable real sequences

$$\ell^2(\mathbb{N}) = \left\{(x_1, x_2, x_3, \dots, x_n, \dots) : x_n \in \mathbb{R} \text{ and } \textstyle\sum_{n\in\mathbb{N}} x_n^2 < \infty\right\}$$

with the standard inner product. Any infinite-dimensional, separable Hilbert space is isomorphic to ℓ^2.

4.B.1. Compact operators.

Definition 4.33. A linear operator $T \in \mathcal{L}(\mathcal{H})$ is compact if it maps bounded sets to precompact sets.

That is, T is compact if $\{Tx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$ in $\mathcal{H}$.

Example 4.34. A bounded linear map with finite-dimensional range is compact. In particular, every linear operator on a finite-dimensional Hilbert space is compact.

Example 4.35. The identity map $I \in \mathcal{L}(\mathcal{H})$ given by $I : x \mapsto x$ is compact if and only if $\mathcal{H}$ is finite-dimensional.

Example 4.36. The map $K \in \mathcal{L}\left(\ell^2\right)$ given by

$$K : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots, \frac{1}{n}x_n, \dots\right)$$

is compact (and self-adjoint).

We have the following spectral theorem for compact self-adjoint operators.

Theorem 4.37. *Let $T : \mathcal{H} \to \mathcal{H}$ be a compact, self-adjoint operator. Then T has a finite or countably infinite number of distinct nonzero, real eigenvalues. If there are infinitely many eigenvalues $\{\lambda_n \in \mathbb{R} : n \in \mathbb{N}\}$ then $\lambda_n \to 0$ as $n \to \infty$. The eigenspace associated with each nonzero eigenvalue is finite-dimensional, and eigenvectors associated with distinct eigenvalues are orthogonal. Furthermore, $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T, including those (if any) with eigenvalue zero.*

4.B.2. Fredholm operators. We summarize the definition and properties of Fredholm operators and give some examples. For proofs, see

Definition 4.38. A linear operator $T \in \mathcal{L}(\mathcal{H})$ is Fredholm if: (a) $\ker T$ has finite dimension; (b) $\operatorname{ran} T$ is closed and has finite codimension.

Condition (b) and the projection theorem for Hilbert spaces imply that $\mathcal{H} = \operatorname{ran} T \oplus (\operatorname{ran} T)^\perp$ where the dimension of $\operatorname{ran} T^\perp$ is finite, and

$$\operatorname{codim} \operatorname{ran} T = \dim(\operatorname{ran} T)^\perp.$$

Definition 4.39. If $T \in \mathcal{L}(\mathcal{H})$ is Fredholm, then the index of T is the integer

$$\operatorname{ind} T = \dim \ker T - \operatorname{codim} \operatorname{ran} T.$$

Example 4.40. Every linear operator $T : \mathcal{H} \to \mathcal{H}$ on a finite-dimensional Hilbert space $\mathcal{H}$ is Fredholm and has index zero. The range is closed since every finite-dimensional linear space is closed, and the dimension formula

$$\dim \ker T + \dim \operatorname{ran} T = \dim \mathcal{H}$$

implies that the index is zero.

Example 4.41. The identity map I on a Hilbert space of any dimension is Fredholm, with $\dim \ker P = \operatorname{codim} \operatorname{ran} P = 0$ and $\operatorname{ind} I = 0$.

Example 4.42. The self-adjoint projection P on ℓ^2 given by

$$P : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (0, x_2, x_3, \dots, x_n, \dots)$$

is Fredholm, with $\dim \ker P = \operatorname{codim} \operatorname{ran} P = 1$ and $\operatorname{ind} P = 0$. The complementary projection

$$Q : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (x_1, 0, 0, \dots, 0, \dots)$$

is not Fredholm, although the range of Q is closed, since $\dim \ker Q$ and $\operatorname{codim} \operatorname{ran} Q$ are infinite.

Example 4.43. The left and right shift maps on ℓ^2, given by

$$S : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (x_2, x_3, x_4, \dots, x_{n+1}, \dots),$$
$$T : (x_1, x_2, x_3, \dots, x_n, \dots) \mapsto (0, x_1, x_2, \dots, x_{n-1}, \dots),$$

are Fredholm. Note that $S^* = T$. We have $\dim\ker S = 1$, $\operatorname{codim}\operatorname{ran} S = 0$, and $\dim\ker T = 0$, $\operatorname{codim}\operatorname{ran} T = 1$, so

$$\operatorname{ind} S = 1, \qquad \operatorname{ind} T = -1.$$

If $n \in \mathbb{N}$, then $\operatorname{ind} S^n = n$ and $\operatorname{ind} T^n = -n$, so the index of a Fredholm operator on an infinite-dimensional space can take all integer values. Unlike the finite-dimensional case, where a linear operator $A : \mathcal{H} \to \mathcal{H}$ is one-to-one if and only if it is onto, S fails to be one-to-one although it is onto, and T fails to be onto although it is one-to-one.

The above example also illustrates the following theorem.

Theorem 4.44. *If $T \in \mathcal{L}(\mathcal{H})$ is Fredholm, then T^* is Fredholm with*

$$\dim\ker T^* = \operatorname{codim}\operatorname{ran} T, \quad \operatorname{codim}\operatorname{ran} T^* = \dim\ker T, \quad \operatorname{ind} T^* = -\operatorname{ind} T.$$

Example 4.45. The compact map K in Example 4.36 is not Fredholm since the range of K,

$$\operatorname{ran} K = \left\{ (y_1, y_2, y_3, \dots, y_n, \dots) \in \ell^2 : \sum_{n\in\mathbb{N}} n^2 y_n^2 < \infty \right\},$$

is not closed. The range is dense in ℓ^2 but, for example,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \ell^2 \setminus \operatorname{ran} K.$$

We denote the set of Fredholm operators by $\mathcal{F}$. Then, according to the next theorem, $\mathcal{F}$ is an open set in $\mathcal{L}(\mathcal{H})$, and

$$\mathcal{F} = \bigcup_{n\in\mathbb{Z}} \mathcal{F}_n$$

is the union of connected components $\mathcal{F}_n$ consisting of the Fredholm operators with index n. Moreover, if $T \in \mathcal{F}_n$, then $T + K \in \mathcal{F}_n$ for any compact operator K.

Theorem 4.46. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is Fredholm and $K \in \mathcal{L}(\mathcal{H})$ is compact.*

(1) *There exists $\epsilon > 0$ such that $T + H$ is Fredholm for any $H \in \mathcal{L}(\mathcal{H})$ with $\|H\| < \epsilon$. Moreover,* $\operatorname{ind}(T + H) = \operatorname{ind} T$.
(2) *$T + K$ is Fredholm and* $\operatorname{ind}(T + K) = \operatorname{ind} T$.

Solvability conditions for Fredholm operators are a consequences of following theorem.

Theorem 4.47. *If $T \in \mathcal{L}(\mathcal{H})$, then $\mathcal{H} = \overline{\operatorname{ran} T} \oplus \ker T^*$ and $\overline{\operatorname{ran} T} = (\ker T)^\perp$.*

Thus, if $T \in \mathcal{L}(\mathcal{H})$ has closed range, then $Tx = y$ has a solution $x \in \mathcal{H}$ if and only if $y \perp z$ for every $z \in \mathcal{H}$ such that $T^*z = 0$. For a Fredholm operator, this is finitely many linearly independent solvability conditions.

Example 4.48. If S, T are the shift maps defined in Example 4.43, then $\ker S^* = \ker T = 0$ and the equation $Sx = y$ is solvable for every $y \in \ell^2$. Solutions are not, however, unique since $\ker S \neq 0$. The equation $Tx = y$ is solvable only if $y \perp \ker S$. If it exists, the solution is unique.

EXAMPLE 4.49. The compact map K in Example 4.36 is self adjoint, $K = K^*$, and $\ker K = 0$. Thus, every element $y \in \ell^2$ is orthogonal to $\ker K^*$, but this condition is not sufficient to imply the solvability of $Kx = y$ because the range of K os not closed. For example,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \in \ell^2 \setminus \operatorname{ran} K.$$

For Fredholm operators with index zero, we get the following Fredholm alternative, which states that the corresponding linear equation has solvability properties which are similar to those of a finite-dimensional linear system.

THEOREM 4.50. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a Fredholm operator and* $\operatorname{ind} T = 0$. *Then one of the following two alternatives holds:*

(1) $\ker T^* = 0$*;* $\ker T = 0$*;* $\operatorname{ran} T = \mathcal{H}$*,* $\operatorname{ran} T^* = \mathcal{H}$*;*
(2) $\ker T^* \neq 0$*;* $\ker T$*,* $\ker T^*$ *are finite-dimensional spaces with the same dimension;* $\operatorname{ran} T = (\ker T^*)^\perp$*,* $\operatorname{ran} T^* = (\ker T)^\perp$*.*

4.C. Difference quotients

Difference quotients provide a useful method for proving the weak differentiability of functions. The main result, in Theorem 4.53 below, is that the uniform boundedness of the difference quotients of a function is sufficient to imply that the function is weakly differentiable.

DEFINITION 4.51. If $u : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R} \setminus \{0\}$, the ith difference quotient of u of size h is the function $D_i^h u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}$$

where e_i is the unit vector in the ith direction. The vector of difference quotient is

$$D^h u = \left(D_1^h u, D_2^h u, \dots, D_n^h u\right).$$

The next proposition gives some elementary properties of difference quotients that are analogous to those of derivatives.

PROPOSITION 4.52. *The difference quotient has the following properties.*

(1) *Commutativity with weak derivatives: if* $u, \partial_i u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*, then*

$$\partial_i D_j^h u = D_j^h \partial_i u.$$

(2) *Integration by parts: if* $u \in L^p(\mathbb{R}^n)$ *and* $v \in L^{p'}(\mathbb{R}^n)$*, where* $1 \leq p \leq \infty$*, then*

$$\int (D_i^h u) v \, dx = -\int u (D_i^h v) \, dx.$$

(3) *Product rule:*

$$D_i^h (uv) = u_i^h \left(D_i^h v\right) + \left(D_i^h u\right) v = u \left(D_i^h v\right) + \left(D_i^h u\right) v_i^h.$$

where $u_i^h(x) = u(x + he_i)$*.*

Proof. Property (1) follows immediately from the linearity of the weak derivative. For (2), note that

$$\begin{aligned}
\int (D_i^h u) v\, dx &= \frac{1}{h} \int \left[u(x+he_i) - u(x)\right] v(x)\, dx \\
&= \frac{1}{h} \int u(x') v(x' - he_i)\, dx' - \frac{1}{h} \int u(x) v(x)\, dx \\
&= \frac{1}{h} \int u(x) \left[v(x - he_i) - v(x)\right]\, dx \\
&= - \int u \left(D_i^{-h} v\right)\, dx.
\end{aligned}$$

For (3), we have

$$\begin{aligned}
u_i^h \left(D_i^h v\right) + \left(D_i^h u\right) v &= u(x+he_i) \left[\frac{v(x+he_i) - v(x)}{h}\right] + \left[\frac{u(x+he_i) - u(x)}{h}\right] v(x) \\
&= \frac{u(x+he_i)v(x+he_i) - u(x)v(x)}{h} \\
&= D_i^h(uv),
\end{aligned}$$

and the same calculation with u and v exchanged. □

Theorem 4.53. *Suppose that Ω is an open set in $\mathbb{R}^n$ and $\Omega' \Subset \Omega$. Let*

$$d = \operatorname{dist}(\Omega', \partial\Omega) > 0.$$

(1) *If $Du \in L^p(\Omega)$ where $1 \le p < \infty$, and $0 < |h| < d$, then*

$$\left\|D^h u\right\|_{L^p(\Omega')} \le \|Du\|_{L^p(\Omega)}.$$

(2) *If $u \in L^p(\Omega)$ where $1 < p < \infty$, and there exists a constant C such that*

$$\left\|D^h u\right\|_{L^p(\Omega')} \le C$$

for all $0 < |h| < d/2$, then $u \in W^{1,p}(\Omega')$ and

$$\|Du\|_{L^p(\Omega')} \le C.$$

Proof. To prove (1), we may assume by an approximation argument that u is smooth. Then

$$u(x+he_i) - u(x) = h \int_0^1 \partial_i u(x + te_i)\, dt,$$

and, by Jensen's inequality,

$$|u(x+he_i) - u(x)|^p \le |h|^p \int_0^1 |\partial_i u(x+te_i)|^p\, dt.$$

Integrating this inequality with respect to x, and using Fubini's theorem, together with the fact that $x + te_i \in \Omega$ if $x \in \Omega'$ and $|t| \le h < d$, we get

$$\int_{\Omega'} |u(x+he_i) - u(x)|^p\, dx \le |h|^p \int_\Omega |\partial_i u(x+te_i)|^p\, dx.$$

Thus, $\|D_i^h u\|_{L^p(\Omega')} \le \|D_i^h u\|_{L^p(\Omega)}$, and (1) follows.

To prove (2), note that since

$$\left\{D_i^h u : 0 < |h| < d\right\}$$

is bounded in $L^p(\Omega')$, the Banach-Alaoglu theorem implies that there is a sequence $\{h_k\}$ such that $h_k \to 0$ as $k \to \infty$ and a function $v_i \in L^p(\Omega')$ such that

$$D_i^{h_k} u \rightharpoonup v_i \qquad \text{as } k \to \infty \text{ in } L^p(\Omega').$$

Suppose that $\phi \in C_c^\infty(\Omega')$. Then, for sufficiently small h_k,

$$\int_{\Omega'} u D_i^{-h_k} \phi \, dx = \int_{\Omega'} \left(D_i^{h_k} u \right) \phi \, dx.$$

Taking the limit as $k \to \infty$, when $D_i^{-h_k}\phi$ converges uniformly to $\partial_i \phi$, we get

$$\int_{\Omega'} u \partial_i \phi \, dx = \int_{\Omega'} v_i \phi \, dx.$$

Hence u is weakly differentiable and $\partial_i u = v_i \in L^p(\Omega')$, which proves (2). $\square$

CHAPTER 5

The Heat and Schrödinger Equations

The heat, or diffusion, equation is

$$u_t = \Delta u. \tag{5.1}$$

Section 4.A derives (5.1) as a model of heat flow.

Steady solutions of the heat equation satisfy Laplace's equation. Using (2.4), we have for smooth functions that

$$\begin{aligned}
\Delta u(x) &= \lim_{r\to 0^+} \fint_{B_r(x)} \Delta u \, dx \\
&= \lim_{r\to 0^+} \frac{n}{r}\frac{\partial}{\partial r}\left[\fint_{\partial B_r(x)} u \, dS\right] \\
&= \lim_{r\to 0^+} \frac{2n}{r^2}\left[\fint_{\partial B_r(x)} u \, dS - u(x)\right].
\end{aligned}$$

Thus, if u is a solution of the heat equation, then the rate of change of $u(x,t)$ with respect to t at a point x is proportional to the difference between the value of u at x and the average of u over nearby spheres centered at x. The solution decreases in time if its value at a point is greater than the nearby mean and increases if its value is less than the nearby averages. The heat equation therefore describes the evolution of a function towards its mean. As $t \to \infty$ solutions of the heat equation typically approach functions with the mean value property, which are solutions of Laplace's equation.

We will also consider the Schrödinger equation

$$iu_t = -\Delta u.$$

This PDE is a dispersive wave equation, which describes a complex wave-field that oscillates with a frequency proportional to the difference between the value of the function and its nearby means.

5.1. The initial value problem for the heat equation

Consider the initial value problem for $u(x,t)$ where $x \in \mathbb{R}^n$

$$\begin{aligned}
u_t &= \Delta u \qquad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\
u(x,0) &= f(x) \qquad \text{for } x \in \mathbb{R}^n.
\end{aligned} \tag{5.2}$$

We will solve (5.2) explicitly for smooth initial data by use of the Fourier transform, following the presentation in [**34**]. Some of the main qualitative features illustrated by this solution are the smoothing effect of the heat equation, the irreversibility of its semiflow, and the need to impose a growth condition as $|x| \to \infty$ in order to pick out a unique solution.

5.1.1. Schwartz solutions. Assume first that the initial data $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth, rapidly decreasing, real-valued Schwartz function $f \in \mathcal{S}$ (see Section 5.6.2). The solution we construct is also a Schwartz function of x at later times $t > 0$, and we will regard it as a function of time with values in $\mathcal{S}$. This is analogous to the geometrical interpretation of a first-order system of ODEs, in which the finite-dimensional phase space of the ODE is replaced by the infinite-dimensional function space $\mathcal{S}$; we then think of a solution of the heat equation as a parametrized curve in the vector space $\mathcal{S}$. A similar viewpoint is useful for many evolutionary PDEs, where the Schwartz space may be replaced other function spaces (for example, Sobolev spaces).

By a convenient abuse of notation, we use the same symbol u to denote the scalar-valued function $u(x,t)$, where $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, and the associated vector-valued function $u(t)$, where $u : [0,\infty) \to \mathcal{S}$. We write the vector-valued function corresponding to the associated scalar-valued function as $u(t) = u(\cdot, t)$.

Definition 5.1. Suppose that (a,b) is an open interval in $\mathbb{R}$. A function $u : (a,b) \to \mathcal{S}$ is continuous at $t \in (a,b)$ if

$$u(t+h) \to u(t) \qquad \text{in } \mathcal{S} \text{ as } h \to 0,$$

and differentiable at $t \in (a,b)$ if there exists a function $v \in \mathcal{S}$ such that

$$\frac{u(t+h) - u(t)}{h} \to v \qquad \text{in } \mathcal{S} \text{ as } h \to 0.$$

The derivative v of u at t is denoted by $u_t(t)$, and if u is differentiable for every $t \in (a,b)$, then $u_t : (a,b) \to \mathcal{S}$ denotes the map $u_t : t \mapsto u_t(t)$.

In other words, u is continuous at t if

$$u(t) = \mathcal{S}\text{-}\lim_{h\to 0} u(t+h),$$

and u is differentiable at t with derivative $u_t(t)$ if

$$u_t(t) = \mathcal{S}\text{-}\lim_{h\to 0} \frac{u(t+h) - u(t)}{h}.$$

We will refer to this derivative as a strong derivative if it is understood that we are considering $\mathcal{S}$-valued functions and we want to emphasize that the derivative is defined as the limit of difference quotients in $\mathcal{S}$.

We define spaces of differentiable Schwartz-valued functions in the natural way. For half-open or closed intervals, we make the obvious modifications to left or right limits at an endpoint.

Definition 5.2. The space $C([a,b];\mathcal{S})$ consists of the continuous functions

$$u : [a,b] \to \mathcal{S}.$$

The space $C^k(a,b;\mathcal{S})$ consists of functions $u : (a,b) \to \mathcal{S}$ that are k-times strongly differentiable in (a,b) with continuous strong derivatives $\partial_t^j u \in C(a,b;\mathcal{S})$ for $0 \le j \le k$, and $C^\infty(a,b;\mathcal{S})$ is the space of functions with continuous strong derivatives of all orders.

Here we write $C(a,b;\mathcal{S})$ rather than $C((a,b);\mathcal{S})$ when we consider functions defined on the open interval (a,b). The next proposition describes the relationship between the C^1-strong derivative and the pointwise time-derivative.

PROPOSITION 5.3. *Suppose that* $u \in C(a, b; \mathcal{S})$ *where* $u(t) = u(\cdot, t)$. *Then* $u \in C^1(a, b; \mathcal{S})$ *if and only if:*

(1) *the pointwise partial derivative* $\partial_t u(x, t)$ *exists for every* $x \in \mathbb{R}^n$ *and* $t \in (a, b)$*;*
(2) $\partial_t u(\cdot, t) \in \mathcal{S}$ *for every* $t \in (a, b)$*;*
(3) *the map* $t \mapsto \partial_t u(\cdot, t)$ *belongs* $C(a, b; \mathcal{S})$.

PROOF. The convergence of functions in $\mathcal{S}$ implies uniform pointwise convergence. Thus, if $u(t) = u(\cdot, t)$ is strongly continuously differentiable, then the pointwise partial derivative $\partial_t u(x, t)$ exists for every $x \in \mathbb{R}^n$ and $\partial_t u(\cdot, t) = u_t(t) \in \mathcal{S}$, so $\partial_t u \in C(a, b; \mathcal{S})$.

Conversely, if a pointwise partial derivative with the given properties exist, then for each $x \in \mathbb{R}^n$

$$\frac{u(x, t+h) - u(x, t)}{h} - \partial_t u(x, t) = \frac{1}{h} \int_t^{t+h} \left[\partial_s u(x, s) - \partial_t u(x, t)\right] ds.$$

Since the integrand is a smooth rapidly decreasing function, it follows from the dominated convergence theorem that we may differentiate under the integral sign with respect to x, to get

$$x^\alpha \partial^\beta \left[\frac{u(x, t+h) - u(x, t)}{h}\right] = \frac{1}{h} \int_t^{t+h} x^\alpha \partial^\beta \left[\partial_s u(x, s) - \partial_t u(x, t)\right] ds.$$

Hence, if $\|\cdot\|_{\alpha,\beta}$ is a Schwartz seminorm (5.72), we have

$$\begin{aligned} \left\| \frac{u(t+h) - u(t)}{h} - \partial_t u(\cdot, t) \right\|_{\alpha,\beta} &\le \frac{1}{|h|} \left| \int_t^{t+h} \|\partial_s u(\cdot, s) - \partial_t u(\cdot, t)\|_{\alpha,\beta}\, ds \right| \\ &\le \max_{t \le s \le t+h} \|\partial_s u(\cdot, s) - \partial_t u(\cdot, t)\|_{\alpha,\beta}, \end{aligned}$$

and since $\partial_t u \in C(a, b; \mathcal{S})$

$$\lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} - \partial_t u(\cdot, t) \right\|_{\alpha,\beta} = 0.$$

It follows that

$$\mathcal{S}\text{-}\lim_{h \to 0} \left[\frac{u(t+h) - u(t)}{h}\right] = \partial_t u(\cdot, t),$$

so u is strongly differentiable and $u_t = \partial_t u \in C(a, b; \mathcal{S})$. □

We interpret the initial value problem (5.2) for the heat equation as follows: A solution is a function $u : [0, \infty) \to \mathcal{S}$ that is continuous for $t \ge 0$, so that it makes sense to impose the initial condition at $t = 0$, and continuously differentiable for $t > 0$, so that it makes sense to impose the PDE pointwise in t. That is, for every $t > 0$, the strong derivative $u_t(t)$ is required to exist and equal $\Delta u(t)$ where $\Delta : \mathcal{S} \to \mathcal{S}$ is the Laplacian operator.

THEOREM 5.4. *If* $f \in \mathcal{S}$*, there is a unique solution*

$$u \in C([0, \infty); \mathcal{S}) \cap C^1(0, \infty; \mathcal{S}) \tag{5.3}$$

of (5.2). Furthermore, $u \in C^\infty([0, \infty); \mathcal{S})$. *The spatial Fourier transform of the solution is given by*

$$\hat{u}(k, t) = \hat{f}(k) e^{-t|k|^2}, \tag{5.4}$$

and for $t > 0$ the solution is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t) f(y)\, dy \tag{5.5}$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t}. \tag{5.6}$$

PROOF. Since the spatial Fourier transform $\mathcal{F}$ is a continuous linear map on $\mathcal{S}$ with continuous inverse, the time-derivative of u exists if and only if the time derivative of $\hat{u} = \mathcal{F}u$ exists, and

$$\mathcal{F}(u_t) = (\mathcal{F}u)_t\, .$$

Moreover, $u \in C([0,\infty);\mathcal{S})$ if and only if $\hat{u} \in C([0,\infty);\mathcal{S})$, and $u \in C^k(0,\infty;\mathcal{S})$ if and only if $\hat{u} \in C^k(0,\infty;\mathcal{S})$.

Taking the Fourier transform of (5.2) with respect to x, we find that $u(x,t)$ is a solution with the regularity in (5.3) if and only if $\hat{u}(k,t)$ satisfies

$$\hat{u}_t = -|k|^2 \hat{u}, \qquad \hat{u}(0) = \hat{f}, \qquad \hat{u} \in C([0,\infty);\mathcal{S}) \cap C^1(0,\infty;\mathcal{S})\, . \tag{5.7}$$

Equation (5.7) has the unique solution (5.4).

To show this in detail, suppose first that $\hat{u}$ satisfies (5.7). Then, from Proposition 5.3, the scalar-valued function $\hat{u}(k,t)$ is pointwise-differentiable with respect to t in $t > 0$ and continuous in $t \ge 0$ for each fixed $k \in \mathbb{R}^n$. Solving the ODE (5.7) with k as a parameter, we find that $\hat{u}$ must be given by (5.4).

Conversely, we claim that the function defined by (5.4) is strongly differentiable with derivative

$$\hat{u}_t(k,t) = -|k|^2 \hat{f}(k) e^{-t|k|^2}. \tag{5.8}$$

To prove this claim, note that if $\alpha, \beta \in \mathbb{N}_0^n$ are any multi-indices, the function

$$k^\alpha \partial^\beta \left[\hat{u}(k,t+h) - \hat{u}(k,t)\right]$$

has the form

$$\hat{a}(k,t)\left[e^{-h|k|^2} - 1\right] e^{-t|k|^2} + h \sum_{i=0}^{|\beta|-1} h^i \hat{b}_i(k,t) e^{-(t+h)|k|^2}$$

where $\hat{a}(\cdot,t), \hat{b}_i(\cdot,t) \in \mathcal{S}$, so taking the supremum of this expression we see that

$$\|\hat{u}(t+h) - \hat{u}(t)\|_{\alpha,\beta} \to 0 \qquad \text{as } h \to 0.$$

Thus, $\hat{u}(\cdot,t)$ is a continuous $\mathcal{S}$-valued function in $t \ge 0$ for every $\hat{f} \in \mathcal{S}$. By a similar argument, the pointwise partial derivative $\hat{u}_t(\cdot,t)$ in (5.8) is a continuous $\mathcal{S}$-valued function. Thus, Proposition 5.3 implies that $\hat{u}$ is a strongly continuously differentiable function that satisfies (5.7). Hence $u = \mathcal{F}^{-1}[\hat{u}]$ satisfies (5.3) and is a solution of (5.2). Moreover, using induction and Proposition 5.3 we see in a similar way that $u \in C^\infty([0,\infty);\mathcal{S})$.

Finally, from Example 5.65, we have

$$\mathcal{F}^{-1}\left[e^{-t|k|^2}\right] = \left(\frac{\pi}{t}\right)^{n/2} e^{-|x|^2/4t}.$$

Taking the inverse Fourier transform of (5.4) and using the convolution theorem, Theorem 5.67, we get (5.5)–(5.6). $\square$

The function $\Gamma(x,t)$ in (5.6) is called the Green's function or fundamental solution of the heat equation in $\mathbb{R}^n$. It is a C^∞-function of (x,t) in $\mathbb{R}^n \times (0,\infty)$, and one can verify by direct computation that

$$\Gamma_t = \Delta\Gamma \qquad \text{if } t > 0. \tag{5.9}$$

Also, since $\Gamma(\cdot,t)$ is a family of Gaussian mollifiers, we have

$$\Gamma(\cdot,t) \rightharpoonup \delta \qquad \text{in } \mathcal{S}' \text{ as } t \to 0^+.$$

Thus, we can interpret $\Gamma(x,t)$ as the solution of the heat equation due to an initial point source located at $x = 0$. The solution is a spherically symmetric Gaussian with spatial integral equal to one which spreads out and decays as t increases; its width is of the order $\sqrt{t}$ and its height is of the order $t^{-n/2}$.

The solution at time t is given by convolution of the initial data with $\Gamma(\cdot,t)$. For any $f \in \mathcal{S}$, this gives a smooth classical solution $u \in C^\infty(\mathbb{R}^n \times [0,\infty))$ of the heat equation which satisfies it pointwise in $t \geq 0$.

5.1.2. Smoothing. Equation (5.5) also gives solutions of (5.2) for initial data that is not smooth. To be specific, we suppose that $f \in L^p$, although one can also consider more general data that does not grow too rapidly at infinity.

THEOREM 5.5. *Suppose that $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Define*

$$u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$$

by (5.5) where Γ is given in (5.6). Then $u \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and $u_t = \Delta u$ in $t > 0$. If $1 \leq p < \infty$, then $u(\cdot,t) \to f$ in L^p as $t \to 0^+$.

PROOF. The Green's function Γ in (5.6) satisfies (5.9), and $\Gamma(\cdot,t) \in L^q$ for every $1 \leq q \leq \infty$, together with all of its derivatives. The dominated convergence theorem and Hölder's inequality imply that if $f \in L^p$ and $t > 0$, we can differentiate under the integral sign in (5.5) arbitrarily often with respect to (x,t) and that all of these derivatives approach zero as $|x| \to \infty$. Thus, u is a smooth, decaying solution of the heat equation in $t > 0$. Moreover, $\Gamma^t(x) = \Gamma(x,t)$ is a family of Gaussian mollifiers and therefore for $1 \leq p < \infty$ we have from Theorem 1.28 that $u(\cdot,t) = \Gamma^t * f \to f$ in L^p as $t \to 0^+$. □

The heat equation therefore immediately smooths any initial data $f \in L^p(\mathbb{R}^n)$ to a function $u(\cdot,t) \in C_0^\infty(\mathbb{R}^n)$. From the Fourier perspective, the smoothing is a consequence of the very rapid damping of the high-wavenumber modes at a rate proportional to $e^{-t|k|^2}$ for wavenumbers $|k|$, which physically is caused by the diffusion of thermal energy from hot to cold parts of spatial oscillations.

Once the solution becomes smooth in space it also becomes smooth in time. In general, however, the solution is not (right) differentiable with respect to t at $t = 0$, and for rough initial data it satisfies the initial condition in an L^p-sense, but not necessarily pointwise.

5.1.3. Irreversibility. For general 'final' data $f \in \mathcal{S}$, we cannot solve the heat equation backward in time to obtain a solution $u : [-T,0] \to \mathcal{S}$, however small we choose $T > 0$. The same argument as the one in the proof of Theorem 5.4 implies that any such solution would be given by (5.4). If, for example, we take $f \in \mathcal{S}$ such that

$$\hat{f}(k) = e^{-\sqrt{1+|k|^2}}$$

then the corresponding solution

$$\hat{u}(k,t) = e^{-t|k|^2 - \sqrt{1+|k|^2}}$$

grows exponentially as $|k| \to \infty$ for every $t < 0$, and therefore $u(t)$ does not belong to $\mathcal{S}$ (or even $\mathcal{S}'$). Physically, this means that the temperature distribution f cannot arise by thermal diffusion from any previous temperature distribution in $\mathcal{S}$ (or $\mathcal{S}'$). The heat equation does, however, have a backward uniqueness property, meaning that if f arises from a previous temperature distribution, then (under appropriate assumptions) that distribution is unique [**9**].

Equivalently, making the time-reversal $t \mapsto -t$, we see that Schwartz-valued solutions of the initial value problem for the backward heat equation

$$u_t = -\Delta u \quad t > 0, \qquad u(x,0) = f(x)$$

do not exist for every $f \in \mathcal{S}$. Moreover, there is a loss of continuous dependence of the solution on the data.

Example 5.6. Consider the one-dimensional heat equation $u_t = u_{xx}$ with initial data

$$f_n(x) = e^{-n}\sin(nx)$$

and corresponding solution

$$u_n(x,t) = e^{-n}\sin(nx)e^{n^2 t}.$$

Then $f_n \to 0$ uniformly together with of all its spatial derivatives as $n \to \infty$, but

$$\sup_{x\in\mathbb{R}} |u_n(x,t)| \to \infty$$

as $n \to \infty$ for any $t > 0$. Thus, the solution does not depend continuously on the initial data in $C_b^\infty(\mathbb{R}^n)$. Multiplying the initial data f_n by e^{-x^2}, we can get an example of the loss of continuous dependence in $\mathcal{S}$.

It is possible to obtain a well-posed initial value problem for the backward heat equation by restricting the initial data to a small enough space with a strong enough norm — for example, to a suitable Gevrey space of C^∞-functions whose spatial derivatives decay at a sufficiently fast rate as their order tends to infinity. These restrictions, however, limit the size of derivatives of all orders, and they are too severe to be useful in applications.

Nevertheless, the backward heat equation is of interest as an inverse problem, namely: Find the temperature distribution at a previous time that gives rise to an observed temperature distribution at the present time. There is a loss of continuous dependence in any reasonable function space for applications, because thermal diffusion damps out large, rapid variations in a previous temperature distribution leading to an imperceptible effect on an observed distribution. Special methods — such as Tychonoff regularization — must be used to formulate such ill-posed inverse problems and develop numerical schemes to solve them.[1]

[1]J. B. Keller, Inverse Problems, *Amer. Math. Month.* **83** (1976) illustrates the difficulty of inverse problems in comparison with the corresponding direct problems by the example of guessing the question to which the answer is "Nine W." The solution is given at the end of this section.

5.1.4. Nonuniqueness. A solution $u(x,t)$ of the initial value problem for the heat equation on $\mathbb{R}^n$ is not unique without the imposition of a suitable growth condition as $|x| \to \infty$. In the above analysis, this was provided by the requirement that $u(\cdot,t) \in \mathcal{S}$, but the much weaker condition that u grows more slowly than $Ce^{a|x|^2}$ as $|x| \to \infty$ for some constants C, a is sufficient to imply uniqueness [**9**].

Example 5.7. Consider, for simplicity, the one-dimensional heat equation

$$u_t = u_{xx}.$$

As observed by Tychonoff (*c.f.* [**21**]), a formal power series expansion with respect to x gives the solution

$$u(x,t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(t)x^{2n}}{(2n)!}$$

for some function $g \in C^\infty(\mathbb{R}^+)$. We can construct a nonzero solution with zero initial data by choosing $g(t)$ to be a nonzero C^∞-function all of whose derivatives vanish at $t = 0$ in such a way that this series converges uniformly for x in compact subsets of $\mathbb{R}$ and $t > 0$ to a solution of the heat equation. This is the case, for example, if

$$g(t) = \exp\left(-\frac{1}{t^2}\right).$$

The resulting solution, however, grows very rapidly as $|x| \to \infty$.

A physical interpretation of this nonuniqueness it is that heat can diffuse from infinity into an unbounded region of initially zero temperature if the solution grows sufficiently quickly. Mathematically, the nonuniqueness is a consequence of the the fact that the initial condition is imposed on a characteristic surface $t = 0$ of the heat equation, meaning that the heat equation does not determine the second-order normal (time) derivative u_{tt} on $t = 0$ in terms of the second-order tangential (spatial) derivatives u, Du, D^2u.

According to the Cauchy-Kowalewski theorem [**14**], any non-characteristic Cauchy problem with analytic initial data has a unique local analytic solution. If $t \in \mathbb{R}$ denotes the normal variable and $x \in \mathbb{R}^n$ the transverse variable, then in solving the PDE by a power series expansion in t we exchange one t-derivative for one x-derivative and the convergence of the Taylor series in x for the analytic initial data implies the convergence of the series for the solution in t. This existence and uniqueness fails for a characteristic initial value problem, such as the one for the heat equation.

The Cauchy-Kowalewski theorem is not as useful as its apparent generality suggests because it does not imply anything about the stability or existence of solutions under non-analytic perturbations, even arbitrarily smooth ones. For example, the Cauchy-Kowalewski theorem is equally applicable to the initial value problem for the wave equation

$$u_{tt} = u_{xx}, \qquad u(x,0) = f(x),$$

which is well-posed in every Sobolev space $H^s(\mathbb{R})$, and the initial value problem for the Laplace equation

$$u_{tt} = -u_{xx}, \qquad u(x,0) = f(x),$$

which is ill-posed in every Sobolev space $H^s(\mathbb{R})$.[2]

5.2. Generalized solutions

In this section we obtain generalized solutions of the initial value problem of the heat equation as a limit of the smooth solutions constructed above. In order to do this, we require estimates on the smooth solutions which ensure that the convergence of initial data in suitable norms implies the convergence of the corresponding solution.

5.2.1. Estimates for the Heat equation. Solutions of the heat equation satisfy two basic spatial estimates, one in L^2 and the L^∞. The L^2 estimate follows from the Fourier representation, and the L^1 estimate follows from the spatial representation. For $1 \le p < \infty$, we let

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f|^p \, dx\right)^{1/p}$$

denote the spatial L^p-norm of a function f; also $\|f\|_{L^\infty}$ denotes the maximum or essential supremum of $|f|$.

Theorem 5.8. *Let $u : [0,\infty) \to \mathcal{S}(\mathbb{R}^n)$ be the solution of (5.2) constructed in Theorem 5.4 and $t > 0$. Then*

$$\|u(t)\|_{L^2} \le \|f\|_{L^2}, \qquad \|u(t)\|_{L^\infty} \le \frac{1}{(4\pi t)^{n/2}} \|f\|_{L^1}.$$

Proof. By Parseval's inequality and (5.4),

$$\|u(t)\|_{L^2} = (2\pi)^n \|\hat{u}(t)\|_{L^2} = (2\pi)^n \left\| e^{-t|k|^2} \hat{f} \right\|_{L^2} \le (2\pi)^n \|\hat{f}\|_{L^2} = \|f\|_{L^2},$$

which gives the first inequality. From (5.5),

$$|u(x,t)| \le \left(\sup_{x\in\mathbb{R}^n} |\Gamma(x,t)|\right) \int_{\mathbb{R}^n} |f(y)| \, dy,$$

and from (5.6)

$$|\Gamma(x,t)| = \frac{1}{(4\pi t)^{n/2}}.$$

The second inequality then follows. □

Using the Riesz-Thorin theorem, Theorem 5.72, it follows by interpolation between $(p,p') = (2,2)$ and $(p,p') = (\infty,1)$, that for $2 \le p \le \infty$

$$\|u(t)\|_{L^p} \le \frac{1}{(4\pi t)^{n(1/2-1/p)}} \|f\|_{L^{p'}}. \tag{5.10}$$

This estimate is not particularly useful for the heat equation, because we can derive stronger parabolic estimates for $\|Du\|_{L^2}$, but the analogous estimate for the Schrödinger equation is very useful.

A generalization of the L^2-estimate holds in any Sobolev space H^s of functions with s spatial L^2-derivatives (see Section 5.C for their definition). Such estimates of L^2-norms of solutions or their derivative are typically referred to as energy estimates, although the corresponding L^2-norms may not correspond to a physical

[2]Finally, here is the question to the answer posed above: Do you spell your name with a "V," Herr Wagner?

energy. In the case of the heat equation, the thermal energy (measured from a zero-point energy at $u = 0$) is proportional to the integral of u.

THEOREM 5.9. *Suppose that $f \in \mathcal{S}$ and $u \in C^\infty([0,\infty);\mathcal{S})$ is the solution of (5.2). Then for any $s \in \mathbb{R}$ and $t \geq 0$*

$$\|u(t)\|_{H^s} \leq \|f\|_{H^s}.$$

PROOF. Using (5.4) and Parseval's identity, and writing $\langle k\rangle = (1+|k|^2)^{1/2}$, we find that

$$\|u(t)\|_{H^s} = (2\pi)^n \left\|\langle k\rangle^s e^{-t|k|^2}\hat{f}\right\|_{L^2} \leq (2\pi)^n \left\|\langle k\rangle^s \hat{f}\right\|_{L^2} = \|f\|_{H^s}.$$

□

We can also derive this H^s-estimate, together with an additional a space-time estimate for Du, directly from the equation without using the explicit solution. We will use this estimate later to construct solutions of a general parabolic PDE by the Galerkin method, so we derive it here directly.

For $1 \leq p < \infty$ and $T > 0$, the L^p-in-time-H^s-in-space norm of a function $u \in C([0,T];\mathcal{S})$ is given by

$$\|u\|_{L^p([0,T];H^s)} = \left(\int_0^T \|u(t)\|_{H^s}^p \, dt\right)^{1/p}.$$

The maximum-in-time-H^s-in-space norm of u is

$$\|u\|_{C([0,T];H^s)} = \max_{t\in[0,T]} \|u(t)\|_{H^s}. \tag{5.11}$$

In particular, if $\Lambda = (I-\Delta)^{1/2}$ is the spatial operator defined in (5.75), then

$$\|u\|_{L^2([0,T];H^s)} = \left(\int_0^T \int_{\mathbb{R}^n} |\Lambda^s u(x,t)|^2 \, dxdt\right)^{1/2}.$$

THEOREM 5.10. *Suppose that $f \in \mathcal{S}$ and $u \in C^\infty([0,T];\mathcal{S})$ is the solution of (5.2). Then for any $s \in \mathbb{R}$*

$$\|u\|_{C([0,T];H^s)} \leq \|f\|_{H^s}, \qquad \|Du\|_{L^2([0,T];H^s)} \leq \frac{1}{\sqrt{2}}\|f\|_{H^s}.$$

PROOF. Let $v = \Lambda^s u$. Then, since $\Lambda^s : \mathcal{S} \to \mathcal{S}$ is continuous and commutes with Δ,

$$v_t = \Delta v, \qquad v(0) = g$$

where $g = \Lambda^s f$. Multiplying this equation by v, integrating the result over $\mathbb{R}^n$, and using the divergence theorem (justified by the continuous differentiability in time and the smoothness and decay in space of v), we get

$$\frac{1}{2}\frac{d}{dt}\int v^2 \, dx = -\int |Dv|^2 \, dx.$$

Integrating this equation with respect to t, we obtain for any $T > 0$ that

$$\frac{1}{2}\int v^2(T)\, dx + \int_0^T \int |Dv(t)|^2 \, dxdt = \frac{1}{2}\int g^2 \, dx. \tag{5.12}$$

Thus,

$$\max_{t\in[0,T]} \int v^2(t)\,dx \leq \int g^2\,dx, \qquad \int_0^T \int |Dv(t)|^2\,dxdt \leq \frac{1}{2}\int g^2\,dx,$$

and the result follows. $\square$

5.2.2. H^s-solutions. In this section we use the above estimates to obtain generalized solutions of the heat equation as a limit of smooth solutions (5.5). In defining generalized solutions, it is convenient to restrict attention to a finite, but arbitrary, time-interval $[0,T]$ where $T>0$. For $s\in\mathbb{R}$, let $C([0,T];H^s)$ denote the Banach space of continuous H^s-valued functions $u:[0,T]\to H^s$ equipped with the norm (5.11).

DEFINITION 5.11. Suppose that $T>0$, $s\in\mathbb{R}$ and $f\in H^s$. A function

$$u\in C\left([0,T];H^s\right)$$

is a generalized solution of (5.2) if there exists a sequence of Schwartz-solutions $u_n:[0,T]\to\mathcal{S}$ such that $u_n\to u$ in $C([0,T];H^s)$ as $n\to\infty$.

According to the next theorem, there is a unique generalized solution defined on any time interval $[0,T]$ and therefore on $[0,\infty)$.

THEOREM 5.12. *Suppose that $T>0$, $s\in\mathbb{R}$ and $f\in H^s(\mathbb{R}^n)$. Then there is a unique generalized solution $u\in C([0,T];H^s)$ of (5.2). The solution is given by (5.4).*

PROOF. Since $\mathcal{S}$ is dense in H^s, there is a sequence of functions $f_n\in\mathcal{S}$ such that $f_n\to f$ in H^s. Let $u_n\in C([0,T];\mathcal{S})$ be the solution of (5.2) with initial data f_n. Then, by linearity, u_n-u_m is the solution with initial data f_n-f_m, and Theorem 5.9 implies that

$$\sup_{t\in[0,T]} \|u_n(t)-u_m(t)\|_{H^s} \leq \|f_n-f_m\|_{H^s}.$$

Hence, $\{u_n\}$ is a Cauchy sequence in $C([0,T];H^s)$ and therefore there exists a generalized solution $u\in C([0,T];H^s)$ such that $u_n\to u$ as $n\to\infty$.

Suppose that $f,g\in H^s$ and $u,v\in C([0,T];H^s)$ are generalized solutions with $u(0)=f$, $v(0)=g$. If $u_n,v_n\in C([0,T];\mathcal{S})$ are approximate solutions with $u_n(0)=f_n$, $v_n(0)=g_n$, then

$$\begin{aligned}\|u(t)-v(t)\|_{H^s} &\leq \|u(t)-u_n(t)\|_{H^s}+\|u_n(t)-v_n(t)\|_{H^s}+\|v_n(t)-v(t)\|_{H^s}\\ &\leq \|u(t)-u_n(t)\|_{H^s}+\|f_n-g_n\|_{H^s}+\|v_n(t)-v(t)\|_{H^s}\end{aligned}$$

Taking the limit of this inequality as $n\to\infty$, we find that

$$\|u(t)-v(t)\|_{H^s}\leq\|f-g\|_{H^s}.$$

In particular, if $f=g$ then $u=v$, so a generalized solution is unique.

Finally, from (5.4) we have

$$\hat{u}_n(k,t)=e^{-t|k|^2}\hat{f}_n(k).$$

Taking the limit of this expression in $C([0,T];\hat{H}^s)$ as $n\to\infty$, where $\hat{H}^s$ is the weighted L^2-space (5.74), we get the same expression for $\hat{u}$. $\square$

We may obtain additional regularity of generalized solutions in time by use of the equation; roughly speaking, we can trade two space-derivatives for one time-derivative.

PROPOSITION 5.13. *Suppose that* $T > 0$, $s \in \mathbb{R}$ *and* $f \in H^s(\mathbb{R}^n)$. *If* $u \in C([0,T];H^s)$ *is a generalized solution of (5.2), then* $u \in C^1([0,T];H^{s-2})$ *and*

$$u_t = \Delta u \qquad \text{in } C([0,T];H^{s-2}).$$

PROOF. Since u is a generalized solution, there is a sequence of smooth solutions $u_n \in C^\infty([0,T];\mathcal{S})$ such that $u_n \to u$ in $C([0,T];H^s)$ as $n \to \infty$. These solutions satisfy $u_{nt} = \Delta u_n$. Since $\Delta : H^s \to H^{s-2}$ is bounded and $\{u_n\}$ is Cauchy in H^s, we see that $\{u_{nt}\}$ is Cauchy in $C([0,T];H^{s-2})$. Hence there exists $v \in C([0,T];H^{s-2})$ such that $u_{nt} \to v$ in $C([0,T];H^{s-2})$. We claim that $v = u_t$. For each $n \in \mathbb{N}$ and $h \neq 0$ we have

$$\frac{u_n(t+h) - u_n(t)}{h} = \frac{1}{h}\int_t^{t+h} u_{ns}(s)\, ds \qquad \text{in } C([0,T];\mathcal{S}),$$

and in the limit $n \to \infty$, we get that

$$\frac{u(t+h) - u(t)}{h} = \frac{1}{h}\int_t^{t+h} v(s)\, ds \qquad \text{in } C([0,T];H^{s-2}).$$

Taking the limit as $h \to 0$ of this equation we find that $u_t = v$ and

$$u \in C([0,T];H^s) \cap C^1([0,T];H^{s-2}).$$

Moreover, taking the limit of $u_{nt} = \Delta u_n$ we get $u_t = \Delta u$ in $C([0,T];H^{s-2})$. □

More generally, a similar argument shows that $u \in C^k([0,T];H^{s-2k})$ for any $k \in \mathbb{N}$. In contrast with the case of ODEs, the time derivative of the solution lies in a different space than the solution itself: u takes values in H^s, but u_t takes values in H^{s-2}. This feature is typical for PDEs when — as is usually the case — one considers solutions that take values in Banach spaces whose norms depend on only finitely many derivatives. It did not arise for Schwartz-valued solutions, since differentiation is a continuous operation on $\mathcal{S}$.

The above proposition did not use any special properties of the heat equation. For $t > 0$, solutions have greatly improved regularity as a result of the smoothing effect of the evolution.

PROPOSITION 5.14. *If* $u \in C([0,T];H^s)$ *is a generalized solution of (5.2), where* $f \in H^s$ *for some* $s \in \mathbb{R}$, *then* $u \in C^\infty((0,T];H^\infty)$ *where* H^∞ *is defined in (5.76).*

PROOF. If $s \in \mathbb{R}$, $f \in H^s$, and $t > 0$, then (5.4) implies that $\hat{u}(t) \in \hat{H}^r$ for every $r \in \mathbb{R}$, and therefore $u(t) \in H^\infty$. It follows from the equation that $u \in C^\infty(0,\infty;H^\infty)$. □

For general H^s-initial data, however, we cannot expect any improved regularity in time at $t = 0$ beyond $u \in C^k([0,T);H^{s-2k})$. The H^∞ spatial regularity stated here is not optimal; for example, one can prove [**9**] that the solution is a real-analytic function of x for $t > 0$, although it is not necessarily a real-analytic function of t.

5.3. The Schrödinger equation

The initial value problem for the Schrödinger equation is

$$
\begin{aligned}
iu_t &= -\Delta u \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}, \\
u(x,0) &= f(x) \qquad \text{for } x \in \mathbb{R}^n,
\end{aligned}
\tag{5.13}
$$

where $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is a complex-valued function. A solution of the Schrödinger equation is the amplitude function of a quantum mechanical particle moving freely in $\mathbb{R}^n$. The function $|u(\cdot,t)|^2$ is proportional to the spatial probability density of the particle.

More generally, a particle moving in a potential $V : \mathbb{R}^n \to \mathbb{R}$ satisfies the Schrödinger equation

$$
iu_t = -\Delta u + V(x)u. \tag{5.14}
$$

Unlike the free Schrödinger equation (5.13), this equation has variable coefficients and it cannot be solved explicitly for general potentials V.

Formally, the Schrödinger equation (5.13) is obtained by the transformation $t \mapsto -it$ of the heat equation to 'imaginary time.' The analytical properties of the heat and Schrödinger equations are, however, completely different and it is interesting to compare them. The proofs are similar, and we leave them as an exercise (or see [**34**]).

The Fourier solution of (5.13) is

$$
\hat{u}(k,t) = e^{-it|k|^2} \hat{f}(k). \tag{5.15}
$$

The key difference from the heat equation is that these Fourier modes oscillate instead of decay in time, and higher wavenumber modes oscillate faster in time. As a result, there is no smoothing of the initial data (measuring smoothness in the L^2-scale of Sobolev spaces H^s) and we can solve the Schrödinger equation both forward and backward in time.

Theorem 5.15. *For any $f \in \mathcal{S}$ there is a unique solution $u \in C^\infty(\mathbb{R};\mathcal{S})$ of (5.13). The spatial Fourier transform of the solution is given by (5.15), and*

$$
u(x,t) = \int \Gamma(x-y,t) f(y)\, dy
$$

where

$$
\Gamma(x,t) = \frac{1}{(4\pi i t)^{n/2}}\, e^{-i|x|^2/4t}.
$$

We get analogous L^p estimates for the Schrödinger equation to the ones for the heat equation.

Theorem 5.16. *Suppose that $f \in \mathcal{S}$ and $u \in C^\infty(\mathbb{R};\mathcal{S})$ is the solution of (5.13). Then for all $t \in \mathbb{R}$,*

$$
\|u(t)\|_{L^2} \le \|f\|_{L^2}, \qquad \|u(t)\|_{L^\infty} \le \frac{1}{(4\pi|t|)^{n/2}} \|f\|_{L^1},
$$

and for $2 < p < \infty$,

$$
\|u(t)\|_{L^p} \le \frac{1}{(4\pi|t|)^{n(1/2-1/p)}} \|f\|_{L^{p'}}. \tag{5.16}
$$

Solutions of the Schrödinger equation do not satisfy a space-time estimate analogous to the parabolic estimate (5.12) in which we 'gain' a spatial derivative. Instead, we get only that the H^s-norm is conserved. Solutions do satisfy a weaker space-time estimate, called a Strichartz estimate, which we derive in Section 5.6.1.

The conservation of the H^s-norm follows from the Fourier representation (5.15), but let us prove it directly from the equation.

THEOREM 5.17. *Suppose that $f \in \mathcal{S}$ and $u \in C^\infty(\mathbb{R};\mathcal{S})$ is the solution of (5.13). Then for any $s \in \mathbb{R}$*

$$\|u(t)\|_{H^s} = \|f\|_{H^s} \qquad \text{for every } t \in \mathbb{R}.$$

PROOF. Let $v = \Lambda^s u$, so that $\|u(t)\|_{H^s} = \|v(t)\|_{L^2}$. Then

$$iv_t = -\Delta v$$

and $v(0) = \Lambda^s f$. Multiplying this PDE by the conjugate $\bar{v}$ and subtracting the complex conjugate of the result, we get

$$i\left(\bar{v}v_t + v\bar{v}_t\right) = v\Delta\bar{v} - \bar{v}\Delta v.$$

We may rewrite this equation as

$$\partial_t|v|^2 + \nabla\cdot\left[i\left(vD\bar{v} - \bar{v}Dv\right)\right] = 0.$$

If $v = u$, this is the equation of conservation of probability where $|u|^2$ is the probability density and $i\left(uD\bar{u} - \bar{u}Du\right)$ is the probability flux. Integrating the equation over $\mathbb{R}^n$ and using the spatial decay of v, we get

$$\frac{d}{dt}\int |v|^2\,dx = 0,$$

and the result follows. □

We say that a function $u \in C\left(\mathbb{R}; H^s\right)$ is a generalized solution of (5.13) if it is the limit of smooth Schwartz-valued solutions uniformly on compact time intervals. The existence of such solutions follows from the preceding H^s-estimates for smooth solutions.

THEOREM 5.18. *Suppose that $s \in \mathbb{R}$ and $f \in H^s(\mathbb{R}^n)$. Then there is a unique generalized solution $u \in C\left(\mathbb{R}; H^s\right)$ of (5.13) given by*

$$\hat{u}(k) = e^{-it|k|^2}\hat{f}(k).$$

Moreover, for any $k \in \mathbb{N}$, we have $u \in C^k\left(\mathbb{R}; H^{s-2k}\right)$.

Unlike the heat equation, there is no smoothing of the solution and there is no H^s-regularity for $t \neq 0$ beyond what is stated in this theorem.

5.4. Semigroups and groups

The solution of an $n \times n$ linear first-order system of ODEs for $\vec{u}(t) \in \mathbb{R}^n$,

$$\vec{u}_t = A\vec{u},$$

may be written as

$$\vec{u}(t) = e^{tA}\vec{u}(0) \qquad -\infty < t < \infty$$

where $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ is the matrix exponential of tA. The finite-dimensionality of the phase space $\mathbb{R}^n$ is not crucial here. As we discuss next, similar results hold for any linear ODE in a Banach space generated by a bounded linear operator.

5.4.1. Uniformly continuous groups. Suppose that X is a Banach space. We denote by $\mathcal{L}(X)$ the Banach space of bounded linear operators $A : X \to X$ equipped with the operator norm

$$\|A\|_{\mathcal{L}(X)} = \sup_{u \in X\setminus\{0\}} \frac{\|Au\|_X}{\|u\|_X}.$$

We say that a sequence of bounded linear operators converges uniformly if it converges with respect to the operator norm.

For $A \in \mathcal{L}(X)$ and $t \in \mathbb{R}$, we define the operator exponential by the series

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \cdots + \frac{1}{n!}A^n + \ldots. \tag{5.17}$$

This operator is well-defined. Its properties are similar to those of the real-valued exponential function e^{at} for $a \in \mathbb{R}$ and are proved in the same way.

THEOREM 5.19. *If $A \in \mathcal{L}(X)$ and $t \in \mathbb{R}$, then the series in (5.17) converges uniformly in $\mathcal{L}(X)$. Moreover, the function $t \mapsto e^{tA}$ belongs to $C^\infty(\mathbb{R}; \mathcal{L}(X))$ and for every $s, t \in \mathbb{R}$*

$$e^{sA}e^{tA} = e^{(s+t)A}, \qquad \frac{d}{dt}e^{tA} = Ae^{tA}.$$

Consider a linear homogeneous initial value problem

$$u_t = Au, \qquad u(0) = f \in X, \qquad u \in C^1(\mathbb{R}; X). \tag{5.18}$$

The solution is given by the operator exponential.

THEOREM 5.20. *The unique solution $u \in C^\infty(\mathbb{R}; X)$ of (5.18) is given by*

$$u(t) = e^{tA}f.$$

EXAMPLE 5.21. For $1 \le p < \infty$, let $A : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be the bounded translation operator

$$Af(x) = f(x+1).$$

The solution $u \in C^\infty(\mathbb{R}; L^p)$ of the differential-difference equation

$$u_t(x,t) = u(x+1,t), \qquad u(x,0) = f(x)$$

is given by

$$u(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f(x+n).$$

EXAMPLE 5.22. Suppose that $a \in L^1(\mathbb{R}^n)$ and define the bounded convolution operator $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by $Af = a * f$. Consider the IVP

$$u_t(x,t) = \int_{\mathbb{R}^n} a(x-y)u(y)\,dy, \qquad u(x,0) = f(x) \in L^2(\mathbb{R}^n).$$

Taking the Fourier transform of this equation and using the convolution theorem, we get

$$\hat{u}_t(k,t) = (2\pi)^n \hat{a}(k)\hat{u}(k,t), \qquad \hat{u}(k,0) = \hat{f}(k).$$

The solution is

$$\hat{u}(k,t) = e^{(2\pi)^n \hat{a}(k)t}\hat{f}(k).$$

It follows that

$$u(x,t) = \int g(x-y,t)f(y)\,dy$$

where the Fourier transform of $g(x,t)$ is given by

$$\hat{g}(k,t) = \frac{1}{(2\pi)^n} e^{(2\pi)^n \hat{a}(k) t}.$$

Since $a \in L^1(\mathbb{R}^n)$, the Riemann-Lebesgue lemma implies that $\hat{a} \in C_0(\mathbb{R}^n)$, and therefore $\hat{g}(\cdot,t) \in C_b(\mathbb{R}^n)$ for every $t \in \mathbb{R}$. Since convolution with g corresponds to multiplication of the Fourier transform by a bounded multiplier, it defines a bounded linear map on $L^2(\mathbb{R}^n)$.

The solution operators $\mathrm{T}(t) = e^{tA}$ of (5.18) form a uniformly continuous one-parameter group. Conversely, any uniformly continuous one-parameter group of transformations on a Banach space is generated by a bounded linear operator.

Definition 5.23. Let X be a Banach space. A one-parameter, uniformly continuous group on X is a family $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ of bounded linear operators $\mathrm{T}(t) : X \to X$ such that:

(1) $\mathrm{T}(0) = I$;
(2) $\mathrm{T}(s)\mathrm{T}(t) = \mathrm{T}(s+t)$ for all $s, t \in \mathbb{R}$;
(3) $\mathrm{T}(h) \to I$ uniformly in $\mathcal{L}(X)$ as $h \to 0$.

Theorem 5.24. *If $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ is a uniformly continuous group on a Banach space X, then:*

(1) $\mathrm{T} \in C^\infty(\mathbb{R}; \mathcal{L}(X))$;
(2) $A = \mathrm{T}_t(0)$ *is a bounded linear operator on* X;
(3) $\mathrm{T}(t) = e^{tA}$ *for every* $t \in \mathbb{R}$.

Note that the differentiability (and, in fact, the analyticity) of $\mathrm{T}(t)$ with respect to t is implied by its continuity and the group property $\mathrm{T}(s)\mathrm{T}(t) = \mathrm{T}(s+t)$. This is analogous to what happens for the real exponential function: The only continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation

$$f(0) = 1, \qquad f(s)f(t) = f(s+t) \quad \text{for all } s, t \in \mathbb{R} \tag{5.19}$$

are the exponential functions $f(t) = e^{at}$ for $a \in \mathbb{R}$, and these functions are analytic.

Some regularity assumption on f is required in order for (5.19) to imply that f is an exponential function. If we drop the continuity assumption, then the function defined by $f(0) = 1$ and $f(t) = 0$ for $t \neq 0$ also satisfies (5.19). This function and the exponential functions are the only Lebesgue measurable solutions of (5.19). If we drop the measurability requirement, then we get many other solutions.

Example 5.25. If $f = e^g$ where $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$g(0) = 0, \qquad g(s) + g(t) = g(s+t),$$

then f satisfied (5.19). The linear functions $g(t) = at$ satisfy this functional equation for any $a \in \mathbb{R}$, but there are many other non-measurable solutions. To "construct" examples, consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ of rational numbers, and let $\{e_\alpha \in \mathbb{R} : \alpha \in I\}$ denote an algebraic basis. Given any values $\{c_\alpha \in \mathbb{R} : \alpha \in I\}$ define $g : \mathbb{R} \to \mathbb{R}$ such that $g(e_\alpha) = c_\alpha$ for each $\alpha \in I$, and if $x = \sum x_\alpha e_\alpha$ is the finite expansion of $x \in \mathbb{R}$ with respect to the basis, then

$$g\left(\sum x_\alpha e_\alpha\right) = \sum x_\alpha c_\alpha.$$

5.4.2. Strongly continuous semigroups. We may consider the heat equation and other linear evolution equations from a similar perspective to the Banach space ODEs discussed above. Significant differences arise, however, as a result of the fact that the Laplacian and other spatial differential operators are unbounded maps of a Banach space into itself. In particular, the solution operators associated with unbounded operators are strongly but not uniformly continuous functions of time, and we get solutions that are, in general, continuous but not continuously differentiable. Moreover, as in the case of the heat equation, we may only be able to solve the equation forward in time, which gives us a semigroup of solution operators instead of a group.

Abstracting the notion of a family of solution operators with continuous trajectories forward in time, we are led to the following definition.

DEFINITION 5.26. Let X be a Banach space. A one-parameter, strongly continuous (or C_0) semigroup on X is a family $\{\mathrm{T}(t) : t \geq 0\}$ of bounded linear operators $\mathrm{T}(t) : X \to X$ such that

(1) $\mathrm{T}(0) = I$;
(2) $\mathrm{T}(s)\mathrm{T}(t) = \mathrm{T}(s+t)$ for all $s, t \geq 0$;
(3) $\mathrm{T}(h)f \to f$ strongly in X as $h \to 0^+$ for every $f \in X$.

The semigroup is said to be a contraction semigroup if $\|\mathrm{T}(t)\| \leq 1$ for all $t \geq 0$, where $\|\cdot\|$ denotes the operator norm.

The semigroup property (2) holds for the solution maps of any well-posed autonomous evolution equation: it says simply that we can solve for time $s+t$ by solving for time t and then for time s. Condition (3) means explicitly that

$$\|\mathrm{T}(t)f - f\|_X \to 0 \qquad \text{as } t \to 0^+.$$

If this holds, then the semigroup property (2) implies that $\mathrm{T}(t+h)f \to \mathrm{T}(t)f$ in X as $h \to 0$ for every $t > 0$, not only for $t = 0$ [**8**]. The term 'contraction' in Definition 5.26 is not used in a strict sense, and the norm of the solution of a contraction semigroup is not required to be strictly decreasing in time; it may for example, remain constant.

The heat equation

$$u_t = \Delta u, \qquad u(x,0) = f(x) \tag{5.20}$$

is one of the primary motivating examples for the theory of semigroups. For definiteness, we suppose that $f \in L^2$, but we could also define a heat-equation semigroup on other Hilbert or Banach spaces, such as H^s or L^p for $1 < p < \infty$.

From Theorem 5.12 with $s = 0$, for every $f \in L^2$ there is a unique generalized solution $u : [0,\infty) \to L^2$ of (5.20), and therefore for each $t \geq 0$ we may define a bounded linear map $\mathrm{T}(t) : L^2 \to L^2$ by $\mathrm{T}(t) : f \mapsto u(t)$. The operator $\mathrm{T}(t)$ is defined explicitly by

$$\begin{aligned} &\mathrm{T}(0) = I, \quad \mathrm{T}(t)f = \Gamma^t * f \qquad \text{for } t > 0, \\ &\widehat{\mathrm{T}(t)f}(k) = e^{-t|k|^2}\hat{f}(k). \end{aligned} \tag{5.21}$$

where the $*$ denotes spatial convolution with the Green's function $\Gamma^t(x) = \Gamma(x,t)$ given in (5.6).

We also use the notation

$$\mathrm{T}(t) = e^{t\Delta}$$

and interpret $\mathrm{T}(t)$ as the operator exponential of $t\Delta$. The semigroup property then becomes the usual exponential formula

$$e^{(s+t)\Delta} = e^{s\Delta} e^{t\Delta}.$$

THEOREM 5.27. *The solution operators* $\{\mathrm{T}(t) : t \geq 0\}$ *of the heat equation defined in (5.21) form a strongly continuous contraction semigroup on* $L^2(\mathbb{R}^n)$.

PROOF. This theorem is a restatement of results that we have already proved, but let us verify it explicitly. The semigroup property follows from the Fourier representation, since

$$e^{-(s+t)|k|^2} = e^{-s|k|^2} e^{-t|k|^2}.$$

It also follows from the spatial representation, since

$$\Gamma^{s+t} = \Gamma^s * \Gamma^t.$$

The probabilistic interpretation of this identity is that the sum of independent Gaussian random variables is a Gaussian random variable, and the variance of the sum is the sum of the variances.

Theorem 5.12, with $s = 0$, implies that the semigroup is strongly continuous since $t \mapsto \mathrm{T}(t)f$ belongs to $C\left([0,\infty); L^2\right)$ for every $f \in L^2$. Finally, it is immediate from (5.21) and Parseval's theorem that $\|\mathrm{T}(t)\| \leq 1$ for every $t \geq 0$, so the semigroup is a contraction semigroup. □

An alternative way to view this result is that the solution maps

$$\mathrm{T}(t) : \mathcal{S} \subset L^2 \to \mathcal{S} \subset L^2$$

constructed in Theorem 5.4 are defined on a dense subspace $\mathcal{S}$ of L^2, and are bounded on L^2, so they extend to bounded linear maps $\mathrm{T}(t) : L^2 \to L^2$, which form a strongly continuous semigroup.

Although for every $f \in L^2$ the trajectory $t \mapsto \mathrm{T}(t)f$ is a continuous function from $[0,\infty)$ into L^2, it is *not* true that $t \mapsto \mathrm{T}(t)$ is a continuous map from $[0,\infty)$ into the space $\mathcal{L}(L^2)$ of bounded linear maps on L^2 since $\mathrm{T}(t+h)$ does not converge to $\mathrm{T}(t)$ as $h \to 0$ uniformly with respect to the operator norm.

Proposition 5.13 implies a solution $t \mapsto \mathrm{T}(t)f$ belongs to $C^1\left([0,\infty); L^2\right)$ if $f \in H^2$, but for $f \in L^2 \setminus H^2$ the solution is not differentiable with respect to t in L^2 at $t = 0$. For every $t > 0$, however, we have from Proposition 5.14 that the solution belongs to $C^\infty(0,\infty; H^\infty)$. Thus, the the heat equation semiflow maps the entire phase space L^2 forward in time into a dense subspace H^∞ of smooth functions. As a result of this smoothing, we cannot reverse the flow to obtain a map backward in time of L^2 into itself.

5.4.3. Strongly continuous groups. Conservative wave equations do not smooth solutions in the same way as parabolic equations like the heat equation, and they typically define a group of solution maps both forward and backward in time.

DEFINITION 5.28. Let X be a Banach space. A one-parameter, strongly continuous (or C_0) group on X is a family $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ of bounded linear operators $\mathrm{T}(t) : X \to X$ such that

(1) $\mathrm{T}(0) = I$;
(2) $\mathrm{T}(s)\mathrm{T}(t) = \mathrm{T}(s+t)$ for all $s, t \in \mathbb{R}$;
(3) $\mathrm{T}(h)f \to f$ strongly in X as $h \to 0$ for every $f \in X$.

If X is a Hilbert space and each $\mathrm{T}(t)$ is a unitary operator on X, then the group is said to be a unitary group.

Thus $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ is a strongly continuous group if and only if $\{\mathrm{T}(t) : t \geq 0\}$ is a strongly continuous semigroup of invertible operators and $\mathrm{T}(-t) = \mathrm{T}^{-1}(t)$.

THEOREM 5.29. *Suppose that $s \in \mathbb{R}$. The solution operators $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ of the Schrödinger equation (5.13) defined by*

$$(\widehat{\mathrm{T}(t)f})(k) = e^{-it|k|^2}\hat{f}(k). \tag{5.22}$$

form a strongly continuous, unitary group on $H^s(\mathbb{R}^n)$.

Unlike the heat equation semigroup, the Schrödinger equation is a dispersive wave equation which does not smooth solutions. The solution maps $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ form a group of unitary operators on L^2 which map H^s onto itself (*c.f.* Theorem 5.17). A trajectory $u(t)$ belongs to $C^1(\mathbb{R}; L^2)$ if and only if $u(0) \in H^2$, and $u \in C^k(\mathbb{R}; L^2)$ if and only if $u(0) \in H^{1+k}$. If $u(0) \in L^2 \setminus H^2$, then $u \in C(\mathbb{R}; L^2)$ but u is nowhere strongly differentiable in L^2 with respect to time. Nevertheless, the continuous non-differentiable trajectories remain close in L^2 to the differentiable trajectories. This dense intertwining of smooth trajectories and continuous, non-differentiable trajectories in an infinite-dimensional phase space is not easy to imagine and has no analog for ODEs.

The Schrödinger operators $\mathrm{T}(t) = e^{it\Delta}$ do *not* form a strongly continuous group on $L^p(\mathbb{R}^n)$ when $p \neq 2$. Suppose, for contradiction, that $\mathrm{T}(t) : L^p \to L^p$ is bounded for some $1 \leq p < \infty$, $p \neq 2$ and $t \in \mathbb{R} \setminus \{0\}$. Then since $\mathrm{T}(-t) = T^*(t)$, duality implies that $\mathrm{T}(-t) : L^{p'} \to L^{p'}$ is bounded, and we can assume that $1 \leq p < 2$ without loss of generality. From Theorem 5.16, $\mathrm{T}(t) : L^p \to L^{p'}$ is bounded, and thus for every $f \in L^p \cap L^{p'} \subset L^2$

$$\|f\|_{L^p} = \|\mathrm{T}(t)\mathrm{T}(-t)f\|_{L^p} \leq C_1 \|\mathrm{T}(-t)f\|_{L^{p'}} \leq C_1 C_2 \|f\|_{L^{p'}}.$$

This estimate is false if $p \neq 2$, so $\mathrm{T}(t)$ cannot be bounded on L^p.

5.4.4. Generators. Given an operator A that generates a semigroup, we may define the semigroup $\mathrm{T}(t) = e^{tA}$ as the collection of solution operators of the equation $u_t = Au$. Alternatively, given a semigroup, we may ask for an operator A that generates it.

DEFINITION 5.30. Suppose that $\{\mathrm{T}(t) : t \geq 0\}$ is a strongly continuous semigroup on a Banach space X. The generator A of the semigroup is the linear operator in X with domain $\mathcal{D}(A)$,

$$A : \mathcal{D}(A) \subset X \to X,$$

defined as follows:

(1) $f \in \mathcal{D}(A)$ if and only if the limit

$$\lim_{h \to 0^+} \left[\frac{\mathrm{T}(h)f - f}{h}\right]$$

exists with respect to the strong (norm) topology of X;

(2) if $f \in \mathcal{D}(A)$, then

$$Af = \lim_{h \to 0^+} \left[\frac{\mathrm{T}(h)f - f}{h}\right].$$

To describe which operators are generators of a semigroup, we recall some definitions and results from functional analysis. See [**8**] for further discussion and proofs of the results.

Definition 5.31. An operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X is closed if whenever $\{f_n\}$ is a sequence of points in $\mathcal{D}(A)$ such that $f_n \to f$ and $Af_n \to g$ in X as $n \to \infty$, then $f \in \mathcal{D}(A)$ and $Af = g$.

Equivalently, A is closed if its graph

$$\mathcal{G}(A) = \{(f, g) \in X \times X : f \in \mathcal{D}(A) \text{ and } Af = g\}$$

is a closed subset of $X \times X$.

Theorem 5.32. *If A is the generator of a strongly continuous semigroup $\{\mathrm{T}(t)\}$ on a Banach space X, then A is closed and its domain $\mathcal{D}(A)$ is dense in X.*

Example 5.33. If $\mathrm{T}(t)$ is the heat-equation semigroup on L^2, then the L^2-limit

$$\lim_{h \to 0^+} \left[\frac{\mathrm{T}(h)f - f}{h} \right]$$

exists if and only if $f \in H^2$, and then it is equal to Δf. The generator of the heat equation semigroup on L^2 is therefore the unbounded Laplacian operator with domain H^2,

$$\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

If $f_n \to f$ in L^2 and $\Delta f_n \to g$ in L^2, then the continuity of distributional derivatives implies that $\Delta f = g$ and elliptic regularity theory (or the explicit Fourier representation) implies that $f \in H^2$. Thus, the Laplacian with domain $H^2(\mathbb{R}^n)$ is a closed operator in $L^2(\mathbb{R}^n)$. It is also self-adjoint.

Not every closed, densely defined operator generates a semigroup: the powers of its resolvent must satisfy suitable estimates.

Definition 5.34. Suppose that $A : \mathcal{D}(A) \subset X \to X$ is a closed linear operator in a Banach space X and $\mathcal{D}(A)$ is dense in X. A complex number $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(A)$ of A if $\lambda I - A : \mathcal{D}(A) \subset X \to X$ is one-to-one and onto. If $\lambda \in \rho(A)$, the inverse

$$R(\lambda, A) = (\lambda I - A)^{-1} : X \to X \tag{5.23}$$

is called the resolvent of A.

The open mapping (or closed graph) theorem implies that if A is closed, then the resolvent $R(\lambda, A)$ is a bounded linear operator on X whenever it is defined. This is because $(f, Af) \mapsto \lambda f - Af$ is a one-to-one, onto map from the graph $\mathcal{G}(A)$ of A to X, and $\mathcal{G}(A)$ is a Banach space since it is a closed subset of the Banach space $X \times X$.

The resolvent of an operator A may be interpreted as the Laplace transform of the corresponding semigroup. Formally, if

$$\tilde{u}(\lambda) = \int_0^\infty u(t) e^{-\lambda t}\, dt$$

is the Laplace transform of $u(t)$, then taking the Laplace transform with respect to t of the equation

$$u_t = Au \qquad u(0) = f,$$

we get

$$\lambda\tilde{u} - f = A\tilde{u}.$$

For $\lambda \in \rho(A)$, the solution of this equation is

$$\tilde{u}(\lambda) = R(\lambda, A)f.$$

This solution is the Laplace transform of the time-domain solution

$$u(t) = \mathrm{T}(t)f$$

with $R(\lambda, A) = \widetilde{\mathrm{T}(t)}$, or

$$(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t}e^{tA}\,dt.$$

This identity can be given a rigorous sense for the generators A of a semigroup, and it explains the connection between semigroups and resolvents. The Hille-Yoshida theorem provides a necessary and sufficient condition on the resolvents for an operator to generate a strongly continuous semigroup.

THEOREM 5.35. *A linear operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X is the generator of a strongly continuous semigroup $\{\mathrm{T}(t); t \geq 0\}$ on X if and only if there exist constants $M \geq 1$ and $a \in \mathbb{R}$ such that the following conditions are satisfied:*

(1) *the domain $\mathcal{D}(A)$ is dense in X and A is closed;*
(2) *every $\lambda \in \mathbb{R}$ such that $\lambda > a$ belongs to the resolvent set of A;*
(3) *if $\lambda > a$ and $n \in \mathbb{N}$, then*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - a)^n} \tag{5.24}$$

where the resolvent $R(\lambda, A)$ is defined in (5.23).

In that case,

$$\|\mathrm{T}(t)\| \leq Me^{at} \qquad \textit{for all } t \geq 0. \tag{5.25}$$

This theorem is often not useful in practice because the condition on arbitrary powers of the resolvent is difficult to check. For contraction semigroups, we have the following simpler version.

COROLLARY 5.36. *A linear operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X is the generator of a strongly continuous contraction semigroup $\{\mathrm{T}(t); t \geq 0\}$ on X if and only if:*

(1) *the domain $\mathcal{D}(A)$ is dense in X and A is closed;*
(2) *every $\lambda \in \mathbb{R}$ such that $\lambda > 0$ belongs to the resolvent set of A;*
(3) *if $\lambda > 0$, then*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}. \tag{5.26}$$

This theorem follows from the previous one since

$$\|R(\lambda, A)^n\| \leq \|R(\lambda, A)\|^n \leq \frac{1}{\lambda^n}.$$

The crucial condition here is that $M = 1$. We can always normalize $a = 0$, since if A satisfies Theorem 5.35 with $a = \alpha$, then $A - \alpha I$ satisfies Theorem 5.35 with $a =$

0. Correspondingly, the substitution $u = e^{\alpha t} v$ transforms the evolution equation $u_t = Au$ to $v_t = (A - \alpha I)v$.

The Lumer-Phillips theorem provides a more easily checked condition (that A is 'm-dissipative') for A to generate a contraction semigroup. This condition often follows for PDEs from a suitable energy estimate.

Definition 5.37. A closed, densely defined operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X is dissipative if for every $\lambda > 0$

$$\lambda \|f\| \leq \|(\lambda I - A) f\| \qquad \text{for all } f \in \mathcal{D}(A). \tag{5.27}$$

The operator A is maximally dissipative, or m-dissipative for short, if it is dissipative and the range of $\lambda I - A$ is equal to X for some $\lambda > 0$.

The estimate (5.27) implies immediately that $\lambda I - A$ is one-to-one. It also implies that the range of $\lambda I - A : \mathcal{D}(A) \subset X \to X$ is closed. To see this, suppose that g_n belongs to the range of $\lambda I - A$ and $g_n \to g$ in X. If $g_n = (\lambda I - A) f_n$, then (5.27) implies that $\{f_n\}$ is Cauchy since $\{g_n\}$ is Cauchy, and therefore $f_n \to f$ for some $f \in X$. Since A is closed, it follows that $f \in \mathcal{D}(A)$ and $(\lambda I - A)f = g$. Hence, g belongs to the range of $\lambda I - A$.

The range of $\lambda I - A$ may be a proper closed subspace of X for every $\lambda > 0$; if, however, A is m-dissipative, so that $\lambda I - A$ is onto X for some $\lambda > 0$, then one can prove that $\lambda I - A$ is onto for every $\lambda > 0$, meaning that the resolvent set of A contains the positive real axis $\{\lambda > 0\}$. The estimate (5.27) is then equivalent to (5.26). We therefore get the following result, called the Lumer-Phillips theorem.

Theorem 5.38. *An operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X is the generator of a contraction semigroup on X if and only if:*

1. *A is closed and densely defined;*
2. *A is m-dissipative.*

Example 5.39. Consider $\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. If $f \in H^2$, then using the integration-by-parts property of the weak derivative on H^2 we have for $\lambda > 0$ that

$$\begin{aligned}
\|(\lambda I - \Delta) f\|_{L^2}^2 &= \int (\lambda f - \Delta f)^2 \, dx \\
&= \int \left[\lambda^2 f^2 - 2\lambda f \Delta f + (\Delta f)^2\right] dx \\
&= \int \left[\lambda^2 f^2 + 2\lambda Df \cdot Df + (\Delta f)^2\right] dx \\
&\geq \lambda^2 \int f^2 \, dx.
\end{aligned}$$

Hence,

$$\lambda \|f\|_{L^2} \leq \|(\lambda I - \Delta) f\|_{L^2}$$

and Δ is dissipative. The range of $\lambda I - \Delta$ is equal to L^2 for any $\lambda > 0$, as one can see by use of the Fourier transform (in fact, $I - \Delta$ is an isometry of H^2 onto L^2). Thus, Δ is m-dissipative. The Lumer-Phillips theorem therefore implies that $\Delta : H^2 \subset L^2 \to L^2$ generates a strongly continuous semigroup on $L^2(\mathbb{R}^n)$, as we have seen explicitly by use of the Fourier transform.

Thus, in order to show that an evolution equation

$$u_t = Au \qquad u(0) = f$$

in a Banach space X generates a strongly continuous contraction semigroup, it is sufficient to check that $A : \mathcal{D}(A) \subset X \to X$ is a closed, densely defined, dissipative operator and that for some $\lambda > 0$ the resolvent equation

$$\lambda f - Af = g$$

has a solution $f \in X$ for every $g \in X$.

EXAMPLE 5.40. The linearized Kuramoto-Sivashinsky (KS) equation is

$$u_t = -\Delta u - \Delta^2 u.$$

This equation models a system with long-wave instability, described by the backward heat-equation term $-\Delta u$, and short wave stability, described by the forth-order diffusive term $-\Delta^2 u$. The operator

$$A : H^4(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad Au = -\Delta u - \Delta^2 u$$

generates a strongly continuous semigroup on $L^2(\mathbb{R}^n)$, or $H^s(\mathbb{R}^n)$. One can verify this directly from the Fourier representation,

$$\widehat{\left[e^{tA} f\right]}(k) = e^{t(|k|^2 - |k|^4)} \hat{f}(k),$$

but let us check the hypotheses of the Lumer-Phillips theorem instead. Note that

$$|k|^2 - |k|^4 \leq \frac{3}{16} \qquad \text{for all } |k| \geq 0. \tag{5.28}$$

We claim that $\tilde{A} = A - \alpha I$ is m-dissipative for $\alpha \leq 3/16$. First, $\tilde{A}$ is densely defined and closed, since if $f_n \in H^4$ and $f_n \to f$, $\tilde{A} f_n \to g$ in L^2, the Fourier representation implies that $f \in H^4$ and $\tilde{A} f = g$. If $f \in H^4$, then using (5.28), we have

$$\begin{aligned}
\left\|\lambda f - \tilde{A} f\right\|^2 &= \int_{\mathbb{R}^n} \left(\lambda + \alpha - |k|^2 + |k|^4\right)^2 \left|\hat{f}(k)\right|^2 \, dk \\
&\geq \lambda \int_{\mathbb{R}^n} \left|\hat{f}(k)\right|^2 \, dk \\
&\geq \lambda \|f\|_{L^2}^2,
\end{aligned}$$

which means that $\tilde{A}$ is dissipative. Moreover, $\lambda I - \tilde{A} : H^4 \to L^2$ is one-to-one and onto for any $\lambda > 0$, since $(\lambda I - \tilde{A}) f = g$ if and only if

$$\hat{f}(k) = \frac{\hat{g}(k)}{\lambda + \alpha - |k|^2 + |k|^4}.$$

Thus, $\tilde{A}$ is m-dissipative, so it generates a contraction semigroup on L^2. It follows that A generates a semigroup on $L^2(\mathbb{R}^n)$ such that

$$\left\|e^{tA}\right\|_{\mathcal{L}(L^2)} \leq e^{3t/16},$$

corresponding to $M = 1$ and $a = 3/16$ in (5.25).

Finally, we state Stone's theorem, which gives an equivalence between self-adjoint operators acting in a Hilbert space and strongly continuous unitary groups. Before stating the theorem, we give the definition of an unbounded self-adjoint operator. For definiteness, we consider complex Hilbert spaces.

Definition 5.41. Let $\mathcal{H}$ be a complex Hilbert space with inner-product

$$(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}.$$

An operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint if:

(1) the domain $\mathcal{D}(A)$ is dense in $\mathcal{H}$;
(2) $x \in \mathcal{D}(A)$ if and only if there exists $z \in \mathcal{H}$ such that $(x, Ay) = (z, y)$ for every $y \in \mathcal{D}(A)$;
(3) $(x, Ay) = (Ax, y)$ for all $x, y \in \mathcal{D}(\mathcal{A})$.

Condition (2) states that $\mathcal{D}(A) = \mathcal{D}(A^*)$ where A^* is the Hilbert space adjoint of A, in which case $z = Ax$, while (3) states that A is symmetric on its domain. A precise characterization of the domain of a self-adjoint operator is essential; for differential operators acting in L^p-spaces, the domain can often be described by the use of Sobolev spaces. The next result is Stone's theorem (see *e.g.* [**44**] for a proof).

Theorem 5.42. *An operator $iA : \mathcal{D}(iA) \subset \mathcal{H} \to \mathcal{H}$ in a complex Hilbert space $\mathcal{H}$ is the generator of a strongly continuous unitary group on $\mathcal{H}$ if and only if A is self-adjoint.*

Example 5.43. The generator of the Schrödinger group on $H^s(\mathbb{R}^n)$ is the self-adjoint operator

$$i\Delta : \mathcal{D}(i\Delta) \subset H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \qquad \mathcal{D}(i\Delta) = H^{s+2}(\mathbb{R}^n).$$

Example 5.44. Consider the Klein-Gordon equation

$$u_{tt} - \Delta u + u = 0$$

in $\mathbb{R}^n$. We rewrite this as a first-order system

$$u_t = v, \qquad v_t = \Delta u,$$

which has the form $w_t = Aw$ where

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ \Delta - I & 0 \end{pmatrix}.$$

We let

$$\mathcal{H} = H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$$

with the inner product of $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2)$ defined by

$$(w_1, w_2)_{\mathcal{H}} = (u_1, u_2)_{H^1} + (v_1, v_2)_{L^2}, \quad (u_1, u_2)_{H^1} = \int (u_1 u_2 + Du_1 \cdot Du_2)\, dx.$$

Then the operator

$$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}, \qquad \mathcal{D}(A) = H^2(\mathbb{R}^n) \oplus H^1(\mathbb{R}^n)$$

is self-adjoint and generates a unitary group on $\mathcal{H}$.

We can instead take

$$\mathcal{H} = L^2(\mathbb{R}^n) \oplus H^{-1}(\mathbb{R}^n), \qquad \mathcal{D}(A) = H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$$

and get a unitary group on this larger space.

5.4.5. Nonhomogeneous equations. The solution of a linear nonhomogeneous ODE

$$u_t = Au + g, \qquad u(0) = f \tag{5.29}$$

may be expressed in terms of the solution operators of the homogeneous equation by the variation of parameters, or Duhamel, formula.

THEOREM 5.45. *Suppose that $A : X \to X$ is a bounded linear operator on a Banach space X and $\mathrm{T}(t) = e^{tA}$ is the associated uniformly continuous group. If $f \in X$ and $g \in C(\mathbb{R}; X)$, then the solution $u \in C^1(\mathbb{R}; X)$ of (5.29) is given by*

$$u(t) = \mathrm{T}(t)f + \int_0^t \mathrm{T}(t-s)g(s)\,ds. \tag{5.30}$$

This solution is continuously strongly differentiable and satisfies the ODE (5.29) pointwise in t for every $t \in \mathbb{R}$. We refer to such a solution as a classical solution. For a strongly continuous group with an unbounded generator, however, the Duhamel formula (5.30) need not define a function $u(t)$ that is differentiable at any time t even if $g \in C(\mathbb{R}; X)$.

EXAMPLE 5.46. Let $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ be a strongly continuous group on a Banach space X with generator $A : \mathcal{D}(A) \subset X \to X$, and suppose that there exists $g_0 \in X$ such that $\mathrm{T}(t)g_0 \notin \mathcal{D}(A)$ for every $t \in \mathbb{R}$. For example, if $\mathrm{T}(t) = e^{it\Delta}$ is the Schrödinger group on $L^2(\mathbb{R}^n)$ and $g_0 \notin H^2(\mathbb{R}^n)$, then $\mathrm{T}(t)g_0 \notin H^2(\mathbb{R}^n)$ for every $t \in \mathbb{R}$. Taking $g(t) = \mathrm{T}(t)g_0$ and $f = 0$ in (5.30) and using the semigroup property, we get

$$u(t) = \int_0^t \mathrm{T}(t-s)\mathrm{T}(s)g_0\,ds = \int_0^t \mathrm{T}(t)g_0\,ds = t\mathrm{T}(t)g_0.$$

This function is continuous but not differentiable with respect to t, since $\mathrm{T}(t)f$ is differentiable at t_0 if and only if $\mathrm{T}(t_0)f \in \mathcal{D}(A)$.

It may happen that the function $u(t)$ defined in (5.30) is is differentiable with respect to t in a distributional sense and satisfies (5.29) pointwise almost everywhere in time. We therefore introduce two other notions of solution that are weaker than that of a classical solution.

DEFINITION 5.47. Suppose that A be the generator of a strongly continuous semigroup $\{\mathrm{T}(t) : t \geq 0\}$, $f \in X$ and $g \in L^1([0,T]; X)$. A function $u : [0,T] \to X$ is a strong solution of (5.29) on $[0,T]$ if:

(1) u is absolutely continuous on $[0,T]$ with distributional derivative $u_t \in L^1(0,T; X)$;
(2) $u(t) \in \mathcal{D}(A)$ pointwise almost everywhere for $t \in (0,T)$;
(3) $u_t(t) = Au(t) + g(t)$ pointwise almost everywhere for $t \in (0,T)$;
(4) $u(0) = f$.

A function $u : [0,T] \to X$ is a mild solution of (5.29) on $[0,T]$ if u is given by (5.30) for $t \in [0,T]$.

Every classical solution is a strong solution and every strong solution is a mild solution. As Example 5.46 shows, however, a mild solution need not be a strong solution.

The Duhamel formula provides a useful way to study semilinear evolution equations of the form

$$u_t = Au + g(u) \tag{5.31}$$

where the linear operator A generates a semigroup on a Banach space X and

$$g : \mathcal{D}(F) \subset X \to X$$

is a nonlinear function. For semilinear PDEs, $g(u)$ typically depends on u but none of its spatial derivatives and then (5.31) consists of a linear PDE perturbed by a zeroth-order nonlinear term.

If $\{\mathrm{T}(t)\}$ is the semigroup generated by A, we may replace (5.31) by an integral equation for $u : [0, T] \to X$

$$u(t) = \mathrm{T}(t)u(0) + \int_0^t \mathrm{T}(t-s) g\left(u(s)\right)\, ds. \tag{5.32}$$

We then try to show that solutions of this integral equation exist. If these solutions have sufficient regularity, then they also satisfy (5.31).

In the standard Picard approach to ODEs, we would write (5.31) as

$$u(t) = u(0) + \int_0^t \left[Au(s) + g\left(u(s)\right)\right]\, ds. \tag{5.33}$$

The advantage of (5.32) over (5.33) is that we have replaced the unbounded operator A by the bounded solution operators $\{\mathrm{T}(t)\}$. Moreover, since $\mathrm{T}(t-s)$ acts on $g(u(s))$ it is possible for the regularizing properties of the linear operators T to compensate for the destabilizing effects of the nonlinearity F. For example, in Section 5.5 we study a semilinear heat equation, and in Section 5.6 to prove the existence of solutions of a nonlinear Schrödinger equation.

5.4.6. Non-autonomous equations. The semigroup property $\mathrm{T}(s)\mathrm{T}(t) = \mathrm{T}(s+t)$ holds for autonomous evolution equations that do not depend explicitly on time. One can also consider time-dependent linear evolution equations in a Banach space X of the form

$$u_t = A(t)u$$

where $A(t) : \mathcal{D}\left(A(t)\right) \subset X \to X$. The solution operators $\mathrm{T}(t; s)$ from time s to time t of a well-posed nonautonomous equation depend separately on the initial and final times, not just on the time difference; they satisfy

$$\mathrm{T}(t; s)\mathrm{T}(s; r) = \mathrm{T}(t; r) \qquad \text{for } r \leq s \leq t.$$

The time-dependence of A makes such equations more difficult to analyze from the semigroup viewpoint than autonomous equations. First, since the domain of $A(t)$ depends in general on t, one must understand how these domains are related and for what times a solution belongs to the domain. Second, the operators $A(s)$, $A(t)$ may not commute for $s \neq t$, meaning that one must order them correctly with respect to time when constructing solution operators $\mathrm{T}(t; s)$.

Similar issues arise in using semigroup theory to study quasi-linear evolution equations of the form

$$u_t = A(u)u$$

in which, for example, $A(u)$ is a differential operator acting on u whose coefficients depend on u (see *e.g.* [**44**] for further discussion). Thus, while semigroup theory is an effective approach to the analysis of autonomous semilinear problems, its application to nonautonomous or quasilinear problems often leads to considerable technical difficulties.

5.5. A semilinear heat equation

Consider the following initial value problem for $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$:

$$u_t = \Delta u + \lambda u - \gamma u^m, \qquad u(x, 0) = g(x) \tag{5.34}$$

where $\lambda, \gamma \in \mathbb{R}$ and $m \in \mathbb{N}$ are parameters. This PDE is a scalar, semilinear reaction diffusion equation. The solution $u = 0$ is linearly stable when $\lambda < 0$ and linearly unstable when $\lambda > 0$. The nonlinear reaction term is potentially stabilizing if $\gamma > 0$ and m is odd or m is even and solutions are nonnegative (they remain nonegative by the maximum principle). For example, if $m = 3$ and $\gamma > 0$, then the spatially-independent reaction ODE $u_t = \lambda u - \gamma u^3$ has a supercritical pitchfork bifurcation at $u = 0$ as λ passes through 0. Thus, (5.34) provides a model equation for the study of bifurcation and loss of stability of equilbria in PDEs.

We consider (5.34) on $\mathbb{R}^n$ since this allows us to apply the results obtained earlier in the Chapter for the heat equation on $\mathbb{R}^n$. In some respects, the behavior this IBVP on a bounded domain is simpler to analyze. The negative Laplacian on $\mathbb{R}^n$ does not have a compact resolvent and has a purely continuous spectrum $[0, \infty)$. By contrast, negative Laplacian on a bounded domain, with say homogeneous Dirichlet boundary conditions, has compact resolvent and a discrete set of eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$. As a result, only finitely many modes become unstable as λ increases, and the long time dynamics of (5.34) is essentially finite-dimensional in nature.

Equations of the form

$$u_t = \Delta u + f(u)$$

on a bounded one-dimensional domain were studied by Chafee and Infante (1974), so this equation is sometimes called the Chafee-Infante equation. We consider here the special case with

$$f(u) = \lambda u - \gamma u^m \tag{5.35}$$

so that we can focus on the essential ideas. We do not attempt to obtain an optimal result; our aim is simply to illustrate how one can use semigroup theory to prove the existence of solutions of semilinear parabolic equations such as (5.34). Moreover, semigroup theory is not the only possible approach to such problems. For example, one can also use a Galerkin method.

5.5.1. Motivation.

We will use the linear heat equation semigroup to reformulate (5.34) as a nonlinear integral equation in an appropriate function space and apply a contraction mapping argument.

To motivate the following analysis, we proceed formally at first. Suppose that $A = -\Delta$ generates a semigroup e^{-tA} on some space X, and let F be the nonlinear operator $F(u) = f(u)$, meaning that F is composition with f regarded as an operator on functions. Then (5.34) maybe written as the abstract evolution equation

$$u_t = -Au + F(u), \qquad u(0) = g.$$

Using Duhamel's formula, we get

$$u(t) = e^{-tA} g + \int_0^t e^{-(t-s)A} F(u(s))\, ds.$$

We use this integral equation to *define* mild solutions of the equation.

We want to formulate the integral equation as a fixed point problem $u = \Phi(u)$ on a space of Y-valued functions $u : [0, T] \to Y$. There are many ways to achieve this. In the framework we use here, we choose spaces $Y \subset X$ such that: (a) $F : Y \to X$ is locally Lipschitz continuous; (b) $e^{-tA} : X \to Y$ for $t > 0$ with integrable operator norm as $t \to 0^+$. This allows the smoothing of the semigroup to compensate for a loss of regularity in the nonlinearity.

As we will show, one appropriate choice in $1 \leq n \leq 3$ space dimensions is $X = L^2(\mathbb{R}^n)$ and $Y = H^{2\alpha}(\mathbb{R}^n)$ for $n/4 < \alpha < 1$. Here $H^{2\alpha}(\mathbb{R}^n)$ is the L^2-Sobolev space of fractional order 2α defined in Section 5.C. We write the order of the Sobolev space as 2α because $H^{2\alpha}(\mathbb{R}^n) = \mathcal{D}(A^\alpha)$ is the domain of the αth-power of the generator of the semigroup.

5.5.2. Mild solutions. Let A denote the negative Laplacian operator in L^2,

$$A : \mathcal{D}(A) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad A = -\Delta, \qquad \mathcal{D}(A) = H^2(\mathbb{R}^n). \tag{5.36}$$

We define A as an operator acting in L^2 because we can study it explicitly by use of the Fourier transform.

As discussed in Section 5.4.2, A is a closed, densely defined positive operator, and $-A$ is the generator of a strongly continuous contraction semigroup

$$\{e^{-tA} : t \geq 0\}$$

on $L^2(\mathbb{R}^n)$. The Fourier representation of the semigroup operators is

$$e^{-tA} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad (\widehat{e^{-tA}h})(k) = e^{-t|k|^2}\hat{h}(k). \tag{5.37}$$

If $t > 0$ we have for any $\alpha > 0$ that

$$e^{-tA} : L^2(\mathbb{R}^n) \to H^{2\alpha}(\mathbb{R}^n).$$

This property expresses the instantaneous smoothing of solutions of the heat equation *c.f.* Proposition 5.14.

We define the nonlinear operator

$$F : H^{2\alpha}(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \qquad F(h)(x) = \lambda h(x) - \gamma h^m(x). \tag{5.38}$$

In order to ensure that F takes values in L^2 and has good continuity properties, we assume that $\alpha > n/4$. The Sobolev embedding theorem (Theorem 5.79) implies that $H^{2\alpha}(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n)$. Hence, if $h \in H^{2\alpha}$, then $h \in L^2 \cap C_0$, so $h \in L^p$ for every $2 \leq p \leq \infty$, and $F(h) \in L^2 \cap C_0$. We then define mild $H^{2\alpha}$-valued solutions of (5.34) as follows.

Definition 5.48. Suppose that $T > 0$, $\alpha > n/4$, and $g \in H^{2\alpha}(\mathbb{R}^n)$. A mild $H^{2\alpha}$-valued solution of (5.34) on $[0, T]$ is a function

$$u \in C\left([0, T]; H^{2\alpha}(\mathbb{R}^n)\right)$$

such that

$$u(t) = e^{-tA}g + \int_0^t e^{-(t-s)A}F(u(s))\, ds \qquad \text{for every } 0 \leq t \leq T, \tag{5.39}$$

where e^{-tA} is given by (5.37), and F is given by (5.38).

5.5.3. Existence. In order to prove a local existence result, we choose α large enough that the nonlinear term is well-behaved by Sobolev embedding, but small enough that the norm of the semigroup maps from L^2 into $H^{2\alpha}$ is integrable as $t \to 0^+$. As we will see, this is the case if $n/4 < \alpha < 1$, so we restrict attention to $1 \le n \le 3$ space dimensions.

THEOREM 5.49. *Suppose that $1 \le n \le 3$ and $n/4 < \alpha < 1$. Then there exists $T > 0$, depending only on α, n, $\|g\|_{H^{2\alpha}}$, and the coefficients of f, such that (5.34) has a unique mild solution $u \in C\left([0,T]; H^{2\alpha}\right)$ in the sense of Definition 5.48.*

PROOF. We write (5.39) as

$$
\begin{aligned}
&u = \Phi(u), \\
&\Phi : C\left([0,T]; H^{2\alpha}\right) \to C\left([0,T]; H^{2\alpha}\right), \\
&\Phi(u)(t) = e^{-tA} g + \int_0^t e^{-(t-s)A} F\left(u(s)\right)\, ds.
\end{aligned}
\tag{5.40}
$$

We will show that Φ defined in (5.40) is a contraction mapping on a suitable ball in $C\left([0,T]; H^{2\alpha}\right)$. We do this in a series of Lemmas. The first Lemma is an estimate of the norm of the semigroup operators on the domain of a fractional power of the generator.

LEMMA 5.50. *Let e^{-tA} be the semigroup operator defined in (5.37) and $\alpha > 0$. If $t > 0$, then*

$$e^{-tA} : L^2(\mathbb{R}^n) \to H^{2\alpha}(\mathbb{R}^n)$$

and there is a constant $C = C(\alpha, n)$ such that

$$\left\|e^{-tA}\right\|_{\mathcal{L}(L^2, H^{2\alpha})} \le \frac{Ce^t}{t^\alpha}.$$

PROOF. Suppose that $h \in L^2(\mathbb{R}^n)$. Using the Fourier representation (5.37) of e^{-tA} as multiplication by $e^{-t|k|^2}$ and the definition of the $H^{2\alpha}$-norm, we get that

$$
\begin{aligned}
\left\|e^{-tA} h\right\|_{H^{2\alpha}}^2 &= (2\pi)^n \int_{\mathbb{R}^n} \left(1 + |k|^2\right)^{2\alpha} e^{-2t|k|^2} \left|\hat{h}(k)\right|^2 \, dk \\
&\le (2\pi)^n \sup_{k \in \mathbb{R}^n} \left[\left(1 + |k|^2\right)^{2\alpha} e^{-2t|k|^2}\right] \int_{\mathbb{R}^n} \left|\hat{h}(k)\right|^2 \, dk.
\end{aligned}
$$

Hence, by Parseval's theorem,

$$\left\|e^{-tA} h\right\|_{H^{2\alpha}} \le M \|h\|_{L^2}$$

where

$$M = (2\pi)^{n/2} \sup_{k \in \mathbb{R}^n} \left[\left(1 + |k|^2\right)^{2\alpha} e^{-2t|k|^2}\right]^{1/2}.$$

Writing $1 + |k|^2 = x$, we have

$$M = (2\pi)^{n/2} e^t \sup_{x \ge 1} \left[x^\alpha e^{-tx}\right] \le \frac{Ce^t}{t^\alpha}.$$

and the result follows. □

Next, we show that Φ is a locally Lipschitz continuous map on the space $C\left([0,T]; H^{2\alpha}(\mathbb{R}^n)\right)$.

LEMMA 5.51. *Suppose that $\alpha > n/4$. Let Φ be the map defined in (5.40) where F is given by (5.38), A is given by (5.36) and $g \in H^{2\alpha}(\mathbb{R}^n)$. Then*

$$\Phi : C\left([0,T]; H^{2\alpha}(\mathbb{R}^n)\right) \to C\left([0,T]; H^{2\alpha}(\mathbb{R}^n)\right) \tag{5.41}$$

and there exists a constant $C = C(\alpha, m, n)$ such that

$$\begin{aligned}&\|\Phi(u) - \Phi(v)\|_{C([0,T];H^{2\alpha})}\\ &\leq CT^{1-\alpha}\left(1 + \|u\|^{m-1}_{C([0,T];H^{2\alpha})} + \|v\|^{m-1}_{C([0,T];H^{2\alpha})}\right)\|u - v\|_{C([0,T];H^{2\alpha})}\end{aligned}$$

for every $u, v \in C\left([0,T]; H^{2\alpha}\right)$.

PROOF. We write Φ in (5.40) as

$$\Phi(u)(t) = e^{-tA}g + \Psi(u)(t), \qquad \Psi(u)(t) = \int_0^t e^{-(t-s)A}F\left(u(s)\right)\, ds.$$

Since $g \in H^{2\alpha}$ and $\{e^{-tA} : t \geq 0\}$ is a strongly continuous semigroup on $H^{2\alpha}$, the map $t \mapsto e^{-tA}g$ belongs to $C\left([0,T]; H^{2\alpha}\right)$. Thus, we only need to prove the result for Ψ.

The fact that $\Psi(u) \in C\left([0,T]; H^{2\alpha}\right)$ if $u \in C\left([0,T]; H^{2\alpha}\right)$ follows from the Lipschitz continuity of Ψ and a density argument. Thus, we only need to prove the Lipschitz estimate.

If $u, v \in C\left([0,T]; H^{2\alpha}\right)$, then using Lemma 5.50 we find that

$$\begin{aligned}\|\Psi(u)(t) - \Psi(v)(t)\|_{H^{2\alpha}} &\leq C\int_0^t \frac{e^{(t-s)}}{|t-s|^\alpha}\|F\left(u(s)\right) - F\left(v(s)\right)\|_{L^2}\, ds\\ &\leq C\sup_{0\leq s\leq T}\|F\left(u(s)\right) - F\left(v(s)\right)\|_{L^2}\int_0^t \frac{1}{|t-s|^\alpha}\, ds.\end{aligned}$$

Evaluating the s-integral, with $\alpha < 1$, and taking the supremum of the result over $0 \leq t \leq T$, we get

$$\|\Psi(u) - \Psi(v)\|_{L^\infty(0,T;H^{2\alpha})} \leq CT^{1-\alpha}\|F\left(u\right) - F\left(v\right)\|_{L^\infty(0,T;L^2)}. \tag{5.42}$$

From (5.35), if $g, h \in C_0 \subset H^{2\alpha}$ we have

$$\|F\left(g\right) - F\left(h\right)\|_{L^2} \leq |\lambda|\,\|g - h\|_{L^2} + |\gamma|\,\|g^m - h^m\|_{L^2}$$

and

$$\|g^m - h^m\|_{L^2} \leq C\left(\|g\|^{m-1}_{L^\infty} + \|h\|^{m-1}_{L^\infty}\right)\|g - h\|_{L^2}.$$

Hence, using the Sobolev inequality $\|g\|_{L^\infty} \leq C\|g\|_{H^{2\alpha}}$ for $\alpha > n/4$ and the fact that $\|g\|_{L^2} \leq \|g\|_{H^{2\alpha}}$, we get that

$$\|F(g) - F(h)\|_{L^2} \leq C\left(1 + \|g\|^{m-1}_{H^{2\alpha}} + \|h\|^{m-1}_{H^{2\alpha}}\right)\|g - h\|_{H^{2\alpha}},$$

which means that $F : H^{2\alpha} \to L^2$ is locally Lipschitz continuous.[3] The use of this result in (5.42) proves the Lemma. □

[3]Actually, under the assumptions we make here, $F : H^{2\alpha} \to H^{2\alpha}$ is locally Lipschitz continuous as a map from $H^{2\alpha}$ into itself, and we don't need to use the smoothing properties of the heat equation semigroup to obtain a fixed point problem in $C([0,T]; H^{2\alpha})$, so perhaps this wasn't the best example to choose! For stronger nonlinearities, however, it would be necessary to use the smoothing.

The existence theorem now follows by a standard contraction mapping argument. If $\|g\|_{H^{2\alpha}} = R$, then

$$\left\|e^{-tA}g\right\|_{H^{2\alpha}} \le R \qquad \text{for every } 0 \le t \le T$$

since $\{e^{-tA}\}$ is a contraction semigroup on $H^{2\alpha}$. Therefore, if we choose

$$E = \left\{u \in C([0,T]; H^{2\alpha} : \|u\|_{C([0,T];H^{2\alpha})} \le 2R\right\}$$

we see from Lemma 5.51 that $\Phi : E \to E$ if we choose $T > 0$ such that

$$CT^{1-\alpha}\left(1 + 2R^{m-1}\right) = \theta R$$

where $0 < \theta < 1$. Moreover, in that case

$$\|\Phi(u) - \Phi(v)\|_{C([0,T];H^{2\alpha})} \le \theta \|u - v\|_{C([0,T];H^{2\alpha})} \qquad \text{for every } u, v \in E.$$

The contraction mapping theorem then implies the existence of a unique solution $u \in E$. □

This result can be extended and improved in many directions. In particular, if A is the negative Laplacian acting in $L^p(\mathbb{R}^n)$,

$$A : W^{2,p}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \qquad A = -\Delta.$$

then one can prove that $-A$ is the generator of a strongly continuous semigroup on L^p for every $1 < p < \infty$. Moreover, we can define fractional powers of A

$$A^\alpha : \mathcal{D}(A^\alpha) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

If we choose $2p > n$ and $n/2p < \alpha < 1$, then Sobolev embedding implies that $\mathcal{D}(A^\alpha) \hookrightarrow C_0$ and the same argument as the one above applies. This gives the existence of local mild solutions with values in $\mathcal{D}(A^\alpha)$ in any number of space dimensions. The proof of the necessary estimates and embedding theorems is more involved that the proofs above if $p \ne 2$, since we cannot use the Fourier transform to obtain out explicit solutions.

More generally, this local existence proof extends to evolution equations of the form ([**41**], §15.1)

$$u_t + Au = F(u),$$

where we look for mild solutions $u \in C([0,T]; X)$ taking values in a Banach space X and there is a second Banach spaces Y such that:

(1) $e^{-tA} : X \to X$ is a strongly continuous semigroup for $t \ge 0$;
(2) $F : X \to Y$ is locally Lipschitz continuous;
(3) $e^{-tA} : Y \to X$ for $t > 0$ and for some $\alpha < 1$

$$\left\|e^{-tA}\right\|_{\mathcal{L}(X,Y)} \le \frac{C}{t^\alpha} \qquad \text{for } 0 < t \le T.$$

In the above example, we used $X = H^{2\alpha}$ and $Y = L^2$. If A is a sectorial operator that generates an analytic semigroup on Y, then one can define fractional powers A^α of A, and the semigroup $\{e^{-tA}\}$ satisfies the above properties with $X = \mathcal{D}(A^\alpha)$ for $0 \le \alpha < 1$ [**36**]. Thus, one gets a local existence result provided that $F : \mathcal{D}(A^\alpha) \to L^2$ is locally Lipschitz, with an existence-time that depends on the X-norm of the initial data.

In general, the X-norm of the solution may blow up in finite time, and one gets only a local solution. If, however, one has an *a priori* estimate for $\|u(t)\|_X$ that is global in time, then global existence follows from the local existence result.

5.6. The nonlinear Schrödinger equation

The nonlinear Schrödinger (NLS) equation is

$$iu_t = -\Delta u - \lambda |u|^\alpha u \tag{5.43}$$

where $\lambda \in \mathbb{R}$ and $\alpha > 0$ are constants. In many applications, such as the asymptotic description of weakly nonlinear dispersive waves, we get $\alpha = 2$, leading to the cubically nonlinear NLS equation.

A physical interpretation of (5.43) is that it describes the motion of a quantum mechanical particle in a potential $V = -\lambda|u|^\alpha$ which depends on the probability density $|u|^2$ of the particle *c.f.* (5.14). If $\lambda \neq 0$, we can normalize $\lambda = \pm 1$ so the magnitude of λ is not important; the sign of λ is, however, crucial.

If $\lambda > 0$, then the potential becomes large and negative when $|u|^2$ becomes large, so the particle 'digs' its own potential well; this tends to trap the particle and further concentrate is probability density, possibly leading to the formation of singularities in finite time if $n \geq 2$ and $\alpha \geq 4/n$. The resulting equation is called the focusing NLS equation.

If $\lambda < 0$, then the potential becomes large and positive when $|u|^2$ becomes large; this has a repulsive effect and tends to make the probability density spread out. The resulting equation is called the defocusing NLS equation. The local L^2-existence result that we obtain here for subcritical nonlinearities $0 < \alpha < 4/n$ is, however, not sensitive to the sign of λ.

The one-dimensional cubic NLS equation

$$iu_t + u_{xx} + \lambda|u|^2 u = 0$$

is completely integrable. If $\lambda > 0$, this equation has localized traveling wave solutions called solitons in which the effects of nonlinear self-focusing balance the tendency of linear dispersion to spread out the the wave. Moreover, these solitons preserve their identity under nonlinear interactions with other solitons. Such localized solutions exist for the focusing NLS equation in higher dimensions, but the NLS equation is not integrable if $n \geq 2$, and in that case the soliton solutions are not preserved under nonlinear interactions.

In this section, we obtain an existence result for the NLS equation. The linear Schrödinger equation group is not smoothing, so we cannot use it to compensate for the nonlinearity at a fixed time as we did in Section 5.5 for the semilinear equation. Instead, we use some rather delicate space-time estimates for the linear Schrödinger equation, called Strichartz estimates, to recover the powers lost by the nonlinearity. We derive these estimates first.

5.6.1. Strichartz estimates.

5.6.1. Strichartz estimates. The Strichartz estimates for the Schrödinger equation (5.13) may be derived by use of the interpolation estimate in Theorem 5.16 and the Hardy-Littlewood-Sobolev inequality in Theorem 5.77. The space-time norm in the Strichartz estimate is $L^q(\mathbb{R})$ in time and $L^r(\mathbb{R}^n)$ in space for suitable exponents (q, r), which we call an admissible pair.

DEFINITION 5.52. The pair of exponents (q, r) is an admissible pair if

$$\frac{2}{q} = \frac{n}{2} - \frac{n}{r} \tag{5.44}$$

where $2 < q < \infty$ and

$$2 < r < \frac{2n}{n-2} \qquad \text{if } n \geq 3 \tag{5.45}$$

or $2 < r < \infty$ if $n = 1, 2$.

The Strichartz estimates continue to hold for some endpoints with $q = 2$ or $q = \infty$, but we will not consider these cases here.

Theorem 5.53. *Suppose that $\{\mathrm{T}(t) : t \in \mathbb{R}\}$ is the unitary group of solution operators of the Schrödinger equation on $\mathbb{R}^n$ defined in (5.22) and (q, r) is an admissible pair as in Definition 5.52.*

(1) *For $f \in L^2(\mathbb{R}^n)$, let $u(t) = \mathrm{T}(t)f$. Then $u \in L^q(\mathbb{R}; L^r)$, and there is a constant $C(n, r)$ such that*

$$\|u\|_{L^q(\mathbb{R};L^r)} \leq C\|f\|_{L^2}. \tag{5.46}$$

(2) *For $g \in L^{q'}(\mathbb{R}; L^{r'})$, let*

$$v(t) = \int_{-\infty}^{\infty} \mathrm{T}(t-s)g(s)\, ds.$$

Then $v \in L^{q'}(\mathbb{R}; L^{r'}) \cap C(\mathbb{R}; L^2)$ and there is a constant $C(n, r)$ such that

$$\|v\|_{L^\infty(\mathbb{R};L^2)} \leq C\, \|g\|_{L^{q'}(\mathbb{R};L^{r'})}\,, \tag{5.47}$$

$$\|v\|_{L^q(\mathbb{R};L^r)} \leq C\, \|g\|_{L^{q'}(\mathbb{R};L^{r'})}\,. \tag{5.48}$$

Proof. By a density argument, it is sufficient to prove the result for smooth functions. We therefore assume that $g \in C_c^\infty(\mathbb{R}; \mathcal{S})$ is a smooth Schwartz-valued function with compact support in time and $f \in \mathcal{S}$. We prove the inequalities in reverse order.

Using the interpolation estimate Theorem 5.16, we have for $2 < r < \infty$ that

$$\|v(t)\|_{L^r} \leq \int_{-\infty}^{\infty} \frac{\|g(s)\|_{L^{r'}}}{(4\pi|t-s|)^{n(1/2-1/r)}}\, ds.$$

If r is admissible, then $0 < n(1/2 - 1/r) < 1$. Thus, taking the L^q-norm of this inequality with respect to t and using the Hardy-Littlewood-Sobolev inequality (Theorem 5.77) in the result, we find that

$$\|v\|_{L^q(\mathbb{R};L^r)} \leq C\, \|g\|_{L^p(\mathbb{R};L^{r'})}$$

where p is given by

$$\frac{1}{p} = 1 + \frac{1}{q} + \frac{n}{r} - \frac{n}{2}.$$

If q, r satisfy (5.44), then $p = q'$, and we get (5.48).

Using Fubini's theorem and the unitary group property of $\mathrm{T}(t)$, we have

$$\begin{aligned}
(v(t), v(t))_{L^2(\mathbb{R}^n)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\mathrm{T}(t-r)g(r), \mathrm{T}(t-s)g(s)\right)_{L^2(\mathbb{R}^n)}\, drds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\mathrm{T}(s-r)g(r), g(s)\right)_{L^2(\mathbb{R}^n)}\, drds \\
&= \int_{-\infty}^{\infty} \left(v(s), g(s)\right)_{L^2(\mathbb{R}^n)}\, ds.
\end{aligned}$$

Using Hölder's inequality and (5.48) in this equation, we get

$$\|v(t)\|^2_{L^2(\mathbb{R}^n)} \leq \|v\|_{L^q(\mathbb{R};L^r)} \|g\|_{L^{q'}(\mathbb{R};L^{r'})} \leq C \|g\|^2_{L^{q'}(\mathbb{R};L^{r'})}.$$

Taking the supremum of this inequality over $t \in \mathbb{R}$, we obtain (5.47). In fact, since

$$v(t) = \mathrm{T}(t) \int_{-\infty}^{\infty} \mathrm{T}(-s)g(s)\, ds$$

$v \in C(\mathbb{R}; L^2)$ is an L^2-solution of the homogeneous Schrödinger equation and $\|v(t)\|_{L^2(\mathbb{R}^n)}$ is independent of t.

If $f \in \mathcal{S}$, $u(t) = \mathrm{T}(t)f$, and $g \in C_c^\infty(\mathbb{R}; \mathcal{S})$, then using (5.48) we get

$$\begin{aligned}
\int_{-\infty}^{\infty} (u(t), g(t))_{L^2}\ dt &= \int_{-\infty}^{\infty} (\mathrm{T}(t)f, g(t))_{L^2}\ dt \\
&= \left(f, \int_{-\infty}^{\infty} \mathrm{T}(-t)g(t)\, dt\right)_{L^2} \\
&\leq \|f\|_{L^2} \left\| \int_{-\infty}^{\infty} \mathrm{T}(-t)g(t)\, dt \right\|_{L^2} \\
&\leq C \|f\|_{L^2} \|g\|_{L^{q'}(\mathbb{R};L^{r'})}.
\end{aligned}$$

It then follows by duality and density that

$$\|u\|_{L^q(\mathbb{R};L^r)} = \sup_{g \in C_c^\infty(\mathbb{R};\mathcal{S})} \frac{\int_{-\infty}^{\infty} (u(t), g(t))_{L^2}\ dt}{\|g\|_{L^{q'}(\mathbb{R};L^{r'})}} \leq C \|f\|_{L^2},$$

which proves (5.46). □

This estimate describes a dispersive smoothing effect of the Schrödinger equation. For example, the L^r-spatial norm of the solution may blow up at some time, but it must be finite almost everywhere in t. Intuitively, this is because if the Fourier modes of the solution are sufficiently in phase at some point in space and time that they combine to form a singularity, then dispersion pulls them apart at later times.

Although the above proof of the Schrödinger equation Strichartz estimates is elementary, in the sense that given the interpolation estimate for the Schrödinger equation and the one-dimensional Hardy-Littlewood-Sobolev inequality it uses only Hölder's inequality, it does not explicitly clarify the role of dispersion (beyond the dispersive decay of solutions in time). An alternative point of view is in terms of restriction theorems for the Fourier transform.

The Fourier solution of the Schrödinger equation (5.13) is

$$u(x,t) = \int_{\mathbb{R}^n} \hat{f}(k) e^{ik\cdot x + i|k|^2 t}\, dk.$$

Thus, the space-time Fourier transform $\hat{u}(k,\tau)$ of $u(x,t)$,

$$\hat{u}(k,\tau) = \frac{1}{(2\pi)^{n+1}} \int u(x,t) e^{ik\cdot x + i\tau t}\, dxdt,$$

is a measure supported on the paraboloid $\tau + |k|^2 = 0$. This surface has non-singular curvature, which is a geometrical expression of the dispersive nature of the Schrödinger equation. The Strichartz estimates describe a boundedness property of the restriction of the Fourier transform to curved surfaces.

As an illustration of this phenomenon, we state the Tomas-Stein theorem on the restriction of the Fourier transform in $\mathbb{R}^{n+1}$ to the unit sphere $\mathbb{S}^n$.

THEOREM 5.54. *Suppose that $f \in L^p(\mathbb{R}^{n+1})$ with*

$$1 \leq p \leq \frac{2n+4}{n+4}$$

and let $\hat{g} = \hat{f}\Big|_{\mathbb{S}^n}$. Then there is a constant $C(p,n)$ such that

$$\|\hat{g}\|_{L^2(\mathbb{S}^n)} \leq C\,\|f\|_{L^p(\mathbb{R}^{n+1})}.$$

5.6.2. Local L^2-solutions. In this section, we use the Strichartz estimates for the linear Schrödinger equation to obtain a local existence result for solutions of the nonlinear Schrödinger equation with initial data in L^2.

If X is a Banach space and $T > 0$, we say that $u \in C([0,T];X)$ is a mild X-valued solution of (5.43) if it satisfies the Duhamel-type integral equation

$$u = \mathrm{T}(t)f + i\lambda \int_0^t \mathrm{T}(t-s)\left\{|u|^\alpha(s)u(s)\right\} ds \qquad \text{for } t \in [0,T] \tag{5.49}$$

where $\mathrm{T}(t) = e^{it\Delta}$ is the solution operator of the linear Schrödinger equation defined by (5.22). If a solution of (5.49) has sufficient regularity then it is also a solution of (5.43), but here we simply take (5.49) as our definition of a solution. We suppose that $t \geq 0$ for definiteness; the same arguments apply for $t \leq 0$.

Before stating an existence theorem, we explain the idea of the proof, which is based on the contraction mapping theorem. We write (5.49) as a fixed-point equation

$$u = \Phi(u) \qquad \Phi(u)(t) = \mathrm{T}(t)f + i\lambda\Psi(u)(t), \tag{5.50}$$

$$\Psi(u)(t) = \int_0^t \mathrm{T}(t-s)\left\{|u|^\alpha(s)u(s)\right\} ds. \tag{5.51}$$

We want to find a Banach space E of functions $u : [0,T] \to L^r$ and a closed ball $B \subset E$ such that $\Phi : B \to B$ is a contraction mapping when $T > 0$ is sufficiently small.

As discussed in Section 5.4.3, the Schrödinger operators $\mathrm{T}(t)$ form a strongly continuous group on L^p only if $p = 2$. Thus if $f \in L^2$, then

$$\Phi : C\left([0,T]; L^{2/(\alpha+1)}\right) \to C\left([0,T]; L^2\right),$$

but Φ does not map the space $C\left([0,T]; L^r\right)$ into itself for any exponent $1 \leq r \leq \infty$.

If α is not too large, however, there are exponents q, r such that

$$\Phi : L^q\left(0,T;L^r\right) \to L^q\left(0,T;L^r\right). \tag{5.52}$$

This happens because, as shown by the Strichartz estimates, the linear solution operator T can regain the space-time regularity lost by the nonlinearity. (For a brief discussion of vector-valued L^p-spaces, see Section 6.A.)

To determine values of q, r for which (5.52) holds, we write

$$L^q\left(0,T;L^r\right) = L^q_t L^r_x$$

for short, and consider the action of Φ defined in (5.50)–(5.51) on such a space.

First, consider the term $\mathrm{T}f$ in (5.50) which is independent of u. Theorem 5.53 implies that $\mathrm{T}f \in L^q_t L^r_x$ if $f \in L^2$ for any admissible pair (q,r).

Second, consider the nonlinear term $\Psi(u)$ in (5.51). We have

$$\begin{aligned}\| |u|^\alpha u \|_{L^q_t L^r_x} &= \left[\int_0^T \left(\int_{\mathbb{R}^n} |u|^{r(\alpha+1)}\, dx\right)^{q/r} dt\right]^{1/q} \\ &= \left[\int_0^T \left(\int_{\mathbb{R}^n} |u|^{r(\alpha+1)}\, dx\right)^{q(\alpha+1)/r(\alpha+1)} dt\right]^{(\alpha+1)/q(\alpha+1)} \\ &= \|u\|^{\alpha+1}_{L_t^{q(\alpha+1)} L_x^{r(\alpha+1)}}.\end{aligned}$$

Thus, if $u \in L^{q_1}_t L^{r_1}_x$ then $|u|^\alpha u \in L^{q_2'}_t L^{r_2'}_x$ where

$$q_1 = q_2'(\alpha+1), \qquad r_1 = r_2'(\alpha+1). \tag{5.53}$$

If (q_2, r_2) is an admissible pair, then the Strichartz estimate (5.48) implies that

$$\Psi(u) \in L^{q_2}_t L^{r_2}_x.$$

In order to ensure that Ψ preserves the L^r_x-norm of u, we need to choose $r = r_1 = r_2$, which implies that $r = r'(\alpha+1)$, or

$$r = \alpha + 2. \tag{5.54}$$

If r is given by (5.54), then it follows from Definition 5.52 that

$$(q_2, r_2) = (q, \alpha+2)$$

is an admissible pair if

$$q = \frac{4(\alpha+2)}{n\alpha} \tag{5.55}$$

and $0 < \alpha < 4/(n-2)$, or $0 < \alpha < \infty$ if $n = 1, 2$. In that case, we have

$$\Psi : L^{q_1}_t L^{\alpha+2}_x \to L^q_t L^{\alpha+2}_x$$

where

$$q_1 = q'(\alpha+1). \tag{5.56}$$

In order for Ψ to map $L^q_t L^{\alpha+2}_x$ into itself, we need $L^{q_1}_t \supset L^q_t$ or $q_1 \le q$. This condition holds if $\alpha + 2 \le q$ or $\alpha \le 4/n$. In order to prove that Φ is a contraction we will interpolate in time from $L^{q_1}_t$ to L^q_t, which requires that $q_1 < q$ or $\alpha < 4/n$. A similar existence result holds in the critical case $\alpha = 4/n$ but the proof requires a more refined argument which we do not describe here.

Thus according to this discussion,

$$\Phi : L^q_t L^{\alpha+2}_x \to L^q_t L^{\alpha+2}_x$$

if q is given by (5.55) and $0 < \alpha < 4/n$. This motivates the hypotheses in the following theorem.

THEOREM 5.55. *Suppose that* $0 < \alpha < 4/n$ *and*

$$q = \frac{4(\alpha+2)}{n\alpha}.$$

For every $f \in L^2(\mathbb{R}^n)$, *there exists*

$$T = T\left(\|f\|_{L^2}, n, \alpha, \lambda\right) > 0$$

and a unique solution u of (5.49) with

$$u \in C\left([0,T]; L^2(\mathbb{R}^n)\right) \cap L^q\left(0,T; L^{\alpha+2}(\mathbb{R}^n)\right).$$

Moreover, the solution map $f \mapsto u$ is locally Lipschitz continuous.

Proof. For $T > 0$, let E be the Banach space

$$E = C\left([0,T]; L^2\right) \cap L^q\left(0,T; L^{\alpha+2}\right)$$

with norm

$$\|u\|_E = \max_{[0,T]} \|u(t)\|_{L^2} + \left(\int_0^T \|u(t)\|_{L^{\alpha+2}}^q \, dt\right)^{1/q} \tag{5.57}$$

and let Φ be the map in (5.50)–(5.51). We claim that $\Phi(u)$ is well-defined for $u \in E$ and $\Phi : E \to E$.

The preceding discussion shows that $\Phi(u) \in L^q_t L^{\alpha+2}_x$ if $u \in L^q_t L^{\alpha+2}_x$. Writing $C_t L^2_x = C\left([0,T]; L^2\right)$, we see that $\mathrm{T}(\cdot)f \in C_t L^2_x$ since $f \in L^2$ and T is a strongly continuous group on L^2. Moreover, (5.47) implies that $\Psi(u) \in C_t L^2_x$ since $\Psi(u)$ is the uniform limit of smooth functions $\Psi(u_k)$ such that $u_k \to u$ in $L^q_t L^{\alpha+2}_x$ *c.f.* (5.71). Thus, $\Phi : E \to E$.

Next, we estimate $\|\Phi(u)\|_E$ and show that there exist positive numbers

$$T = T\left(\|f\|_{L^2}, n, \alpha, \lambda\right), \qquad a = a\left(\|f\|_{L^2}, n, \alpha\right)$$

such that Φ maps the ball

$$B = \{u \in E : \|u\|_E \leq a\} \tag{5.58}$$

into itself.

First, we estimate $\|\mathrm{T}f\|_E$. Since T is a unitary group, we have

$$\|\mathrm{T}f\|_{C_t L^2_x} = \|f\|_{L^2} \tag{5.59}$$

while the Strichartz estimate (5.46) implies that

$$\|\mathrm{T}f\|_{L^q_t L^{\alpha+2}_x} \leq C\|f\|_{L^2}. \tag{5.60}$$

Thus, there is a constant $C = C(n,\alpha)$ such that

$$\|\mathrm{T}f\|_E \leq C\|f\|_{L^2}. \tag{5.61}$$

In the rest of the proof, we use C to denote a generic constant depending on n and α.

Second, we estimate $\|\Psi(u)\|_E$ where Ψ is given by (5.51). The Strichartz estimate (5.47) gives

$$\begin{aligned}
\|\Psi(u)\|_{C_t L^2_x} &\leq C\left\| |u|^{\alpha+1} \right\|_{L^{q'}_t L^{(\alpha+2)'}_x} \\
&\leq C\|u\|^{\alpha+1}_{L^{q'(\alpha+1)}_t L^{(\alpha+2)'(\alpha+1)}_x} \\
&\leq C\|u\|^{\alpha+1}_{L^{q_1}_t L^{\alpha+2}_x}
\end{aligned} \tag{5.62}$$

where q_1 is given by (5.56). If $\phi \in L^p(0,T)$ and $1 \le p \le q$, then Hölder's inequality with $r = q/p \ge 1$ gives

$$\begin{aligned} \|\phi\|_{L^p(0,T)} &= \left(\int_0^T 1 \cdot |\phi(t)|^p \, dt\right)^{1/p} \\ &\le \left[\left(\int_0^T 1^{r'} \, dt\right)^{1/r'} \left(\int_0^T |\phi(t)|^{pr} \, dt\right)^{1/r}\right]^{1/p} \\ &\le T^{1/p-1/q} \|\phi\|_{L^q(0,T)}. \end{aligned} \tag{5.63}$$

Using this inequality with $p = q_1$ in (5.62), we get

$$\|\Psi(u)\|_{C_t L^2_x} \le C T^\theta \|u\|^{\alpha+1}_{L^q_t L^{\alpha+2}_x} \tag{5.64}$$

where $\theta = (\alpha+1)(1/q_1 - 1/q) > 0$ is given by

$$\theta = 1 - \frac{n\alpha}{4}. \tag{5.65}$$

We estimate $\|\Psi(u)\|_{L^q_t L^{\alpha+2}_x}$ in a similar way. The Strichartz estimate (5.48) and the Hölder estimate (5.63) imply that

$$\|\Psi(u)\|_{L^q_t L^{\alpha+2}_x} \le C \|u\|^{\alpha+1}_{L^{q_1}_t L^{\alpha+2}_x} \le C T^\theta \|u\|^{\alpha+1}_{L^q_t L^{\alpha+2}_x}. \tag{5.66}$$

Thus, from (5.64) and (5.66), we have

$$\|\Psi(u)\|_E \le C T^\theta \|u\|^{\alpha+1}_{L^q_t L^{\alpha+2}_x}. \tag{5.67}$$

Using (5.61) and (5.67), we find that there is a constant $C = C(n,\alpha)$ such that

$$\|\Phi(u)\|_E \le \|\mathrm{T} f\|_E + |\lambda| \, \|\Psi(u)\|_E \le C\|f\|_{L^2} + C|\lambda| T^\theta \|u\|^{\alpha+1}_{L^q_t L^{\alpha+2}_x} \tag{5.68}$$

for all $u \in E$. We choose positive constants a, T such that

$$a \ge 2C\|f\|_{L^2}, \qquad 0 < 2C|\lambda| T^\theta a^\alpha \le 1.$$

Then (5.68) implies that $\Phi : B \to B$ where $B \subset E$ is the ball (5.58).

Next, we show that Φ is a contraction on B. From (5.50) we have

$$\Phi(u) - \Phi(v) = i\lambda \left[\Psi(u) - \Psi(v)\right]. \tag{5.69}$$

Using the Strichartz estimates (5.47)–(5.48) in (5.51) as before, we get

$$\|\Psi(u) - \Psi(v)\|_E \le C \left\| \, |u|^\alpha u - |v|^\alpha v \, \right\|_{L^{q'}_t L^{(\alpha+2)'}_x}. \tag{5.70}$$

For any $\alpha > 0$ there is a constant $C(\alpha)$ such that

$$\left| \, |w|^\alpha w - |z|^\alpha z \, \right| \le C \left(|w|^\alpha + |z|^\alpha\right) |w - z| \qquad \text{for all } w, z \in \mathbb{C}.$$

Using the identity

$$(\alpha+2)' = \frac{\alpha+2}{\alpha+1}$$

and Hölder's inequality with $r = \alpha + 1$, $r' = (\alpha+1)/\alpha$, we get that

$$\begin{aligned}
\left\| \, |u|^\alpha u - |v|^\alpha v \, \right\|_{L_x^{(\alpha+2)'}} &= \left(\int \left| \, |u|^\alpha u - |v|^\alpha v \, \right|^{(\alpha+2)'} \, dx \right)^{1/(\alpha+2)'} \\
&\le C \left(\int \left(|u|^\alpha + |v|^\alpha \right)^{(\alpha+2)'} |u - v|^{(\alpha+2)'} \, dx \right)^{1/(\alpha+2)'} \\
&\le C \left(\int \left(|u|^\alpha + |v|^\alpha \right)^{r'(\alpha+2)'} \, dx \right)^{1/r'(\alpha+2)'} \\
&\qquad \left(\int |u - v|^{r(\alpha+2)'} \, dx \right)^{1/r(\alpha+2)'} \\
&\le C \left(\|u\|_{L_x^{\alpha+2}}^\alpha + \|v\|_{L_x^{\alpha+2}}^\alpha \right) \|u - v\|_{L_x^{\alpha+2}}
\end{aligned}$$

We use this inequality in (5.70) followed by Hölder's inequality in time to get

$$\begin{aligned}
\|\Psi(u) - \Psi(v)\|_E &\le C \left(\int_0^T \left[\|u\|_{L_x^{\alpha+2}}^\alpha + \|v\|_{L_x^{\alpha+2}}^\alpha \right]^{q'} \|u - v\|_{L_x^{\alpha+2}}^{q'} \, dt \right)^{1/q'} \\
&\le C \left(\int_0^T \left[\|u\|_{L_x^{\alpha+2}}^\alpha + \|v\|_{L_x^{\alpha+2}}^\alpha \right]^{p'q'} \, dt \right)^{1/p'q'} \\
&\qquad \left(\int_0^T \|u - v\|_{L_x^{\alpha+2}}^{pq'} \, dt \right)^{1/pq'} .
\end{aligned}$$

Taking $p = q/q' > 1$ we get

$$\|\Psi(u) - \Psi(v)\|_E \le C \left(\int_0^T \left[\|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'} + \|v(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'} \right] dt \right)^{1/pq'} \|u - v\|_{L_t^q L_x^{\alpha+2}} .$$

Interpolating in time as in (5.63), we have

$$\int_0^T \|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'} \, dt \le \left(\int_0^T 1^{\alpha p'q'r'} \, dt \right)^{1/r'} \left(\int_0^T \|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'r} \, dt \right)^{1/r}$$

and taking $\alpha p'q'r = q$, which implies that $1/p'q'r' = \theta$ where θ is given by (5.65), we get

$$\left(\int_0^T \|u(t)\|_{L_x^{\alpha+2}}^{\alpha p'q'} \, dt \right)^{1/r} \le T^\theta \|u - v\|_{L_t^q L_x^{\alpha+2}} .$$

It therefore follows that

$$\|\Psi(u) - \Psi(v)\|_E \le C T^\theta \left(\|u\|_{L_t^q L_x^{\alpha+2}}^\alpha + \|v\|_{L_t^q L_x^{\alpha+2}}^\alpha \right) \|u - v\|_{L_t^q L_x^{\alpha+2}} . \tag{5.71}$$

Using this result in (5.69), we get

$$\|\Phi(u) - \Phi(v)\|_E \le C|\lambda| T^\theta \left(\|u\|_E^\alpha + \|v\|_E^\alpha \right) \|u - v\|_E .$$

Thus if $u, v \in B$,

$$\|\Phi(u) - \Phi(v)\|_E \le 2C|\lambda| T^\theta a^\alpha \|u - v\|_E .$$

Choosing $T > 0$ such that $2C|\lambda|T^\theta a^\alpha < 1$, we get that $\Phi : B \to B$ is a contraction, so it has a unique fixed point in B. Since we can choose the radius a of B as large as we wish by taking T small enough, the solution is unique in E.

The Lipshitz continuity of the solution map follows from the contraction mapping theorem. If Φ_f denotes the map in (5.50), $\Phi_{f_1}, \Phi_{f_2} : B \to B$ are contractions, and u_1, u_2 are the fixed points of Φ_{f_1}, Φ_{f_2}, then

$$\|u_1 - u_2\|_E \leq C \|f_1 - f_2\|_{L^2} + K \|u_1 - u_2\|_E$$

where $K < 1$. Thus

$$\|u_1 - u_2\|_E \leq \frac{C}{1-K} \|f_1 - f_2\|_{L^2}.$$

□

This local existence theorem implies the global existence of L^2-solutions for subcritical nonlinearities $0 < \alpha < 4/n$ because the existence time depends only the L^2-norm of the initial data and one can show that the L^2-norm of the solution is constant in time.

For more about the extensive theory of the nonlinear Schrödinger equation and other nonlinear dispersive PDEs see, for example, [**6, 29, 39, 40**].

Appendix

May the Schwartz be with you![4]

In this section, we summarize some results about Schwartz functions, tempered distributions, and the Fourier transform. For complete proofs, see [**24, 34**].

5.A. The Schwartz space

Since we will study the Fourier transform, we consider complex-valued functions.

Definition 5.56. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \in C^\infty(\mathbb{R}^n)$ and

$$x^\alpha \partial^\beta f(x) \to 0 \qquad \text{as } |x| \to \infty$$

for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. For $\alpha, \beta \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$ let

$$\|f\|_{\alpha,\beta} = \sup_{\mathbb{R}^n} \left|x^\alpha \partial^\beta f\right|. \tag{5.72}$$

A sequence of functions $\{f_k : k \in \mathbb{N}\}$ converges to a function f in $\mathcal{S}(\mathbb{R}^n)$ if

$$\|f_n - f\|_{\alpha,\beta} \to 0 \qquad \text{as } k \to \infty$$

for every $\alpha, \beta \in \mathbb{N}_0^n$.

That is, the Schwartz space consists of smooth functions whose derivatives (including the function itself) decay at infinity faster than any power; we say, for short, that Schwartz functions are rapidly decreasing. When there is no ambiguity, we will write $\mathcal{S}(\mathbb{R}^n)$ as $\mathcal{S}$.

Example 5.57. The function $f(x) = e^{-|x|^2}$ belongs to $\mathcal{S}(\mathbb{R}^n)$. More generally, if p is any polynomial, then $g(x) = p(x)\, e^{-|x|^2}$ belongs to $\mathcal{S}$.

Example 5.58. The function

$$f(x) = \frac{1}{(1 + |x|^2)^k}$$

does not belongs to $\mathcal{S}$ for any $k \in \mathbb{N}$ since $|x|^{2k} f(x)$ does not decay to zero as $|x| \to \infty$.

Example 5.59. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-x^2} \sin\left(e^{x^2}\right)$$

does not belong to $\mathcal{S}(\mathbb{R})$ since $f'(x)$ does not decay to zero as $|x| \to \infty$.

The space $\mathcal{D}(\mathbb{R}^n)$ of smooth complex-valued functions with compact support is contained in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. If $f_k \to f$ in $\mathcal{D}$ (in the sense of Definition 3.8), then $f_k \to f$ in $\mathcal{S}$, so $\mathcal{D}$ is continuously embedded in $\mathcal{S}$. Furthermore, if $f \in \mathcal{S}$, and $\eta \in C_c^\infty(\mathbb{R}^n)$ is a cutoff function with $\eta_k(x) = \eta(x/k)$, then $\eta_k f \to f$ in $\mathcal{S}$ as $k \to \infty$, so $\mathcal{D}$ is dense in $\mathcal{S}$.

[4]Spaceballs

The topology of $\mathcal{S}$ is defined by the countable family of semi-norms $\|\cdot\|_{\alpha,\beta}$ given in (5.72). This topology is not derived from a norm, but it is metrizable; for example, we can use as a metric

$$d(f,g) = \sum_{\alpha,\beta\in\mathbb{N}_0^n} \frac{c_{\alpha,\beta}\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}}$$

where the $c_{\alpha,\beta} > 0$ are any positive constants such that $\sum_{\alpha,\beta\in\mathbb{N}_0^n} c_{\alpha,\beta}$ converges. Moreover, $\mathcal{S}$ is complete with respect to this metric. A complete, metrizable topological vector space whose topology may be defined by a countable family of semi-norms is called a Fréchet space. Thus, $\mathcal{S}$ is a Fréchet space.

If we want to make explicit that a limit exists with respect to the Schwartz topology, we write

$$f = \mathcal{S}\text{-}\lim_{k\to\infty} f_k,$$

and call f the $\mathcal{S}$-limit of $\{f_k\}$.

If $f_k \to f$ as $k \to \infty$ in $\mathcal{S}$, then $\partial^\alpha f_k \to \partial^\alpha f$ for any multi-index $\alpha \in \mathbb{N}_0^n$. Thus, the differentiation operator $\partial^\alpha : \mathcal{S} \to \mathcal{S}$ is a continuous linear map on $\mathcal{S}$.

5.A.1. Tempered distributions. Tempered distributions are distributions (*c.f.* Section 3.3) that act continuously on Schwartz functions. Roughly speaking, we can think of tempered distributions as distributions that grow no faster than a polynomial at infinity.[5]

DEFINITION 5.60. A tempered distribution T on $\mathbb{R}^n$ is a continuous linear functional $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The topological vector space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{S}'$. If $\langle T, f\rangle$ denotes the value of $T \in \mathcal{S}'$ acting on $f \in \mathcal{S}$, then a sequence $\{T_k\}$ converges to T in $\mathcal{S}'$, written $T_k \rightharpoonup T$, if

$$\langle T_k, f\rangle \to \langle T, f\rangle \qquad \text{for every } f \in \mathcal{S}.$$

Since $\mathcal{D} \subset \mathcal{S}$ is densely and continuously embedded, we have $\mathcal{S}' \subset \mathcal{D}'$. Moreover, a distribution $T \in \mathcal{D}'$ extends uniquely to a tempered distribution $T \in \mathcal{S}'$ if and only if it is continuous on $\mathcal{D}$ with respect to the topology on $\mathcal{S}$.

Every function $f \in L^1_{\text{loc}}$ defines a regular distribution $T_f \in \mathcal{D}'$ by

$$\langle T_f, \phi\rangle = \int f\phi\, dx \qquad \text{for all } \phi \in \mathcal{D}.$$

If $|f| \le p$ is bounded by some polynomial p, then T_f extends to a tempered distribution $T_f \in \mathcal{S}'$, but this is not the case for functions f that grow too rapidly at infinity.

EXAMPLE 5.61. The locally integrable function $f(x) = e^{|x|^2}$ defines a regular distribution $T_f \in \mathcal{D}'$ but this distribution does not extend to a tempered distribution.

EXAMPLE 5.62. If $f(x) = e^x \cos(e^x)$, then $T_f \in \mathcal{D}'(\mathbb{R})$ extends to a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$ even though the values of $f(x)$ grow exponentially as $x \to \infty$.

[5]The name 'tempered distribution' is short for 'distribution of temperate growth,' meaning polynomial growth.

This tempered distribution is the distributional derivative $T = T_g'$ of the regular distribution T_g where $f = g'$ and $g(x) = \sin(e^x)$:

$$\langle f, \phi \rangle = -\langle g, \phi' \rangle = -\int \sin(e^x)\phi(x)\, dx \qquad \text{for all } \phi \in \mathcal{S}.$$

The distribution T is decreasing in a weak sense at infinity because of the rapid oscillations of f.

EXAMPLE 5.63. The series

$$\sum_{n \in \mathbb{N}} \delta^{(n)}(x - n)$$

where $\delta^{(n)}$ is the nth derivative of the δ-function converges to a distribution in $\mathcal{D}'(\mathbb{R})$, but it does not converge in $\mathcal{S}'(\mathbb{R})$ or define a tempered distribution.

We define the derivative of tempered distributions in the same way as for distributions. If $\alpha \in \mathbb{N}_0^n$ is a multi-index, then

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle.$$

We say that a C^∞-function f is slowly growing if the function and all of its derivatives are of polynomial growth, meaning that for every $\alpha \in \mathbb{N}_0^n$ there exists a constant C_α and an integer N_α such that

$$|\partial^\alpha f(x)| \le C_\alpha \left(1 + |x|^2\right)^{N_\alpha}.$$

If f is C^∞ and slowly growing, then $f\phi \in \mathcal{S}$ whenever $\phi \in \mathcal{S}$, and multiplication by f is a continuous map on $\mathcal{S}$. Thus for $T \in \mathcal{S}'$, we may define the product $fT \in \mathcal{S}'$ by

$$\langle fT, \phi \rangle = \langle T, f\phi \rangle.$$

5.B. The Fourier transform

The Schwartz space is a natural one to use for the Fourier transform. Differentiation and multiplication exchange rôles under the Fourier transform and therefore so do the properties of smoothness and rapid decrease. As a result, the Fourier transform is an automorphism of the Schwartz space. By duality, the Fourier transform is also an automorphism of the space of tempered distributions.

5.B.1. The Fourier transform on $\mathcal{S}$.

DEFINITION 5.64. The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{f}(k) = \frac{1}{(2\pi)^n} \int f(x) e^{-ik\cdot x}\, dx. \tag{5.73}$$

The inverse Fourier transform of f is the function $\check{f} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\check{f}(x) = \int f(k) e^{ik\cdot x}\, dk.$$

We generally use x to denote the variable on which a function f depends and k to denote the variable on which its Fourier transform depends.

EXAMPLE 5.65. For $\sigma > 0$, the Fourier transform of the Gaussian

$$f(x) = \frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-|x|^2/2\sigma^2}$$

is the Gaussian

$$\hat{f}(k) = \frac{1}{(2\pi)^n}\, e^{-\sigma^2|k|^2/2}$$

The Fourier transform maps differentiation to multiplication by a monomial and multiplication by a monomial to differentiation. As a result, $f \in \mathcal{S}$ if and only if $\hat{f} \in \mathcal{S}$, and $f_n \to f$ in $\mathcal{S}$ if and only if $\hat{f}_n \to \hat{f}$ in $\mathcal{S}$.

THEOREM 5.66. *The Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ defined by $\mathcal{F} : f \mapsto \hat{f}$ is a continuous, one-to-one map of $\mathcal{S}$ onto itself. The inverse $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$ is given by $\mathcal{F}^{-1} : f \mapsto \check{f}$. If $f \in \mathcal{S}$, then*

$$\mathcal{F}\left[\partial^\alpha f\right] = (ik)^\alpha \hat{f}, \qquad \mathcal{F}\left[(-ix)^\beta f\right] = \partial^\beta \hat{f}.$$

The Fourier transform maps the convolution product of two functions to the pointwise product of their transforms.

THEOREM 5.67. *If $f, g \in \mathcal{S}$, then the convolution $h = f * g \in \mathcal{S}$, and*

$$\hat{h} = (2\pi)^n \hat{f}\hat{g}.$$

If $f, g \in \mathcal{S}$, then

$$\int f\overline{g}\, dx = (2\pi)^n \int \hat{f}\overline{\hat{g}}\, dk.$$

In particular,

$$\int |f|^2\; dx = (2\pi)^n \int |\hat{f}|^2\, dk.$$

5.B.2. The Fourier transform on $\mathcal{S}'$. The main reason to introduce tempered distributions is that their Fourier transform is also a tempered distribution. If $\phi, \psi \in \mathcal{S}$, then by Fubini's theorem

$$\begin{aligned}
\int \phi\hat{\psi}\, dx &= \int \phi(x) \left[\frac{1}{(2\pi)^n} \int \psi(y) e^{-ix\cdot y}\, dy\right] dx \\
&= \int \left[\frac{1}{(2\pi)^n} \int \phi(x) e^{-ix\cdot y}\, dx\right] \psi(y)\, dy \\
&= \int \hat{\phi}\psi\, dx.
\end{aligned}$$

This motivates the following definition for the Fourier transform of a tempered distribution which is compatible with the one for Schwartz functions.

DEFINITION 5.68. If $T \in \mathcal{S}'$, then the Fourier transform $\hat{T} \in \mathcal{S}'$ is the distribution defined by

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle \qquad \text{for all } \phi \in \mathcal{S}.$$

The inverse Fourier transform $\check{T} \in \mathcal{S}'$ is the distribution defined by

$$\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle \qquad \text{for all } \phi \in \mathcal{S}.$$

We also write $\hat{T} = \mathcal{F}T$ and $\check{T} = \mathcal{F}^{-1}T$. The linearity and continuity of the Fourier transform on $\mathcal{S}$ implies that $\hat{T}$ is a linear, continuous map on $\mathcal{S}$, so the Fourier transform of a tempered distribution is a tempered distribution. The invertibility of the Fourier transform on $\mathcal{S}$ implies that $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is invertible with inverse $\mathcal{F}^{-1} : \mathcal{S}' \to \mathcal{S}'$.

EXAMPLE 5.69. If δ is the delta-function supported at 0, $\langle \delta, \phi \rangle = \phi(0)$, then

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^n} \int \phi(x)\, dx = \left\langle \frac{1}{(2\pi)^n}, \phi \right\rangle.$$

Thus, the Fourier transform of the δ-function is the constant function $(2\pi)^{-n}$. We may write this Fourier transform formally as

$$\delta(x) = \frac{1}{(2\pi)^n} \int e^{ik\cdot x}\, dk.$$

This result is consistent with Example 5.65. We have for the Gaussian δ-sequence that

$$\frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-|x|^2/2\sigma^2} \rightharpoonup \delta \qquad \text{in } \mathcal{S}' \text{ as } \sigma \to 0.$$

The corresponding Fourier transform of this limit is

$$\frac{1}{(2\pi)^n}\, e^{-\sigma^2|k|^2/2} \rightharpoonup \frac{1}{(2\pi)^n} \qquad \text{in } \mathcal{S}' \text{ as } \sigma \to 0.$$

If $T \in \mathcal{S}'$, it follows directly from the definitions and the properties of Schwartz functions that

$$\langle \widehat{\partial^\alpha T}, \phi \rangle = \langle \partial^\alpha T, \hat{\phi} \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \hat{\phi} \rangle = \langle T, \widehat{(ik)^\alpha \phi} \rangle = \langle \hat{T}, (ik)^\alpha \phi \rangle = \langle (ik)^\alpha \hat{T}, \phi \rangle,$$

with a similar result for the inverse transform. Thus,

$$\widehat{\partial^\alpha T} = (ik)^\alpha \hat{T}, \qquad \widehat{(-ix)^\beta T} = \partial^\beta \hat{T}.$$

The Fourier transform does not define a map of the test function space $\mathcal{D}$ into itself, since the Fourier transform of a compactly supported function does not, in general, have compact support. Thus, the Fourier transform of a distribution $T \in \mathcal{D}'$ is not, in general, a distribution $\hat{T} \in \mathcal{D}'$; this explains why we define the Fourier transform for the smaller class of tempered distributions.

The Fourier transform maps the space $\mathcal{D}$ onto a space $\mathcal{Z}$ of real-analytic functions,[6] and one can define the Fourier transform of a general distribution $T \in \mathcal{D}'$ as an ultradistribution $\hat{T} \in \mathcal{Z}'$ acting on $\mathcal{Z}$. We will not consider this theory further here.

[6]A function $\phi : \mathbb{R} \to \mathbb{C}$ belongs to $\mathcal{Z}(\mathbb{R})$ if and only if it extends to an entire function $\phi : \mathbb{C} \to \mathbb{C}$ with the property that, writing $z = x + iy$, there exists $a > 0$ and for each $k = 0, 1, 2, \ldots$ a constant C_k such that

$$\left| z^k \phi(z) \right| \leq C_k e^{a|y|}.$$

5.B.3. The Fourier transform on L^1. If $f \in L^1(\mathbb{R}^n)$, then

$$\left| \int f(x) e^{-ik\cdot x}\, dx \right| \leq \int |f|\, dx,$$

so we may define the Fourier transform $\hat{f}$ directly by the absolutely convergent integral in (5.73). Moreover,

$$\left| \hat{f}(k) \right| \leq \frac{1}{(2\pi)^n} \int |f|\, dx.$$

It follows by approximation of f by Schwartz functions that $\hat{f}$ is a uniform limit of Schwartz functions, and therefore $\hat{f} \in C_0$ is a continuous function that approaches zero at infinity. We therefore get the following Riemann-Lebesgue lemma.

THEOREM 5.70. *The Fourier transform is a bounded linear map $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ and*

$$\left\| \hat{f} \right\|_{L^\infty} \leq \frac{1}{(2\pi)^n} \|f\|_{L^1}.$$

The range of the Fourier transform on L^1 is not all of C_0, however, and it is difficult to characterize.

5.B.4. The Fourier transform on L^2. The next theorem, called Parseval's theorem, states that the Fourier transform preserves the L^2-inner product and norm, up to factors of 2π. It follows that we may extend the Fourier transform by density and continuity from $\mathcal{S}$ to an isomorphism on L^2 with the same properties. Explicitly, if $f \in L^2$, we choose any sequence of functions $f_k \in \mathcal{S}$ such that f_k converges to f in L^2 as $k \to \infty$. Then we define $\hat{f}$ to be the L^2-limit of the $\hat{f}_k$. Note that it is necessary to use a somewhat indirect approach to define the Fourier transform on L^2, since the Fourier integral in (5.73) does not converge if $f \in L^2 \setminus L^1$.

THEOREM 5.71. *The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a one-to-one, onto bounded linear map. If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\int f\overline{g}\, dx = (2\pi)^n \int \hat{f}\overline{\hat{g}}\, dk.$$

In particular,

$$\int |f|^2\, dx = (2\pi)^n \int |\hat{f}|^2\, dk.$$

5.B.5. The Fourier transform on L^p. The boundedness of the Fourier transform $\mathcal{F} : L^p \to L^{p'}$ for $1 < p < 2$ follows from its boundedness for $\mathcal{F} : L^1 \to L^\infty$ and $\mathcal{F} : L^2 \to L^2$ by use of the following Riesz-Thorin interpolation theorem.

THEOREM 5.72. *Let Ω be a measure space and $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$. Suppose that*

$$T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \to L^{q_0}(\Omega) + L^{q_1}(\Omega)$$

is a linear map such that $T : L^{p_i}(\Omega) \to L^{q_i}(\Omega)$ for $i = 0, 1$ and

$$\|Tf\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}}, \qquad \|Tf\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}$$

for some constants M_0, M_1. If $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then $T : L^p(\Omega) \to L^q(\Omega)$ *maps* $L^p(\Omega)$ *into* $L^q(\Omega)$ *and*

$$\|Tf\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}.$$

In this theorem, $L^{p_0}(\Omega) + L^{p_1}(\Omega)$ denotes the vector space of all complex-valued functions of the form $f = f_0 + f_1$ where $f_0 \in L^{p_0}(\Omega)$ and $f_1 \in L^{p_1}(\Omega)$. Note that if $q_0 = p_0'$ and $q_1 = p_1'$, then $q = p'$. An immediate consequence of this theorem and the L^1-L^2 estimates for the Fourier transform is the following Hausdorff-Young theorem.

THEOREM 5.73. *Suppose that* $1 \le p \le 2$. *The Fourier transform is a bounded linear map* $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ *and*

$$\|\mathcal{F}f\|_{L^{p'}} \le \frac{1}{(2\pi)^n} \|f\|_{L^p}.$$

If $1 \le p < 2$, the range of the Fourier transform on L^p is not all of $L^{p'}$, and there exist functions $f \in L^{p'}$ whose inverse Fourier transform is a tempered distribution that is not regular. Correspondingly, if $p > 2$ the range of $\mathcal{F} : L^p \to \mathcal{S}'$ contains non-regular distributions. For example, $1 \in L^\infty$ and $\mathcal{F}(1) = \delta$.

5.C. The Sobolev spaces $H^s(\mathbb{R}^n)$

A function belongs to L^2 if and only if its Fourier transform belongs to L^2, and the Fourier transform preserves the L^2-norm. As a result, the Fourier transform provides a simple way to define L^2-Sobolev spaces on $\mathbb{R}^n$, including ones of fractional and negative order. This approach does not generalize to L^p-Sobolev spaces with $p \ne 2$, since there is no simple way to characterize when a function belongs to L^p in terms of its Fourier transform.

We define a function $\langle\cdot\rangle : \mathbb{R}^n \to \mathbb{R}$ by

$$\langle x \rangle = \left(1 + |x|^2\right)^{1/2}.$$

This function grows linearly at infinity, like $|x|$, but is bounded away from zero. (There should be no confusion with the use of angular brackets to denote a duality pairing.)

DEFINITION 5.74. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transform $\hat{f}$ is a regular distribution such that

$$\int \langle k \rangle^{2s} \left|\hat{f}(k)\right|^2 dk < \infty.$$

The inner product and norm of $f, g \in H^s$ are defined by

$$(f,g)_{H^s} = (2\pi)^n \int \langle k \rangle^{2s} \hat{f}(k)\overline{\hat{g}}(k)\, dk, \qquad \|f\|_{H^s} = (2\pi)^n \left(\int \langle k \rangle^{2s} \left|\hat{f}(k)\right|^2 dk\right)^{1/2}.$$

Thus, under the Fourier transform, $H^s(\mathbb{R}^n)$ is isomorphic to the weighted L^2-space

$$\hat{H}^s(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{C} : \langle k \rangle f \in L^2\right\}, \tag{5.74}$$

with inner product

$$\left(\hat{f}, \hat{g}\right)_{\hat{H}^s} = (2\pi)^n \int \langle k \rangle^{2s} \hat{f}\overline{\hat{g}}\, dk.$$

The Sobolev spaces $\{H^s : s \in \mathbb{R}\}$ form a decreasing scale of Hilbert spaces with H^s continuously embedded in H^r for $s > r$. If $s \in \mathbb{N}$ is a positive integer, then $H^s(\mathbb{R}^n)$ is the usual Sobolev space of functions whose weak derivatives of order less than or equal to s belong to $L^2(\mathbb{R}^n)$, so this notation is consistent with our previous notation.

We may give a spatial description of H^s for general $s \in \mathbb{R}$ in terms of the pseudo-differential operator $\Lambda : \mathcal{S}' \to \mathcal{S}'$ with symbol $\langle k\rangle$ defined by

$$\Lambda = (I - \Delta)^{1/2}, \qquad \widehat{(\Lambda f)}(k) = \langle k\rangle \hat{f}(k). \tag{5.75}$$

Then $f \in H^s$ if and only if $\Lambda^s f \in L^2$, and

$$(f, g)_{\hat{H}^s} = \int (\Lambda^s f)\,(\Lambda^s \bar{g})\ dx, \qquad \|f\|_{\hat{H}^s} = \left(\int |\Lambda^s f|^2\ dx\right)^{1/2}.$$

Thus, roughly speaking, a function belongs to H^s if it has s weak derivatives (or integrals if $s < 0$) that belong to L^2.

Example 5.75. If $\delta \in \mathcal{S}'(\mathbb{R}^n)$, then $\hat{\delta} = (2\pi)^{-n}$ and

$$\int \langle k\rangle^{2s} \hat{\delta}^2\, dk = \frac{1}{(2\pi)^{2n}} \int \langle k\rangle^{2s}\, dk$$

converges if $2s < -n$. Thus, $\delta \in H^s(\mathbb{R}^n)$ if $s < -n/2$, which is precisely when functions in H^s are continuous and pointwise evaluation at 0 is a bounded linear functional. More generally, every compactly supported distribution belongs to H^s for some $s \in \mathbb{R}$.

Example 5.76. The Fourier transform of $1 \in \mathcal{S}'$, given by $\hat{1} = \delta$, is not a regular distribution. Thus, $1 \notin H^s$ for any $s \in \mathbb{R}$.

We let

$$H^\infty = \bigcap_{s\in\mathbb{R}} H^s, \qquad H^{-\infty} = \bigcup_{s\in\mathbb{R}} H^s. \tag{5.76}$$

Then $\mathcal{S} \subset H^\infty \subset H^{-\infty} \subset \mathcal{S}'$ and by the Sobolev embedding theorem $H^\infty \subset C_0^\infty$.

5.D. Fractional integrals

One way to approach fractional integrals and derivatives is through potential theory.

5.D.1. The Riesz potential.

For $0 < \alpha < n$, we define the Riesz potential $I_\alpha : \mathbb{R}^n \to \mathbb{R}$ by

$$I_\alpha(x) = \frac{1}{\gamma_\alpha}\frac{1}{|x|^{n-\alpha}}, \qquad \gamma_\alpha = \frac{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}{\Gamma(n/2 - \alpha/2)}.$$

Since $\alpha > 0$, we have $I_\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The Riesz potential of a function $\phi \in \mathcal{S}$ is defined by

$$I_\alpha * \phi(x) = \frac{1}{\gamma_\alpha}\int \frac{\phi(y)}{|x-y|^{n-\alpha}}\, dy.$$

The Fourier transform of this equation is

$$\widehat{(I_\alpha * \phi)}(k) = \frac{1}{|k|^\alpha}\,\hat{\phi}(k).$$

Thus, we can interpret convolution with I_α as a homogeneous, spherically symmetric fractional integral operator of the order α. We write it symbolically as

$$I_\alpha * \phi = |D|^{-\alpha} \phi,$$

where $|D|$ is the operator with symbol $|k|$. In particular, if $n \geq 3$ and $\alpha = 2$, the potential I_2 is the Green's function of the Laplacian operator,

$$-\Delta I_2 = \delta.$$

If we consider

$$|D|^{-\alpha} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

as a map from L^p to L^q, then a scaling argument similar to the one for the Sobolev embedding theorem implies that the map can be bounded only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \tag{5.77}$$

The following Hardy-Littlewood-Sobolev inequality states that this map is, in fact, bounded for $1 < p < n/\alpha$. The proof (see *e.g.* [**18**] or [**27**]) uses the boundedness of the Hardy-Littlewood maximal function on L^p for $1 < p < \infty$.

THEOREM 5.77. *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$, and q is defined by (5.77). If $f \in L^p(\mathbb{R}^n)$, then $I_\alpha * f \in L^q(\mathbb{R}^n)$ and there exists a constant $C(n, \alpha, p)$ such that*

$$\|I_\alpha * f\|_{L^q} \leq C \|f\|_{L^p} \qquad \text{for every } f \in L^p(\mathbb{R}^n).$$

This inequality may be thought of as a generalization of the Gagliardo-Nirenberg inequality in Theorem 3.28 to fractional derivatives. If $\alpha = 1$, then $q = p^*$ is the Sobolev conjugate of p, and writing $f = |D|g$ we get

$$\|g\|_{L^{p^*}} \leq C \|(|D|g)\|_{L^p}.$$

5.D.2. The Bessel potential. The Bessel potential corresponds to the operator

$$\Lambda^{-\alpha} = (I - \Delta)^{-\alpha/2} = \left(I + |D|^2\right)^{-\alpha/2}.$$

where Λ is defined in (5.75) and $\alpha > 0$. The operator $\Lambda^{-\alpha}$ is a non-homogeneous, spherically symmetric fractional integral operator; it plays an analogous role for non-homogeneous Sobolev spaces to the fractional derivative $|D|^{-\alpha}$ for homogeneous Sobolev spaces.

If $\phi \in \mathcal{S}$, then

$$(\widehat{\Lambda^{-\alpha}\phi})(k) = \frac{1}{(1 + |k|^2)^{\alpha/2}} \hat{\phi}(k).$$

Thus, by the convolution theorem,

$$\Lambda^{-\alpha}\phi = G_\alpha * \phi$$

where

$$G_\alpha = \mathcal{F}^{-1}\left[\frac{1}{(1 + |k|^2)^{\alpha/2}}\right]. \tag{5.78}$$

For any $0 < \alpha < \infty$, this distributional inverse transform defines a positive function that is smooth in $\mathbb{R}^n \setminus \{0\}$. For example, if $\alpha = 2$, then G_2 is the Green's function of the Helmholtz equation

$$-\Delta G_2 + G_2 = \delta.$$

Unlike the kernel I_α of the Riesz transform, however, there is no simple explicit expression for G_α.

For large k, the Fourier transform of the Bessel potential behaves asymptotically like the Riesz potential and the potentials have the same singular behavior at $x \to 0$. For small k, the Bessel potential behaves like $1 - (\alpha/2)|k|^2$, and it decays exponentially as $|x| \to \infty$ rather than algebraically like the Riesz potential. We therefore have the following estimate.

Proposition 5.78. *Suppose that $0 < \alpha < n$ and G_α is the Bessel potential defined in (5.78). Then there exists a constant $C = C(\alpha, n)$ such that*

$$0 < G_\alpha(x) \le \frac{C}{|x|^{n-\alpha}} \quad \text{if } 0 < |x| < 1, \qquad 0 < G_\alpha(x) \le e^{-|x|/2} \quad \text{if } |x| \ge 1.$$

Finally, we state a version of the Sobolev embedding theorem for fractional L^2-Sobolev spaces.

Theorem 5.79. *If $0 < s < n/2$ and*

$$\frac{1}{q} = \frac{1}{2} - \frac{s}{n},$$

then $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and there exists a constant $C = C(n, s)$ such that

$$\|f\|_{L^q} \le \|f\|_{H^s}.$$

If $n/2 < s < \infty$, then $H^s(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n)$ and there exists a constant $C = C(n, s)$ such that

$$\|f\|_{L^\infty} \le \|f\|_{H^s}.$$

Proof. The result for $s < n/2$ follows from Proposition 5.78 and the Hardy-Littlewood-Sobolev inequality *c.f.* [**18**].

If $s > n/2$, we have for $f \in \mathcal{S}$ that

$$\begin{aligned}
\|f\|_{L^\infty} &= \sup_{x \in \mathbb{R}^n} \left| \int \hat{f}(k) e^{ik\cdot x}\, dk \right| \\
&\le \int \left| \hat{f}(k) \right| dk \\
&\le \int \frac{1}{(1+|k|^2)^{s/2}} \cdot \left(1+|k|^2\right)^{s/2} \left| \hat{f}(k) \right| dk \\
&\le \left(\int \frac{1}{(1+|k|^2)^s}\, dk \right)^{1/2} \left(\int \left(1+|k|^2\right)^s \left| \hat{f}(k) \right|^2 dk \right)^{1/2} \\
&\le C \|f\|_{H^s},
\end{aligned}$$

since the first integral converges when $2s > n$. Since $\mathcal{S}$ is dense in H^s, it follows that this inequality holds for every $f \in H^s$ and that $f \in C_0$ since f is the uniform limit of Schwartz functions. $\square$

CHAPTER 6

Parabolic Equations

The theory of parabolic PDEs closely follows that of elliptic PDEs and, like elliptic PDEs, parabolic PDEs have strong smoothing properties. For example, there are parabolic versions of the maximum principle and Harnack's inequality, and a Schauder theory for Hölder continuous solutions [**28**]. Moreover, we may establish the existence and regularity of weak solutions of parabolic PDEs by the use of L^2-energy estimates.

6.1. The heat equation

Just as Laplace's equation is a prototypical example of an elliptic PDE, the heat equation

$$u_t = \Delta u + f \tag{6.1}$$

is a prototypical example of a parabolic PDE. This PDE has to be supplemented by suitable initial and boundary conditions to give a well-posed problem with a unique solution. As an example of such a problem, consider the following IBVP with Dirichlet BCs on a bounded open set $\Omega \subset \mathbb{R}^n$ for $u : \Omega \times [0, \infty) \to \mathbb{R}$:

$$\begin{aligned} &u_t = \Delta u + f(x,t) && \text{for } x \in \Omega \text{ and } t > 0,\\ &u(x,t) = 0 && \text{for } x \in \partial\Omega \text{ and } t > 0,\\ &u(x,0) = g(x) && \text{for } x \in \Omega. \end{aligned} \tag{6.2}$$

Here $f : \Omega \times (0, \infty) \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are a given forcing term and initial condition. This problem describes the evolution in time of the temperature $u(x,t)$ of a body occupying the region Ω containing a heat source f per unit volume, whose boundary is held at fixed zero temperature and whose initial temperature is g.

One important estimate (in L^∞) for solutions of (6.2) follows from the maximum principle. If $f \le 0$, corresponding to 'heat sinks,' then for any $T > 0$,

$$\max_{\overline{\Omega}\times[0,T]} u \le \max\left[0, \max_{\overline{\Omega}} g\right].$$

To derive this inequality, note that if u is a smooth function which attains a maximum at $x \in \Omega$ and $0 < t \le T$, then $u_t = 0$ if $0 < t < T$ or $u_t \ge 0$ if $t = T$ and $\Delta u \le 0$. Thus $u_t - \Delta u \ge 0$ which is impossible if $f < 0$, so u attains its maximum on $\partial\Omega \times [0, T]$, where $u = 0$, or at $t = 0$. The result for $f \le 0$ follows by a perturbation argument. The physical interpretation of this maximum principle in terms of thermal diffusion is that a local "hotspot" cannot develop spontaneously in the interior when no heat sources are present. Similarly, if $f \ge 0$, we have the minimum principle

$$\min_{\overline{\Omega}\times[0,T]} u \ge \min\left[0, \min_{\overline{\Omega}} g\right].$$

Another basic estimate for the heat equation (in L^2) follows from an integration of the equation. We multiply (6.1) by u, integrate over Ω, apply the divergence theorem, and use the BC that $u = 0$ on $\partial\Omega$ to obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |Du|^2\,dx = \int_\Omega fu\,dx.$$

Integrating this equation with respect to time and using the initial condition, we get

$$\frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega |Du|^2\,dxds = \int_0^t\int_\Omega fu\,dxds + \frac{1}{2}\int_\Omega g^2\,dx. \tag{6.3}$$

For $0 \le t \le T$, we have from the Cauchy inequality with ϵ that

$$\begin{aligned}\int_0^t\int_\Omega fu\,dxds &\le \left(\int_0^t\int_\Omega f^2\,dxds\right)^{1/2}\left(\int_0^t\int_\Omega u^2\,dxds\right)^{1/2}\\ &\le \frac{1}{4\epsilon}\int_0^T\int_\Omega f^2\,dxds + \epsilon\int_0^T\int_\Omega u^2\,dxds\\ &\le \frac{1}{4\epsilon}\int_0^T\int_\Omega f^2\,dxds + \epsilon T\max_{0\le t\le T}\int_\Omega u^2\,dx.\end{aligned}$$

Thus, taking the supremum of (6.3) over $t \in [0,T]$ and using this inequality with $\epsilon T = 1/4$ in the result, we get

$$\frac{1}{4}\max_{[0,T]}\int_\Omega u^2(x,t)\,dx + \int_0^T\int_\Omega |Du|^2\,dxdt \le T\int_0^T\int_\Omega f^2\,dxdt + \frac{1}{2}\int_\Omega g^2\,dx.$$

It follows that we have an *a priori* energy estimate of the form

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)} \le C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\right) \tag{6.4}$$

where $C = C(T)$ is a constant depending only on T. We will use this energy estimate to construct weak solutions.[1] The parabolic smoothing of the heat equation is evident from the fact that if $f = 0$, say, we can estimate not only the solution u but its derivative Du in terms of the initial data g.

6.2. General second-order parabolic PDEs

The qualitative properties of (6.1) are almost unchanged if we replace the Laplacian $-\Delta$ by any uniformly elliptic operator L on $\Omega\times(0,T)$. We write L in divergence form as

$$L = -\sum_{i,j=1}^n \partial_i\left(a^{ij}\partial_j u\right) + \sum_{j=1}^n b^j\partial_j u + cu \tag{6.5}$$

where $a^{ij}(x,t)$, $b^i(x,t)$, $c(x,t)$ are coefficient functions with $a^{ij} = a^{ji}$. We assume that there exists $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \ge \theta|\xi|^2 \qquad \text{for all } (x,t)\in\Omega\times(0,T) \text{ and } \xi\in\mathbb{R}^n. \tag{6.6}$$

[1] In fact, we will use a slightly better estimate in which $\|f\|_{L^2(0,T;L^2)}$ is replaced by the weaker norm $\|f\|_{L^2(0,T;H^{-1})}$.

The corresponding parabolic PDE is then

$$u_t + \sum_{j=1}^{n} b^j \partial_j u + cu = \sum_{i,j=1}^{n} \partial_i \left(a^{ij} \partial_j u\right) + f. \tag{6.7}$$

Equation (6.7) describes evolution of a temperature field u under the combined effects of diffusion a^{ij}, advection b^i, linear growth or decay c, and external heat sources f.

The corresponding IBVP with homogeneous Dirichlet BCs is

$$\begin{aligned} &u_t + Lu = f, \\ &u(x,t) = 0 \qquad\qquad \text{for } x \in \partial\Omega \text{ and } t > 0, \\ &u(x,0) = g(x) \qquad\quad \text{for } x \in \overline{\Omega}. \end{aligned} \tag{6.8}$$

Essentially the same estimates hold for this problem as for the heat equation. To begin with, we use the L^2-energy estimates to prove the existence of suitably defined weak solutions of (6.8).

6.3. Definition of weak solutions

To formulate a definition of a weak solution of (6.8), we first suppose that the domain Ω, the coefficients of L, and the solution u are smooth. Multiplying (6.7), by a test function $v \in C_c^\infty(\Omega)$, integrating the result over Ω, and applying the divergence theorem, we get

$$(u_t(t), v)_{L^2} + a\,(u(t), v; t) = (f(t), v)_{L^2} \qquad \text{for } 0 \le t \le T \tag{6.9}$$

where $(\cdot,\cdot)_{L^2}$ denotes the L^2-inner product

$$(u, v)_{L^2} = \int_\Omega u(x) v(x)\, dx,$$

and a is the bilinear form associated with L

$$\begin{aligned} a(u, v; t) = \sum_{i,j=1}^{n} &\int_\Omega a^{ij}(x,t) \partial_i u(x) \partial_j u(x)\, dx \\ &+ \sum_{j=1}^{n} \int_\Omega b^j(x,t) \partial_j u(x) v(x)\, dx + \int_\Omega c(x,t) u(x) v(x)\, dx. \end{aligned} \tag{6.10}$$

In (6.9), we have switched to the "vector-valued" viewpoint, and write $u(t) = u(\cdot, t)$.

To define weak solutions, we generalize (6.9) in a natural way. In order to ensure that the definition makes sense, we make the following assumptions.

Assumption 6.1. *The set $\Omega \subset \mathbb{R}^n$ is bounded and open, $T > 0$, and:*

(1) *the coefficients of a in (6.10) satisfy $a^{ij}, b^j, c \in L^\infty(\Omega \times (0,T))$;*
(2) *$a^{ij} = a^{ji}$ for $1 \le i, j \le n$ and the uniform ellipticity condition (6.6) holds for some constant $\theta > 0$;*
(3) *$f \in L^2\left(0, T; H^{-1}(\Omega)\right)$ and $g \in L^2(\Omega)$.*

Here, we allow f to take values in $H^{-1}(\Omega) = H_0^1(\Omega)'$. We denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ by

$$\langle \cdot, \cdot \rangle : H^{-1}(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

Since the coefficients of a are uniformly bounded in time, it follows from Theorem 4.21 that

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \times (0,T) \to \mathbb{R}.$$

Moreover, there exist constants $C > 0$ and $\gamma \in \mathbb{R}$ such that for every $u, v \in H_0^1(\Omega)$

$$C\|u\|_{H_0^1}^2 \leq a(u,u;t) + \gamma\|u\|_{L^2}^2, \tag{6.11}$$

$$|a(u,v;t)| \leq C\, \|u\|_{H_0^1} \|v\|_{H_0^1}. \tag{6.12}$$

We then define weak solutions of (6.8) as follows.

Definition 6.2. A function $u : [0,T] \to H_0^1(\Omega)$ is a weak solution of (6.8) if:

(1) $u \in L^2\left(0,T; H_0^1(\Omega)\right)$ and $u_t \in L^2\left(0,T; H^{-1}(\Omega)\right)$;

(2) For every $v \in H_0^1(\Omega)$,

$$\langle u_t(t), v\rangle + a\left(u(t), v; t\right) = \langle f(t), v\rangle \tag{6.13}$$

for t pointwise *a.e.* in $[0,T]$ where a is defined in (6.10);

(3) $u(0) = g$.

The PDE is imposed in a weak sense by (6.13) and the boundary condition $u = 0$ on $\partial\Omega$ by the requirement that $u(t) \in H_0^1(\Omega)$. Two points about this definition deserve comment.

First, the time derivative u_t in (6.13) is understood as a distributional time derivative; that is $u_t = w$ if

$$\int_0^T \phi(t)u(t)\,dt = -\int_0^T \phi'(t)w(t)\,dt \tag{6.14}$$

for every $\phi : (0,T) \to \mathbb{R}$ with $\phi \in C_c^\infty(0,T)$. This is a direct generalization of the notion of the weak derivative of a real-valued function. The integrals in (6.14) are vector-valued Lebesgue integrals (Bochner integrals), which are defined in an analogous way to the Lebesgue integral of an integrable real-valued function as the L^1-limit of integrals of simple functions. See Section 6.A for further discussion of such integrals and the weak derivative of vector-valued functions. Equation (6.13) may then be understood in a distributional sense as an equation for the weak derivative u_t on $(0,T)$.

Second, it is not immediately obvious that the initial condition $u(0) = g$ in Definition 6.2 makes sense. We do not explicitly require any continuity on u, and since $u \in L^2\left(0,T; H_0^1(\Omega)\right)$ is defined only up to pointwise everywhere equivalence in $t \in [0,T]$ it is not clear that specifying a pointwise value at $t = 0$ imposes any restriction on u. As shown in Theorem 6.41, however, the conditions that $u \in L^2\left(0,T; H_0^1(\Omega)\right)$ and $u_t \in L^2\left(0,T; H^{-1}(\Omega)\right)$ imply that $u \in C\left([0,T]; L^2(\Omega)\right)$. Therefore, identifying u with its continuous representative, we see that the initial condition makes sense.

We then have the following existence result, whose proof will be given in the following sections.

Theorem 6.3. *Suppose that the conditions in Assumption 6.1 are satisfied. Then for every $f \in L^2\left(0,T; H^{-1}(\Omega)\right)$ and $g \in H_0^1(\Omega)$ there is a unique weak solution*

$$u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$$

of (6.8), in the sense of Definition 6.2, with $u_t \in L^2\left(0, T; H^{-1}(\Omega)\right)$. *Moreover, there is a constant* C*, depending only on* Ω*,* T*, and the coefficients of* L*, such that*

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)} + \|u_t\|_{L^2(0,T;H^{-1})} \leq C\left(\|f\|_{L^2(0,T;H^{-1})} + \|g\|_{L^2}\right).$$

6.4. The Galerkin approximation

The basic idea of the existence proof is to approximate $u : [0, T] \to H_0^1(\Omega)$ by functions $u_N : [0, T] \to E_N$ that take values in a finite-dimensional subspace $E_N \subset H_0^1(\Omega)$ of dimension N. To obtain the u_N, we project the PDE onto E_N, meaning that we require that u_N satisfies the PDE up to a residual which is orthogonal to E_N. This gives a system of ODEs for u_N, which has a solution by standard ODE theory. Each u_N satisfies an energy estimate of the same form as the *a priori* estimate for solutions of the PDE. These estimates are uniform in N, which allows us to pass to the limit $N \to \infty$ and obtain a solution of the PDE.

In more detail, the existence of uniform bounds implies that the sequence $\{u_N\}$ is weakly compact in a suitable space and hence, by the Banach-Alaoglu theorem, there is a weakly convergent subsequence $\{u_{N_k}\}$ such that $u_{N_k} \rightharpoonup u$ as $k \to \infty$. Since the PDE and the approximating ODEs are linear, and linear functionals are continuous with respect to weak convergence, the weak limit of the solutions of the ODEs is a solution of the PDE. As with any similar compactness argument, we get existence but not uniqueness, since it is conceivable that different subsequences of approximate solutions could converge to different weak solutions. We can, however, prove uniqueness of a weak solution directly from the energy estimates. Once we know that the solution is unique, it follows by a compactness argument that we have weak convergence $u_N \rightharpoonup u$ of the full approximate sequence. One can then prove that the sequence, in fact, converges strongly in $L^2(0, T; H_0^1)$.

Methods such as this one, in which we approximate the solution of a PDE by the projection of the solution and the equation into finite dimensional subspaces, are called Galerkin methods. Such methods have close connections with the variational formulation of PDEs. For example, in the time-independent case of an elliptic PDE given by a variational principle, we may approximate the minimization problem for the PDE over an infinite-dimensional function space E by a minimization problem over a finite-dimensional subspace E_N. The corresponding equations for a critical point are a finite-dimensional approximation of the weak formulation of the original PDE. We may then show, under suitable assumptions, that as $N \to \infty$ solutions u_N of the finite-dimensional minimization problem approach a solution u of the original problem.

There is considerable flexibility the finite-dimensional spaces E_N one uses in a Galerkin method. For our analysis, we take

$$E_N = \langle w_1, w_2, \ldots, w_N \rangle \tag{6.15}$$

to be the linear space spanned by the first N vectors in an orthonormal basis $\{w_k : k \in \mathbb{N}\}$ of $L^2(\Omega)$, which we may also assume to be an orthogonal basis of $H_0^1(\Omega)$. For definiteness, take the $w_k(x)$ to be the eigenfunctions of the Dirichlet Laplacian on Ω:

$$-\Delta w_k = \lambda_k w_k \quad w_k \in H_0^1(\Omega) \qquad \text{for } k \in \mathbb{N}. \tag{6.16}$$

From the previous existence theory for solutions of elliptic PDEs, the Dirichlet Laplacian on a bounded open set is a self-adjoint operator with compact resolvent, so that suitably normalized set of eigenfunctions have the required properties.

Explicitly, we have

$$\int_\Omega w_j w_k\,dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \qquad \int_\Omega Dw_j \cdot Dw_k\,dx = \begin{cases} \lambda_j & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

We may expand any $u \in L^2(\Omega)$ in an L^2-convergent series as

$$u(x) = \sum_{k\in\mathbb{N}} c^k w_k(x)$$

where $c^k = (u, w_k)_{L^2}$ and $u \in L^2(\Omega)$ if and only if

$$\sum_{k\in\mathbb{N}} \left|c^k\right|^2 < \infty.$$

Similarly, $u \in H_0^1(\Omega)$, and the series converges in $H_0^1(\Omega)$, if and only if

$$\sum_{k\in\mathbb{N}} \lambda_k \left|c^k\right|^2 < \infty.$$

We denote by $P_N : L^2(\Omega) \to E_N \subset L^2(\Omega)$ the orthogonal projection onto E_N defined by

$$P_N\left(\sum_{k\in\mathbb{N}} c^k w_k\right) = \sum_{k=1}^N c^k w_k. \tag{6.17}$$

We also denote by P_N the orthogonal projections $P_N : H_0^1(\Omega) \to E_N \subset H_0^1(\Omega)$ or $P_N : H^{-1}(\Omega) \to E_N \subset H^{-1}(\Omega)$, which we obtain by restricting or extending P_N from $L^2(\Omega)$ to $H_0^1(\Omega)$ or $H^{-1}(\Omega)$, respectively. Thus, P_N is defined on $H_0^1(\Omega)$ by (6.17) and on $H^{-1}(\Omega)$ by

$$\langle P_N u, v\rangle = \langle u, P_N v\rangle \qquad \text{for all } v \in H_0^1(\Omega).$$

While this choice of E_N is convenient for our existence proof, other choices are useful in different contexts. For example, the finite-element method is a numerical implementation of the Galerkin method which uses a space E_N of piecewise polynomial functions that are supported on simplices, or some other kind of element. Unlike the eigenfunctions of the Laplacian, finite-element basis functions, which are supported on a small number of adjacent elements, are straightforward to construct explicitly. Furthermore, one can approximate functions on domains with complicated geometry in terms of the finite-element basis functions by subdividing the domain into simplices, and one can refine the decomposition in regions where higher resolution is required. The finite-element basis functions are not exactly orthogonal, but they are almost orthogonal since they overlap only if they are supported on nearby elements. As a result, the associated Galerkin equations involve sparse matrices, which is crucial for their efficient numerical solution. One can obtain rigorous convergence proofs for finite-element methods that are similar to the proof discussed here (at least, if the underlying equations are not too complicated).

6.5. Existence of weak solutions

We proceed in three steps:

(1) Construction of approximate solutions;
(2) Derivation of energy estimates for approximate solutions;
(3) Convergence of approximate solutions to a solution.

After proving the existence of weak solutions, we will show that they are unique and make some brief comments on their regularity and continuous dependence on the data. We assume throughout this section, without further comment, that Assumption 6.1 holds.

6.5.1. Construction of approximate solutions. First, we define what we mean by an approximate solution. Let E_N be the N-dimensional subspace of $H_0^1(\Omega)$ given in (6.15)–(6.16) and P_N the orthogonal projection onto E_N given by (6.17).

Definition 6.4. A function $u_N : [0,T] \to E_N$ is an approximate solution of (6.8) if:

(1) $u_N \in L^2(0,T;E_N)$ and $u_{Nt} \in L^2(0,T;E_N)$;
(2) for every $v \in E_N$

$$(u_{Nt}(t), v)_{L^2} + a\left(u_N(t), v; t\right) = \langle f(t), v\rangle \tag{6.18}$$

pointwise *a.e.* in $t \in (0,T)$;

(3) $u_N(0) = P_N g$.

Since $u_N \in H^1(0,T;E_N)$, it follows from the Sobolev embedding theorem for functions of a single variable t that $u_N \in C([0,T];E_N)$, so the initial condition (3) makes sense. Condition (2) requires that u_N satisfies the weak formulation (6.13) of the PDE in which the test functions v are restricted to E_N. This is equivalent to the condition that

$$u_{Nt} + P_N L u_N = P_N f$$

for $t \in (0,T)$ pointwise *a.e.*, meaning that u_N takes values in E_N and satisfies the projection of the PDE onto E_N.[2]

To prove the existence of an approximate solution, we rewrite their definition explicitly as an IVP for an ODE. We expand

$$u_N(t) = \sum_{k=1}^{N} c_N^k(t) w_k \tag{6.19}$$

where the $c_N^k : [0,T] \to \mathbb{R}$ are absolutely continuous scalar coefficient functions. By linearity, it is sufficient to impose (6.18) for $v = w_1, \dots, w_N$. Thus, (6.19) is an approximate solution if and only if

$$c_N^k \in L^2(0,T), \quad c_{Nt}^k \in L^2(0,T) \qquad \text{for } 1 \le k \le N,$$

and $\{c_N^1, \dots, c_N^N\}$ satisfies the system of ODEs

$$c_{Nt}^j + \sum_{k=1}^{N} a^{jk} c_N^k = f^j, \quad c_N^j(0) = g^j \qquad \text{for } 1 \le j \le N \tag{6.20}$$

[2]More generally, one can define approximate solutions which take values in an N-dimensional space E_N and satisfy the projection of the PDE on another N-dimensional space F_N. This flexibility can be useful for problems that are highly non-self adjoint, but it is not needed here.

where

$$a^{jk}(t) = a(w_j, w_k; t), \quad f^j(t) = \langle f(t), w_j \rangle, \quad g^j = (g, w_j)_{L^2}.$$

Equation (6.20) may be written in vector form for $\vec{c} : [0, T] \to \mathbb{R}^N$ as

$$\vec{c}_{Nt} + A(t)\vec{c}_N = \vec{f}(t), \qquad \vec{c}_N(0) = \vec{g} \tag{6.21}$$

where

$$\vec{c}_N = \{c_N^1, \ldots, c_N^N\}^T, \quad \vec{f} = \{f^1, \ldots, f^N\}^T, \quad \vec{g} = \{g^1, \ldots, g^N\}^T,$$

and $A : [0, T] \to \mathbb{R}^{N\times N}$ is a matrix-valued function of t with coefficients $(a^{jk})_{j,k=1,N}$.

Proposition 6.5. *For every $N \in \mathbb{N}$, there exists a unique approximate solution $u_N : [0, T] \to E_N$ of (6.8).*

Proof. This result follows by standard ODE theory. We give the proof since the coefficient functions in (6.21) are bounded but not necessarily continuous functions of t. This is, however, sufficient since the ODE is linear.

From Assumption 6.1 and (6.12), we have

$$A \in L^\infty\left(0, T; \mathbb{R}^{N\times N}\right), \qquad \vec{f} \in L^2\left(0, T; \mathbb{R}^N\right). \tag{6.22}$$

Writing (6.21) as an equivalent integral equation, we get

$$\vec{c}_N = \Phi\left(\vec{c}_N\right), \qquad \Phi\left(\vec{c}_N\right)(t) = \vec{g} - \int_0^t A(s)\vec{c}_N(s)\, ds + \int_0^t \vec{f}(s)\, ds.$$

If follows from (6.22) that $\Phi : C\left([0, T_*]; \mathbb{R}^N\right) \to C\left([0, T_*]; \mathbb{R}^N\right)$ for any $0 < T_* \leq T$. Moreover, if $\vec{p}, \vec{q} \in C\left([0, T_*]; \mathbb{R}^N\right)$ then

$$\|\Phi(\vec{p}) - \Phi(\vec{q})\|_{L^\infty([0,T_*];\mathbb{R}^N)} \leq MT_* \|\vec{p} - \vec{q}\|_{L^\infty([0,T_*];\mathbb{R}^N)}$$

where

$$M = \sup_{0\leq t\leq T} \|A(t)\|.$$

Hence, if $MT_* < 1$, the map Φ is a contraction on $C\left([0, T_*]; \mathbb{R}^N\right)$. The contraction mapping theorem then implies that there is a unique solution on $[0, T_*]$ which extends, after a finite number of applications of this result, to a solution $\vec{c}_N \in C\left([0, T]; \mathbb{R}^N\right)$. The corresponding approximate solution satisfies $u_N \in C([0, T]; E_N)$. Moreover,

$$\vec{c}_{Nt} = \Phi(\vec{c}_N)_t = -A\vec{c}_N + \vec{f} \in L^2\left(0, T; \mathbb{R}^N\right),$$

which implies that $u_{Nt} \in L^2(0, T; E_N)$. □

6.5.2. Energy estimates for approximate solutions. The derivation of energy estimates for the approximate solutions follows the derivation of the *a priori* estimate (6.4) for the heat equation. Instead of multiplying the heat equation by u, we take the test function $v = u_N$ in the Galerkin equations.

Proposition 6.6. *There exists a constant C, depending only on T, Ω, and the coefficient functions a^{ij}, b^j, c, such that for every $N \in \mathbb{N}$ the approximate solution u_N constructed in Proposition 6.5 satisfies*

$$\|u_N\|_{L^\infty(0,T;L^2)} + \|u_N\|_{L^2(0,T;H_0^1)} + \|u_{Nt}\|_{L^2(0,T;H^{-1})} \leq C\left(\|f\|_{L^2(0,T;H^{-1})} + \|g\|_{L^2}\right).$$

PROOF. Taking $v = u_N(t) \in E_N$ in (6.18), we find that

$$(u_{Nt}(t), u_N(t))_{L^2} + a\left(u_N(t), u_N(t); t\right) = \langle f(t), u_N(t)\rangle$$

pointwise *a.e.* in $(0, T)$. Using this equation and the coercivity estimate (6.11), we find that there are constants $\beta > 0$ and $-\infty < \gamma < \infty$ such that

$$\frac{1}{2}\frac{d}{dt}\|u_N\|_{L^2}^2 + \beta\|u_N\|_{H_0^1}^2 \leq \langle f, u_N\rangle + \gamma\|u_N\|_{L^2}^2$$

pointwise *a.e.* in $(0, T)$, which implies that

$$\frac{1}{2}\frac{d}{dt}\left(e^{-2\gamma t}\|u_N\|_{L^2}^2\right) + \beta e^{-2\gamma t}\|u_N\|_{H_0^1}^2 \leq e^{-2\gamma t}\langle f, u_N\rangle.$$

Integrating this inequality with respect to t, using the initial condition $u_N(0) = P_N g$, and the projection inequality $\|P_N g\|_{L^2} \leq \|g\|_{L^2}$, we get for $0 \leq t \leq T$ that

$$\frac{1}{2}e^{-2\gamma t}\|u_N(t)\|_{L^2}^2 + \beta\int_0^t e^{-2\gamma s}\|u_N\|_{H_0^1}^2\,ds \leq \frac{1}{2}\|g\|_{L^2}^2 + \int_0^t e^{-2\gamma s}\langle f, u_N\rangle\,ds. \tag{6.23}$$

It follows from the definition of the H^{-1} norm, the Cauchy-Schwartz inequality, and Cauchy's inequality with ϵ that

$$\begin{aligned}\int_0^t e^{-2\gamma s}\langle f, u_N\rangle\,ds &\leq \int_0^t e^{-2\gamma s}\|f\|_{H^{-1}}\|u_N\|_{H_0^1}\,ds \\ &\leq \left(\int_0^t e^{-2\gamma s}\|f\|_{H^{-1}}^2\,ds\right)^{1/2}\left(\int_0^t e^{-2\gamma s}\|u_N\|_{H_0^1}^2\,ds\right)^{1/2} \\ &\leq C\|f\|_{L^2(0,T;H^{-1})}\left(\int_0^t e^{-2\gamma s}\|u_N\|_{H_0^1}^2\,ds\right)^{1/2} \\ &\leq C\|f\|_{L^2(0,T;H^{-1})}^2 + \frac{\beta}{2}\int_0^t e^{-2\gamma s}\|u_N\|_{H_0^1}^2\,ds,\end{aligned}$$

and using this result in (6.23) we get

$$\frac{1}{2}e^{-2\gamma t}\|u_N(t)\|_{L^2}^2 + \frac{\beta}{2}\int_0^t e^{-2\gamma s}\|u_N\|_{H_0^1}^2\,ds \leq \frac{1}{2}\|g\|_{L^2}^2 + C\|f\|_{L^2(0,T;H^{-1})}^2.$$

Taking the supremum of this equation with respect to t over $[0, T]$, we find that there is a constant C such that

$$\|u_N\|_{L^\infty(0,T;L^2)}^2 + \|u_N\|_{L^2(0,T;H_0^1)}^2 \leq C\left(\|g\|_{L^2}^2 + \|f\|_{L^2(0,T;H^{-1})}^2\right). \tag{6.24}$$

To estimate u_{Nt}, we note that since $u_{Nt}(t) \in E_N$

$$\|u_{Nt}(t)\|_{H^{-1}} = \sup_{v\in E_N\setminus\{0\}}\frac{(u_{Nt}(t), v)_{L^2}}{\|v\|_{H_0^1}}.$$

From (6.18) and (6.12) we have

$$\begin{aligned}(u_{Nt}(t), v)_{L^2} &\leq |a\left(u_N(t), v; t\right)| + |\langle f(t), v\rangle| \\ &\leq C\left(\|u_N(t)\|_{H_0^1} + \|f(t)\|_{H^{-1}}\right)\|v\|_{H_0^1}\end{aligned}$$

for every $v \in H_0^1$, and therefore

$$\|u_{Nt}(t)\|_{H^{-1}}^2 \leq C\left(\|u_N(t)\|_{H_0^1}^2 + \|f(t)\|_{H^{-1}}^2\right).$$

Integrating this equation with respect to t and using (6.24) in the result, we obtain

$$\|u_{Nt}\|^2_{L^2(0,T;H^{-1})} \leq C\left(\|g\|^2_{L^2} + \|f\|^2_{L^2(0,T;H^{-1})}\right). \tag{6.25}$$

Equations (6.24) and (6.25) complete the proof. □

6.5.3. Convergence of approximate solutions. Next we prove that a subsequence of approximate solutions converges to a weak solution. We use a weak compactness argument, so we begin by describing explicitly the type of weak convergence involved.

We identify the dual space of $L^2\left(0,T;H_0^1(\Omega)\right)$ with $L^2\left(0,T;H^{-1}(\Omega)\right)$. The action of $f \in L^2\left(0,T;H^{-1}(\Omega)\right)$ on $u \in L^2\left(0,T;H_0^1(\Omega)\right)$ is given by

$$\langle\langle f, u\rangle\rangle = \int_0^T \langle f, u\rangle\, dt$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the duality pairing between $L^2\left(0,T;H^{-1}\right)$ and $L^2\left(0,T;H_0^1\right)$, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H^{-1} and H_0^1.

Weak convergence $u_N \rightharpoonup u$ in $L^2\left(0,T;H_0^1(\Omega)\right)$ means that

$$\int_0^T \langle f(t), u_N(t)\rangle\, dt \to \int_0^T \langle f(t), u(t)\rangle\, dt \qquad \text{for every } f \in L^2\left(0,T;H^{-1}(\Omega)\right).$$

Similarly, $f_N \rightharpoonup f$ in $L^2\left(0,T;H^{-1}(\Omega)\right)$ means that

$$\int_0^T \langle f_N(t), u(t)\rangle\, dt \to \int_0^T \langle f(t), u(t)\rangle\, dt \qquad \text{for every } u \in L^2\left(0,T;H_0^1(\Omega)\right).$$

If $u_N \rightharpoonup u$ weakly in $L^2\left(0,T;H_0^1(\Omega)\right)$ and $f_N \to f$ strongly in $L^2\left(0,T;H^{-1}(\Omega)\right)$, or conversely, then $\langle f_N, u_N\rangle \to \langle f, u\rangle$.[3]

PROPOSITION 6.7. *A subsequence of approximate solutions converges weakly in* $L^2\left(0,T;H^{-1}(\Omega)\right)$ *to a weak solution*

$$u \in C\left([0,T];L^2(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right)$$

of (6.8) with $u_t \in L^2\left(0,T;H^{-1}(\Omega)\right)$. *Moreover, there is a constant C such that*

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)} + \|u_t\|_{L^2(0,T;H^{-1})} \leq C\left(\|f\|_{L^2(0,T;H^{-1})} + \|g\|_{L^2}\right).$$

PROOF. Proposition 6.6 implies that the approximate solutions $\{u_N\}$ are bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$ and their time derivatives $\{u_{Nt}\}$ are bounded in $L^2\left(0,T;H^{-1}(\Omega)\right)$. It follows from the Banach-Alaoglu theorem (Theorem 1.19) that we can extract a subsequence, which we still denote by $\{u_N\}$, such that

$$u_N \rightharpoonup u \quad \text{in } L^2\left(0,T;H_0^1\right), \qquad u_{Nt} \rightharpoonup u_t \quad \text{in } L^2\left(0,T;H^{-1}\right).$$

Let $\phi \in C_c^\infty(0,T)$ be a real-valued test function and $w \in E_M$ for some $M \in \mathbb{N}$. Taking $v = \phi(t)w$ in (6.18) and integrating the result with respect to t, we find that for $N \geq M$

$$\int_0^T \left\{(u_{Nt}(t), \phi(t)w)_{L^2} + a\left(u_N(t), \phi(t)w; t\right)\right\}\, dt = \int_0^T \langle f(t), \phi(t)w\rangle\, dt.$$

[3]It is, of course, *not* true that $f_N \rightharpoonup f$ and $u_N \rightharpoonup u$ implies $\langle f_N, u_N\rangle \to \langle f, u\rangle$. For example, $\sin N\pi x \rightharpoonup 0$ in $L^2(0,1)$ but $(\sin N\pi x, \sin N\pi x)_{L^2} \to 1/2$.

We take the limit of this equation as $N \to \infty$. Since the function $t \mapsto \phi(t)w$ belongs to $L^2(0,T;H_0^1)$, we have

$$\int_0^T (u_{Nt}, \phi w)_{L^2}\, dt = \langle\langle u_{Nt}, \phi w\rangle\rangle \to \langle\langle u_t, \phi w\rangle\rangle = \int_0^T \langle u_t, \phi w\rangle\, dt.$$

Moreover, the boundedness of a in (6.12) implies similarly that

$$\int_0^T a\left(u_N(t), \phi(t)w; t\right)\, dt \to \int_0^T a\left(u(t), \phi(t)w; t\right)\, dt.$$

It therefore follows that u satisfies

$$\int_0^T \phi\left[\langle u_t, w\rangle + a\left(u, w; t\right)\right]\, dt = \int_0^T \phi\langle f, w\rangle\, dt. \tag{6.26}$$

Since this holds for every $\phi \in C_c^\infty(0,T)$, we have

$$\langle u_t, w\rangle + a\left(u, w; t\right) = \langle f, w\rangle \tag{6.27}$$

pointwise *a.e.* in $(0,T)$ for every $w \in E_M$. Moreover, since

$$\bigcup_{M\in\mathbb{N}} E_M$$

is dense in H_0^1, this equation holds for every $w \in H_0^1$, and therefore u satisfies (6.13).

Finally, to show that the limit satisfies the initial condition $u(0) = g$, we use the integration by parts formula Theorem 6.42 with $\phi \in C^\infty([0,T])$ such that $\phi(0) = 1$ and $\phi(T) = 0$ to get

$$\int_0^T \langle u_t, \phi w\rangle\, dt = \langle u(0), w\rangle - \int_0^T \phi_t\langle u, w\rangle.$$

Thus, using (6.27), we have

$$\langle u(0), w\rangle = \int_0^T \phi_t\langle u, w\rangle + \int_0^T \phi\left[\langle f, w\rangle - a\left(u, w; t\right)\right]\, dt.$$

Similarly, for the Galerkin appoximation with $w \in E_M$ and $N \geq M$, we get

$$\langle g, w\rangle = \int_0^T \phi_t\langle u_N, w\rangle + \int_0^T \phi\left[\langle f, w\rangle - a\left(u_N, w; t\right)\right]\, dt.$$

Taking the limit of this equation as $N \to \infty$, when the right-hand side converges to the right-hand side of the previosus equation, we find that $\langle u(0), w\rangle = \langle g, w\rangle$ for every $w \in E_M$, which implies that $u(0) = g$. □

6.5.4. Uniqueness of weak solutions. If u_1, u_2 are two solutions with the same data f, g, then by linearity $u = u_1 - u_2$ is a solution with zero data $f = 0$, $g = 0$. To show uniqueness, it is therefore sufficient to show that the only weak solution with zero data is $u = 0$.

Since $u(t) \in H_0^1(\Omega)$, we may take $v = u(t)$ as a test function in (6.13), with $f = 0$, to get

$$\langle u_t, u\rangle + a\left(u, u; t\right) = 0,$$

where this equation holds pointwise *a.e.* in $[0,T]$ in the sense of weak derivatives. Using (6.46) and the coercivity estimate (6.11), we find that there are constants $\beta > 0$ and $-\infty < \gamma < \infty$ such that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \beta\|u\|_{H_0^1}^2 \leq \gamma\|u\|_{L^2}^2 .$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \leq \gamma\|u\|_{L^2}^2, \qquad u(0) = 0,$$

and since $\|u(0)\|_{L^2} = 0$, Gronwall's inequality implies that $\|u(t)\|_{L^2} = 0$ for all $t \geq 0$, so $u = 0$.

In a similar way, we get continuous dependence of weak solutions on the data. If u_i is the weak solution with data f_i, g_i for $i = 1, 2$, then there is a constant C independent of the data such that

$$\begin{aligned} &\|u_1 - u_2\|_{L^\infty(0,T;L^2)} + \|u_1 - u_2\|_{L^2(0,T;H_0^1)} \\ &\qquad \leq C\left(\|f_1 - f_2\|_{L^2(0,T;H^{-1})} + \|g_1 - g_2\|_{L^2}\right). \end{aligned}$$

6.5.5. Regularity of weak solutions. For operators with smooth coefficients on smooth domains with smooth data f, g, one can obtain regularity results for weak solutions by deriving energy estimates for higher-order derivatives of the approximate Galerkin solutions u_N and taking the limit as $N \to \infty$. A repeated application of this procedure, and the Sobolev theorem, implies, from the Sobolev embedding theorem, that the weak solutions constructed above are smooth, classical solutions if the data satisfy appropriate compatibility relations. For a discussion of this regularity theory, see §7.1.3 of [**9**].

6.6. A semilinear heat equation

The Galerkin method is not restricted to linear or scalar equations. In this section, we briefly discuss its application to a semilinear heat equation. For more information and examples of the application of Galerkin methods to nonlinear evolutionary PDEs, see Temam [**43**].

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $T > 0$, and consider the semilinear, parabolic IBVP for $u(x,t)$

$$\begin{aligned} &u_t = \Delta u - f(u) && \text{in } \Omega \times (0,T), \\ &u = 0 && \text{on } \partial\Omega \times (0,T), \\ &u(x,0) = g(x) && \text{on } \Omega \times \{0\}. \end{aligned} \tag{6.28}$$

We suppose, for simplicity, that

$$f(u) = \sum_{k=0}^{2p-1} c_k u^k \tag{6.29}$$

is a polynomial of odd degree $2p - 1 \geq 1$. We also assume that the coefficient $c_{2p-1} > 0$ of the highest degree term is positive. We then have the following global existence result.

Theorem 6.8. *Let $T > 0$. For every $g \in L^2(\Omega)$, there is a unique weak solution*

$$u \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right) \cap L^{2p}\left(0,T; L^{2p}(\Omega)\right).$$

of (6.28)–(6.29).

The proof follows the standard Galerkin method for a parabolic PDE. We will not give it in detail, but we comment on the main new difficulty that arises as a result of the nonlinearity.

To obtain the basic *a priori* energy estimate, we multiplying the PDE by u,

$$\left(\frac{1}{2}u^2\right)_t + |Du|^2 + uf(u) = \operatorname{div}(uDu),$$

and integrate the result over Ω, using the divergence theorem and the boundary condition, which gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|Du\|_{L^2}^2 + \int_\Omega uf(u)\,dx = 0.$$

Since $uf(u)$ is an even polynomial of degree $2p$ with positive leading order coefficient, and the measure $|\Omega|$ is finite, there are constants $A > 0$, $C \geq 0$ such that

$$A\|u\|_{L^{2p}_{2p}} \leq \int_\Omega uf(u)\,dx + C.$$

We therefore have that

$$\frac{1}{2}\sup_{[0,T]}\|u\|_{L^2}^2 + \int_0^T \|Du\|_{L^2}^2\,dt + A\int_0^T \|u\|_{2p}^{2p}\,dt \leq CT + \frac{1}{2}\|g\|_{L^2}^2. \tag{6.30}$$

Note that if $\|u\|_{L^{2p}}$ is finite then $\|f(u)\|_{L^q}$ is finite for $q = (2p)'$, since then $q(2p-1) = 2p$ and

$$\int_\Omega |f(u)|^q\,dx \leq A\int_\Omega |u|^{q(2p-1)}\,dx + C \leq A\|u\|_{L^{2p}} + C.$$

Thus, in giving a weak formulation of the PDE, we want to use test functions

$$v \in H_0^1(\Omega) \cap L^{2p}(\Omega)$$

so that both $(Du, Dv)_{L^2}$ and $(f(u), v)_{L^2}$ are well-defined.

The Galerkin approximations $\{u_N\}$ take values in a finite dimensional subspace $E_N \subset H_0^1(\Omega) \cap L^{2p}(\Omega)$ and satisfy

$$u_{Nt} = \Delta u_N + P_N f(u_N),$$

where P_N is the orthogonal projection onto E_N in $L^2(\Omega)$. These approximations satisfy the same estimates as the *a priori* estimates in (6.30). The Galerkin ODEs have a unique local solution since the nonlinear terms are Lipschitz continuous functions of u_N. Moreover, in view of the *a priori* estimates, the local solutions remain bounded, and therefore they exist globally for $0 \leq t < \infty$.

Since the estimates (6.30) hold uniformly in N, we extract a subsequence that converges weakly (or weak-star) $u_N \rightharpoonup u$ in the appropriate topologies to a limiting function

$$u \in L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;H_0^1\right) \cap L^{2p}\left(0,T;L^{2p}\right).$$

Moreover, from the equation

$$u_t \in L^2\left(0,T;H^{-1}\right) + L^q\left(0,T;L^q\right)$$

where $q = (2p)'$ is the Hölder conjugate of $2p$.

In order to prove that u is a solution of the original PDE, however, we have to show that

$$f(u_N) \rightharpoonup f(u) \tag{6.31}$$

in an appropriate sense. This is not immediately clear because of the lack of weak continuity of nonlinear functions; in general, even if $f(u_N) \rightharpoonup \bar{f}$ converges, we may not have $\bar{f} = f(u)$. To show (6.31), we use the compactness Theorem 6.9 stated below. This theorem and the weak convergence properties found above imply that there is a subsequence of approximate solutions such that

$$u_N \to u \qquad \text{strongly in } L^2(0,T;L^2).$$

This is equivalent to strong-L^2 convergence on $\Omega \times (0,T)$. By the Riesz-Fischer theorem, we can therefore extract a subsequence so that $u_N(x,t) \to u(x,t)$ pointwise *a.e.* on $\Omega \times (0,T)$. Using the dominated convergence theorem and the uniform bounds on the approximate solutions, we find that for every $v \in H_0^1(\Omega) \cap L^{2p}(\Omega)$

$$\left(f\left(u_N(t)\right), v\right)_{L^2} \to \left(f\left(u(t)\right), v\right)_{L^2}$$

pointwise a.e. on $[0,T]$.

Finally, we state the compactness theorem used here.

THEOREM 6.9. *Suppose that $X \hookrightarrow Y \hookrightarrow Z$ are Banach spaces, where X, Z are reflexive and X is compactly embedded in Y. Let $1 < p < \infty$. If the functions $u_N : (0,T) \to X$ are such that $\{u_N\}$ is uniformly bounded in $L^2(0,T;X)$ and $\{u_{Nt}\}$ is uniformly bounded in $L^p(0,T;Z)$, then there is a subsequence that converges strongly in $L^2(0,T;Y)$.*

The proof of this theorem is based on Ehrling's lemma.

LEMMA 6.10. *Suppose that $X \hookrightarrow Y \hookrightarrow Z$ are Banach spaces, where X is compactly embedded in Y. For any $\epsilon > 0$ there exists a constant C_ϵ such that*

$$\|u\|_Y \le \epsilon \|u\|_X + C_\epsilon \|u\|_Z .$$

PROOF. If not, there exists $\epsilon > 0$ and a sequence $\{u_n\}$ in X with $\|u_n\|X = 1$ such that

$$\|u_n\|_Y > \epsilon \|u_n\|_X + n \|u_n\|_Z \tag{6.32}$$

for every $n \in \mathbb{N}$. Since $\{u_n\}$ is bounded in X and X is compactly embedded in Y, there is a subsequence, which we still denote by $\{u_n\}$ that converges strongly in Y, to u, say. Then $\{\|u_n\|_Y\}$ is bounded and therefore $u = 0$ from (6.32). However, (6.32) also implies that $\|u_n\|_Y > \epsilon$ for every $n \in \mathbb{N}$, which is a contradiction. □

If we do not impose a sign condition on the nonlinearity, then solutions may 'blow up' in finite time, as for the ODE $u_t = u^3$, and then we do not get global existence.

EXAMPLE 6.11. Consider the following one-dimensional IBVP [**20**] for $u(x,t)$ in $0 < x < 1$, $t > 0$:

$$\begin{aligned} &u_t = u_{xx} + u^3, \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = g(x). \end{aligned} \tag{6.33}$$

Suppose that $u(x,t)$ is smooth solution, and let

$$c(t) = \int_0^1 u(x,t)\sin(\pi x)\,dx$$

denote the first Fourier sine coefficient of u. Multiplying the PDE by $\sin(\pi x)$, integrating with respect to x over $(0,1)$, and using Green's formula to write

$$\begin{aligned}\int_0^1 u_{xx}(x,t)\sin(\pi x)\,dx &= \left[u_x \sin(\pi x) - \pi u \cos(\pi) x\right]_0^1 - \pi^2 \int_0^1 u(x,t)\sin(\pi x)\,dx \\ &= -\pi^2 c,\end{aligned}$$

we get that

$$\frac{dc}{dt} = -\pi^2 c + \int_0^1 u^3 \sin(\pi x)\,dx.$$

Now suppose that $g(x) \geq 0$. Then the maximum principle implies that $u(x,t) \geq 0$ for all $0 < x < 1$, $t > 0$. It then follows from Hölder inequality that

$$\begin{aligned}\int_0^1 u \sin(\pi x)\,dx &= \int_0^1 \left[u^3 \sin(\pi x)\right]^{1/3} \left[\sin(\pi x)\right]^{2/3}\,dx \\ &\leq \left(\int_0^1 u^3 \sin(\pi x)\,dx\right)^{1/3} \left(\int_0^1 \sin(\pi x)\,dx\right)^{2/3} \\ &\leq \left(\frac{2}{\pi}\right)^{2/3} \left(\int_0^1 u^3 \sin(\pi x)\,dx\right)^{1/3}.\end{aligned}$$

Hence

$$\int_0^1 u^3 \sin(\pi x)\,dx \geq \frac{\pi^2}{4} c^3,$$

and therefore

$$\frac{dc}{dt} \geq \pi^2 \left(-c + \frac{1}{4} c^3\right).$$

Thus, if $c(0) > 2$, Gronwall's inequality implies that

$$c(t) \geq y(t)$$

where $y(t)$ is the solution of the ODE

$$\frac{dy}{dt} = \pi^2 \left(-y + \frac{1}{4} y^3\right).$$

This solution is given explicitly by

$$y(t) = \frac{2}{\sqrt{1 - e^{2\pi^2 (t - t_*)}}}$$

This solution approaches infinity as $t \to t_*^-$ where, with $y(0) = c(0)$,

$$t_* = \frac{1}{\pi^2} \log \frac{c(0)}{\sqrt{c(0)^2 - 4}}.$$

Therefore no smooth solution of (6.33) can exist beyond $t = t_*$.

The argument used in the previous example does not prove that $c(t)$ blows up at $t = t_*$. It is conceivable that the solution loses smoothness at an earlier time — for example, because another Fourier coefficient blows up first — thereby invalidating the argument that $c(t)$ blows up. We only get a sharp result if the quantity proven to blow up is a 'controlling norm,' meaning that local smooth solutions exist so long as the controlling norm remains finite.

EXAMPLE 6.12. Beale-Kato-Majda (1984) proved that solutions of the incompressible Euler equations from fluid mechanics in three-space dimensions remain smooth unless

$$\int_0^t \|\omega(s)\|_{L^\infty(\mathbb{R}^3)}\, ds \to \infty \qquad \text{as } t \to t_*^-$$

where $\omega(\cdot, t) = \operatorname{curl} \mathbf{u}(\cdot, t)$ denotes the vorticity (the curl of the fluid velocity $\mathbf{u}(\mathbf{x}, t)$). Thus, the $L^1\left(0, T; L^\infty(\mathbb{R}^3; \mathbb{R}^3)\right)$-norm of ω is a controlling norm for the three-dimensional incompressible Euler equations. It is open question whether or not this norm can blow up in finite time.

6.7. The Navier-Stokes equation

Leray (1934) used a Galerkin method to prove the global existence of weak solutions of the incompressible Navier-Stokes equations. In the case of three space dimensions, Leray's result has not been essentially improved upon since then, and the smoothness and uniqueness of these weak solutions remains an open question.[4] We briefly describe Leray's result here and indicate the main ideas of its proof. For a detailed discussion, see *e.g.* [**38**].

The incompressible Navier-Stokes equations for the velocity $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^n$ and pressure $p(\mathbf{x}, t) \in \mathbb{R}$ of a viscous fluid flowing in n space dimensions, where $n = 2, 3$, and subject to an external body force $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^n$ is the following nonlinear system of PDEs:

$$\begin{aligned} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{f}, \\ \operatorname{div} \mathbf{u} &= 0. \end{aligned} \tag{6.34}$$

These equations express conservation of momentum and incompressibility, respectively. Here, $\nu > 0$ is the kinematic viscosity of the fluid, which we assume is constant. In Cartesian component form, with $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{f} = (f_1, \dots, f_n)$, and $\mathbf{x} = (x_1, \dots, x_n)$, these equations are

$$u_{it} + \sum_{j=1}^n u_j \partial_j u_i + \partial_i p = \nu \sum_{j=1}^n \partial_j \partial_j u_i + f_i,$$

$$\sum_{j=1}^n \partial_j u_j = 0.$$

The analysis described here is based on treating the Navier-Stokes equations as a nonlinear perturbation of the linear parabolic Stokes equations

$$\begin{aligned} \mathbf{u}_t + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{f}. \\ \operatorname{div} \mathbf{u} &= 0. \end{aligned} \tag{6.35}$$

These equations apply to low-Reynolds number (high nondimensionalized viscosity) flows, which is the typical regime for small-scale flows (*e.g.* colloidal particles or spermatoza).

Alternatively, one can think of the Navier-Stokes equations as a parabolic perturbation of the first-order, nonlinear incompressible Euler equations,

$$\begin{aligned} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f}, \\ \operatorname{div} \mathbf{u} &= 0. \end{aligned} \tag{6.36}$$

These equations apply to high-Reynolds number (low viscosity) flows, which is the typical regime for large-scale flows (*e.g.* airplanes or oceans). The nonlinearity of the Euler equations makes them difficult to analyze, especially in three space dimensions.[5] Moreover, the higher-order viscous term $\nu \Delta \mathbf{u}$ in the Navier-Stokes equation is a singular perturbation of the Euler equations, and the limiting behavior of the Navier-Stokes equations as $\nu \to 0$ is a subtle issue, which is not fully understood even now.

[4] Its resolution is one of the seven Clay Mathematics Institute's Millennium Problems.

[5] With the exception of irrotational flows in which $\operatorname{curl} \mathbf{u} = 0$, when the Euler equations reduce to the linear Laplace equation.

Typical initial and boundary conditions for the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ are

$$\mathbf{u} = \mathbf{u}_0 \quad \text{for } t = 0, \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega.$$

The boundary condition $\mathbf{u} = 0$ is the 'no-slip' condition, which states that a viscous fluid 'sticks' to the boundary, assumed here to be stationary. We give an initial condition for the velocity $\mathbf{u}$, only, not the pressure. The Navier-Stokes equations do not give an evolution equation for the pressure; instead, the pressure is determined at each time from the velocity field $\mathbf{u}$ by the elliptic equation

$$-\Delta p = \operatorname{div}(\mathbf{u} \cdot \nabla \mathbf{u}) - \operatorname{div} \mathbf{f}$$

which follows by taking the divergence of the momentum equation.

A convenient way to eliminate the pressure is to project the Navier-Stokes equations onto divergence-free vector fields. Let $L^2(\Omega; \mathbb{R}^n)$ denote the space of square-integrable functions $\mathbf{u} : \Omega \to \mathbb{R}^n$ with inner product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x}.$$

Let $C^\infty_{c,\sigma}(\Omega; \mathbb{R}^n)$ be the space of smooth, compactly supported vector fields $\mathbf{u} : \Omega \to \mathbb{R}^n$ such that $\operatorname{div} \mathbf{u} = 0$, and $G(\Omega; \mathbb{R}^n)$ the space of square-integrable functions $\mathbf{v} : \Omega \to \mathbb{R}^n$ such that $\mathbf{v} = \nabla\phi$ for some $\phi \in H^2(\Omega)$. If $\mathbf{u} \in C^\infty_{c,\sigma}(\Omega; \mathbb{R}^n)$ and $\mathbf{v} \in G(\Omega; \mathbb{R}^n)$, then

$$(\mathbf{u}, \mathbf{v})_{L^2} = \int_\Omega \mathbf{u} \cdot \nabla\phi \, d\mathbf{x} = -\int_\Omega (\operatorname{div} \mathbf{u})\phi \, d\mathbf{x} = 0.$$

Thus, the divergence-free vector-fields are orthogonal to the gradients.

We let

$$L^2_\sigma(\Omega; \mathbb{R}^n) = \overline{C^\infty_{c,\sigma}(\Omega; \mathbb{R}^n)}$$

denote the closure of $C^\infty_{c,\sigma}(\Omega; \mathbb{R}^n)$ in $L^2(\Omega; \mathbb{R}^n)$; that is, $L^2_\sigma(\Omega; \mathbb{R}^n)$ consists of the square-intgerable, divergence-free vector fields. We then have the orthogonal direct sum

$$L^2(\Omega; \mathbb{R}^n) = L^2_\sigma(\Omega; \mathbb{R}^n) \oplus G(\Omega; \mathbb{R}^n),$$

and any $\mathbf{u} \in L^2(\Omega; \mathbb{R}^n)$ may be written uniquely as

$$\mathbf{u} = \mathbf{v} + \nabla\phi \qquad \text{where } \mathbf{v} \in L^2_\sigma(\Omega; \mathbb{R}^n) \text{ and } \nabla\phi \in G(\Omega; \mathbb{R}^n).$$

This is called the Helmholtz decomposition of $\mathbf{u}$. We denote by P the orthogonal projection onto divergence-free vector fields

$$P : L^2(\Omega; \mathbb{R}^n) \to L^2_\sigma(\Omega; \mathbb{R}^n) \subset L^2(\Omega; \mathbb{R}^n), \qquad P : \mathbf{u} \mapsto \mathbf{v}.$$

We may then write the Navier-Stokes equations (6.34) for $\mathbf{u} : (0, T) \to L^2_\sigma(\Omega; \mathbb{R}^n)$ as

$$\mathbf{u}_t + P[\mathbf{u} \cdot \nabla \mathbf{u}] = \nu\Delta\mathbf{u} + P\mathbf{f}.$$

Alternatively, we may formulate the equations in a weak sense by using divergence-free test functions whose integral against ∇p vanishes to get

$$(\mathbf{u}_t, \mathbf{v})_{L^2} + (\mathbf{u} \cdot \nabla\mathbf{u}, \mathbf{v})_{L^2} = \nu(\Delta\mathbf{u}, \mathbf{v}) + (\mathbf{f}, \mathbf{v}) \qquad \text{for all } \mathbf{v} \in C^\infty_{c,\sigma}(\Omega; \mathbb{R}^n).$$

A convenient basis for the Galerkin approximations is provided by the eigenfunctions $\mathbf{w}_k$ of the steady Stokes operator,

$$\begin{aligned} \lambda\mathbf{w} + \nabla p = \nu\Delta\mathbf{w}. \\ \operatorname{div}\mathbf{w} = 0, \\ \mathbf{w} \in H_0^1(\Omega;\mathbb{R}^n). \end{aligned} \tag{6.37}$$

Appendix

In this appendix, we summarize some results about the integration and differentiation of Banach-space valued functions of a single variable. In a rough sense, vector-valued integrals of integrable functions have similar properties, often with similar proofs, to scalar-valued L^1-integrals. Nevertheless, the existence of different topologies (such as the weak and strong topologies) in the range space of integrals that take values in an infinite-dimensional Banach space introduces significant new issues that do not arise in the scalar-valued case.

6.A. Vector-valued functions

Suppose that X is a real Banach space with norm $\|\cdot\|$ and dual space X'. Let $0 < T < \infty$, and consider functions $f : (0,T) \to X$. We will generalize some of the definitions in Section 3.A for real-valued functions of a single variable to vector-valued functions.

6.A.1. Measurability. If $E \subset (0,T)$, let

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \notin E, \end{cases}$$

denote the characteristic function of E.

Definition 6.13. A simple function $f : (0,T) \to X$ is a function of the form

$$f = \sum_{j=1}^{N} c_j \chi_{E_j} \tag{6.38}$$

where $E_1, \dots, E_N$ are Lebesgue measurable subsets of $(0,T)$ and $c_1, \dots, c_N \in X$.

Definition 6.14. A function $f : (0,T) \to X$ is strongly measurable, or measurable for short, if there is a sequence $\{f_n : n \in \mathbb{N}\}$ of simple functions such that $f_n(t) \to f(t)$ strongly in X (*i.e.* in norm) for t *a.e.* in $(0,T)$.

Measurability is preserved under natural operations on functions.

(1) If $f : (0,T) \to X$ is measurable, then $\|f\| : (0,T) \to \mathbb{R}$ is measurable.
(2) If $f : (0,T) \to X$ is measurable and $\phi : (0,T) \to \mathbb{R}$ is measurable, then $\phi f : (0,T) \to X$ is measurable.
(3) If $\{f_n : (0,T) \to X\}$ is a sequence of measurable functions and $f_n(t) \to f(t)$ strongly in X for t pointwise *a.e.* in $(0,T)$, then $f : (0,T) \to X$ is measurable.

We will only use strongly measurable functions, but there are other definitions of measurability. For example, a function $f : (0,T) \to X$ is said to be weakly measurable if the real-valued function $\langle \omega, f\rangle : (0,T) \to \mathbb{R}$ is measurable for every $\omega \in X'$. This amounts to a 'coordinatewise' definition of measurability, in which we represent a vector-valued function by its real-valued coordinate functions. For finite-dimensional, or separable, Banach spaces these definitions coincide, but for non-separable spaces a weakly measurable function need not be strongly measurable. The relationship between weak and strong measurability is given by the following Pettis theorem (1938).

DEFINITION 6.15. A function $f : (0,T) \to X$ taking values in a Banach space X is almost separably valued if there is a set $E \subset (0,T)$ of measure zero such that $f((0,T) \setminus E)$ is separable, meaning that it contains a countable dense subset.

This definition is equivalent to the condition that $f((0,T) \setminus E)$ is included in a closed, separable subspace of X.

THEOREM 6.16. *A function $f : (0,T) \to X$ is strongly measurable if and only if it is weakly measurable and almost separably valued.*

Thus, if X is a separable Banach space, $f : (0,T) \to X$ is strongly measurable if and only $\langle \omega, f \rangle : (0,T) \to \mathbb{R}$ is measurable for every $\omega \in X'$. This theorem therefore reduces the verification of strong measurability to the verification of measurability of real-valued functions.

DEFINITION 6.17. A function $f : [0,T] \to X$ taking values in a Banach space X is weakly continuous if $\langle \omega, f \rangle : [0,T] \to \mathbb{R}$ is continuous for every $\omega \in X'$. The space of such weakly continuous functions is denoted by $C_w([0,T]; X)$.

Since a continuous function is measurable, every almost separably valued, weakly continuous function is strongly measurable.

EXAMPLE 6.18. Suppose that $\mathcal{H}$ is a non-separable Hilbert space whose dimension is equal to the cardinality of $\mathbb{R}$. Let $\{e_t : t \in (0,1)\}$ be an orthonormal basis of $\mathcal{H}$, and define a function $f : (0,1) \to \mathcal{H}$ by $f(t) = e_t$. Then f is weakly but not strongly measurable. If $K \subset [0,1]$ is the standard middle thirds Cantor set and $\{\tilde{e}_t : t \in K\}$ is an orthonormal basis of $\mathcal{H}$, then $g : (0,1) \to \mathcal{H}$ defined by $g(t) = 0$ if $t \notin K$ and $g(t) = \tilde{e}_t$ if $t \in K$ is almost separably valued since $|K| = 0$; thus, g is strongly measurable and equivalent to the zero-function.

EXAMPLE 6.19. Define $f : (0,1) \to L^\infty(0,1)$ by $f(t) = \chi_{(0,t)}$. Then f is not almost separably valued, since $\|f(t) - f(s)\|_{L^\infty} = 1$ for $t \neq s$, so f is not strongly measurable. On the other hand, if we define $g : (0,1) \to L^2(0,1)$ by $g(t) = \chi_{(0,t)}$, then g is strongly measurable. To see this, note that $L^2(0,1)$ is separable and for every $w \in L^2(0,1)$, which is isomorphic to $L^2(0,1)'$, we have

$$(w, g(t))_{L^2} = \int_0^1 w(x)\chi_{(0,t)}(x)\, dx = \int_0^t w(x)\, dx.$$

Thus, $(w, g)_{L^2} : (0,1) \to \mathbb{R}$ is absolutely continuous and therefore measurable.

6.A.2. Integration. The definition of the Lebesgue integral as a supremum of integrals of simple functions does not extend directly to vector-valued integrals because it uses the ordering properties of $\mathbb{R}$ in an essential way. One can use duality to define X-valued integrals $\int f\, dt$ in terms of the corresponding real-valued integrals $\int \langle \omega, f \rangle\, dt$ where $\omega \in X'$, but we will not consider such weak definitions of an integral here.

Instead, we define the integral of vector-valued functions by completing the space of simple functions with respect to the $L^1(0,T; X)$-norm. The resulting integral is called the Bochner integral, and its properties are similar to those of the Lebesgue integral of integrable real-valued functions. For proofs of the results stated here, see *e.g.* [**44**].

DEFINITION 6.20. Let

$$f = \sum_{j=1}^{N} c_j \chi_{E_j}$$

be the simple function in (6.38). The integral of f is defined by

$$\int_0^T f\, dt = \sum_{j=1}^{N} c_j \, |E_j| \in X$$

where $|E_j|$ denotes the Lebesgue measure of E_j.

The value of the integral of a simple function is independent of how it is represented in terms of characteristic functions.

DEFINITION 6.21. A strongly measurable function $f : (0,T) \to X$ is Bochner integrable, or integrable for short, if there is a sequence of simple functions such that $f_n(t) \to f(t)$ pointwise *a.e.* in $(0,T)$ and

$$\lim_{n\to\infty} \int_0^T \|f - f_n\| \; dt = 0.$$

The integral of f is defined by

$$\int_0^T f\, dt = \lim_{n\to\infty} \int_0^T f_n\, dt,$$

where the limit exists strongly in X.

The value of the Bochner integral of f is independent of the sequence $\{f_n\}$ of approximating simple functions, and

$$\left\| \int_0^T f\, dt \right\| \leq \int_0^T \|f\|\, dt.$$

Moreover, if $A : X \to Y$ is a bounded linear operator between Banach spaces X, Y and $f : (0,T) \to X$ is integrable, then $Af : (0,T) \to Y$ is integrable and

$$A\left(\int_0^T f\, dt\right) = \int_0^T Af\, dt. \tag{6.39}$$

More generally, this equality holds whenever $A : \mathcal{D}(A) \subset X \to Y$ is a closed linear operator and $f : (0,T) \to \mathcal{D}(A)$, in which case $\int_0^T f\, dt \in \mathcal{D}(A)$.

EXAMPLE 6.22. If $f : (0,T) \to X$ is integrable and $\omega \in X'$, then $\langle \omega, f\rangle : (0,T) \to \mathbb{R}$ is integrable and

$$\left\langle \omega, \int_0^T f\, dt \right\rangle = \int_0^T \langle \omega, f\rangle\, dt.$$

EXAMPLE 6.23. If $J : X \hookrightarrow Y$ is a continuous embedding of a Banach space X into a Banach space Y, and $f : (0,T) \to X$, then

$$J\left(\int_0^T f\, dt\right) = \int_0^T Jf\, dt.$$

Thus, the X and Y valued integrals agree, and we can identify them.

The following result, due to Bochner (1933), characterizes integrable functions as ones with integrable norm.

THEOREM 6.24. *A function $f : (0, T) \to X$ is Bochner integrable if and only if it is strongly measurable and*

$$\int_0^T \|f\| \, dt < \infty.$$

Thus, in order to verify that a measurable function f is Bochner integrable one only has to check that the real valued function $\|f\| : (0, T) \to \mathbb{R}$, which is necessarily measurable, is integrable.

EXAMPLE 6.25. The functions $f : (0, 1) \to \mathcal{H}$ in Example (6.18) and $f : (0, 1) \to L^\infty(0, 1)$ in Example (6.19) are not Bochner integrable since they are not strongly measurable. The function $g : (0, 1) \to \mathcal{H}$ in Example (6.18) is Bochner integrable, and its integral is equal to zero. The function $g : (0, 1) \to L^2(0, 1)$ in Example (6.19) is Bochner integrable since it is measurable and $\|g(t)\|_{L^2} = t^{1/2}$ is integrable on $(0, 1)$. We leave it as an exercise to compute its integral.

The dominated convergence theorem holds for Bochner integrals. The proof is the same as for the scalar-valued case, and we omit it.

THEOREM 6.26. *Suppose that $f_n : (0, T) \to X$ is Bochner integrable for each $n \in \mathbb{N}$,*

$$f_n(t) \to f(t) \qquad \text{as } n \to \infty \text{ strongly in } X \text{ for } t \text{ a.e. in } (0, T),$$

and there is an integrable function $g : (0, T) \to \mathbb{R}$ such that

$$\|f_n(t)\| \le g(t) \qquad \text{for } t \text{ a.e. in } (0, T) \text{ and every } n \in \mathbb{N}.$$

Then $f : (0, T) \to X$ is Bochner integrable and

$$\int_0^T f_n \, dt \to \int_0^T f \, dt, \qquad \int_0^T \|f_n - f\| \, dt \to 0 \qquad \text{as } n \to \infty.$$

The definition and properties of L^p-spaces of X-valued functions are analogous to the case of real-valued functions.

DEFINITION 6.27. For $1 \le p < \infty$ the space $L^p(0, T; X)$ consists of all strongly measurable functions $f : (0, T) \to X$ such that

$$\int_0^T \|f\|^p \, dt < \infty$$

equipped with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|^p \, dt \right)^{1/p}.$$

The space $L^\infty(0, T; X)$ consists of all strongly measurable functions $f : (0, T) \to X$ such that

$$\|f\|_{L^\infty(0,T;X)} = \sup_{t \in (0,T)} \|f(t)\| < \infty,$$

where sup denotes the essential supremum.

As usual, we regard functions that are equal pointwise *a.e.* as equivalent, and identify a function that is equivalent to a continuous function with its continuous representative.

THEOREM 6.28. *If X is a Banach space and $1 \leq p \leq \infty$, then $L^p(0,T;X)$ is a Banach space.*

Simple functions of the form

$$f(t) = \sum_{i=1}^{n} c_i \chi_{E_i}(t),$$

where $c_i \in X$ and E_i is a measurable subset of $(0,T)$, are dense in $L^p(0,T;X)$. By mollifying these functions with respect to t, we get the following density result.

PROPOSITION 6.29. *If X is a Banach space and $1 \leq p < \infty$, then the collection of functions of the form*

$$f(t) = \sum_{i=1}^{n} c_i \phi_i(t) \qquad \text{where } \phi_i \in C_c^\infty(0,T) \text{ and } c_i \in X$$

is dense in $L^p(0,T;X)$.

The characterization of the dual space of a vector-valued L^p-space is analogous to the scalar-valued case, after we take account of duality in the range space X.

THEOREM 6.30. *Suppose that $1 \leq p < \infty$ and X is a reflexive Banach space with dual space X'. Then the dual of $L^p(0,T;X)$ is isomorphic to $L^{p'}(0,T;X')$ where*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The action of $f \in L^{p'}(0,T;X')$ on $u \in L^p(0,T;X)$ is given by

$$\langle\langle f, u\rangle\rangle = \int_0^T \langle f(t), u(t)\rangle \, dt,$$

where the double brackets denote the $L^{p'}(X')$-$L^p(X)$ duality pairing and the single brackets denote the X'-X duality pairing.

The proof is more complicated than in the scalar case and some condition on X is required. Reflexivity is sufficient (as is the condition that X' is separable).

6.A.3. Differentiability. The definition of continuity and pointwise differentiability of vector-valued functions are the same as in the scalar case. A function $f : (0,T) \to X$ is strongly continuous at $t \in (0,T)$ if $f(s) \to f(t)$ strongly in X as $s \to t$, and f is strongly continuous in $(0,T)$ if it is strongly continuous at every point of $(0,T)$. A function f is strongly differentiable at $t \in (0,T)$, with strong pointwise derivative $f_t(t)$, if

$$f_t(t) = \lim_{h \to 0} \left[\frac{f(t+h) - f(t)}{h}\right]$$

where the limit exists strongly in X, and f is continuously differentiable in $(0,T)$ if its pointwise derivative exists for every $t \in (0,T)$ and $f_t : (0,T) \to X$ is a strongly continuously function.

The assumption of continuous differentiability is often too strong to be useful, so we need a weaker notion of the differentiability of a vector-valued function. As for real-valued functions, such as the step function or the Cantor function, the requirement that the strong pointwise derivative exists *a.e.* in $(0,T)$ does *not* lead to an effective theory. Instead we use the notion of a distributional or weak derivative, which is a natural generalization of the definition for real-valued functions.

Let $L^1_{\mathrm{loc}}(0,T;X)$ denote the space of measurable functions $f:(0,T)\to X$ that are integrable on every compactly supported interval $(a,b)\Subset(0,T)$. Also, as usual, let $C_c^\infty(0,T)$ denote the space of smooth, real-valued functions $\phi:(0,T)\to\mathbb{R}$ with compact support, $\operatorname{supp}\phi\Subset(0,T)$.

Definition 6.31. A function $f\in L^1_{\mathrm{loc}}(0,T;X)$ is weakly differentiable with weak derivative $f_t=g\in L^1_{\mathrm{loc}}(0,T;X)$ if

$$\int_0^T \phi' f\,dt = -\int_0^T \phi g\,dt \qquad \text{for every } \phi\in C_c^\infty(0,T). \tag{6.40}$$

The integrals in (6.40) are understood as Bochner integrals. In the commonly occurring case where $J:X\hookrightarrow Y$ is a continuous embedding, $f\in L^1_{\mathrm{loc}}(0,T;X)$, and $(Jf)_t\in L^1_{\mathrm{loc}}(0,T;Y)$, we have from Example 6.23 that

$$J\left(\int_0^T \phi' f\,dt\right) = \int_0^T \phi' Jf\,dt = -\int_0^T \phi (Jf)_t\,dt.$$

Thus, we can identify f with Jf and use (6.40) to define the Y-valued derivative of an X-valued function. We then write, for example, that $f\in L^p(0,T;X)$ and $f_t\in L^q(0,T;Y)$ if $f(t)$ is L^p in t with values in X and its weak derivative $f_t(t)$ is L^q in t with values in Y.

If $f:(0,T)\to\mathbb{R}$ is a scalar-valued, integrable function, then the Lebesgue differentiation theorem, Theorem 1.21, implies that the limit

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,ds$$

exists and is equal to $f(t)$ for t pointwise *a.e.* in $(0,T)$. The same result is true for vector-valued integrals.

Theorem 6.32. *Suppose that X is a Banach space and $f\in L^1(0,T;X)$, then*

$$f(t) = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,ds$$

for t pointwise a.e. in $(0,T)$.

Proof. Since f is almost separably valued, we may assume that X is separable. Let $\{c_n\in X:n\in\mathbb{N}\}$ be a dense subset of X, then by the Lebesgue differentiation theorem for real-valued functions

$$\|f(t)-c_n\| = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\|f(s)-c_n\|\,ds$$

for every $n \in \mathbb{N}$ and t pointwise *a.e.* in $(0,T)$. Thus, for all such $t \in (0,T)$ and every $n \in \mathbb{N}$, we have

$$\begin{aligned}\limsup_{h\to 0} \frac{1}{h}\int_t^{t+h} \|f(s)-f(t)\|\, ds\\ &\leq \limsup_{h\to 0} \frac{1}{h}\int_t^{t+h} \left(\|f(s)-c_n\| + \|f(t)-c_n\|\right) ds\\ &\leq 2\,\|f(t)-c_n\|.\end{aligned}$$

Since this holds for every c_n, it follows that

$$\limsup_{h\to 0} \frac{1}{h}\int_t^{t+h} \|f(s)-f(t)\|\, ds = 0.$$

Therefore

$$\limsup_{h\to 0} \left\|\frac{1}{h}\int_t^{t+h} f(s)\, ds - f(t)\right\| \leq \limsup_{h\to 0} \frac{1}{h}\int_t^{t+h} \|f(s)-f(t)\|\, ds = 0,$$

which proves the result. □

The following corollary corresponds to the statement that a regular distribution determines the values of its associated locally integrable function pointwise almost everywhere.

Corollary 6.33. *Suppose that $f : (0,T) \to X$ is locally integrable and*

$$\int_0^T \phi f\, dt = 0 \qquad \text{for every } \phi \in C_c^\infty(0,T).$$

Then $f = 0$ pointwise a.e. on $(0,T)$.

Proof. Choose a sequence of test functions $0 \leq \phi_n \leq 1$ whose supports are contained inside a fixed compact subset of $(0,T)$ such that $\phi_n \to \chi_{(t,t+h)}$ pointwise, where $\chi_{(t,t+h)}$ is the characteristic function of the interval $(t,t+h) \subset (0,T)$. If $f \in L^1_{\mathrm{loc}}(0,T;X)$, then by the dominated convergence theorem

$$\int_t^{t+h} f(s)\, ds = \lim_{n\to\infty} \int_0^T \phi_n(s) f(s)\, ds.$$

Thus, if $\int_0^T \phi f\, ds = 0$ for every $\phi \in C_c^\infty(0,T)$, then

$$\int_t^{t+h} f(s)\, ds = 0$$

for every $(t,t+h) \subset (0,T)$. It then follows from the Lebesgue differentiation theorem, Theorem 6.32, that $f = 0$ pointwise *a.e.* in $(0,T)$. □

We also have a vector-valued analog of Proposition 3.6 that the only functions with zero weak derivative are the constant functions. The proof is similar.

Proposition 6.34. *Suppose that $f : (0,T) \to X$ is weakly differentiable and $f' = 0$. Then f is equivalent to a constant function.*

Proof. The condition that the weak derivative f' is zero means that

$$\int_0^T f\phi'\,dt = 0 \qquad \text{for all } \phi \in C_c^\infty(0,T). \tag{6.41}$$

Choose a fixed test function $\eta \in C_c^\infty(0,T)$ whose integral is equal to one, and represent an arbitrary test function $\phi \in C_c^\infty(0,T)$ as

$$\phi = A\eta + \psi'$$

where $A \in \mathbb{R}$ and $\psi \in C_c^\infty(0,T)$ are given by

$$A = \int_0^T \phi\,dt, \qquad \psi(t) = \int_0^t [\phi(s) - A\eta(s)]\, ds.$$

If

$$c = \int_0^T \eta f\,dt \in X,$$

then (6.41) implies that

$$\int_0^T (f-c)\,\phi\,dt = 0 \qquad \text{for all } \phi \in C_c^\infty(0,T), \tag{6.42}$$

and Corollary 6.33 implies that $f = c$ pointwise *a.e.* on $(0,T)$. □

It also follows that a function is weakly differentiable if and only if it is the integral of an integrable function.

Theorem 6.35. *Suppose that X is a Banach space and $f \in L^1(0,T;X)$. Then f is weakly differentiable with integrable derivative $f_t = g \in L^1(0,T;X)$ if and only if*

$$f(t) = c_0 + \int_0^t g(s)\,ds \tag{6.43}$$

pointwise a.e. in $(0,T)$. In that case, f is differentiable pointwise a.e. and its pointwise derivative coincides with its weak derivative.

Proof. If f is given by (6.43), then

$$\frac{f(t+h) - f(t)}{h} = \frac{1}{h}\int_t^{t+h} g(s)\,ds,$$

and the Lebesgue differentiation theorem, Theorem 6.32, implies that the strong derivative of f exists pointwise *a.e.* and is equal to g.

We also have that

$$\left\|\frac{f(t+h) - f(t)}{h}\right\| \le \frac{1}{h}\int_t^{t+h} \|g(s)\|\, ds.$$

Extending f by zero to a function $f : \mathbb{R} \to X$, and using Fubini's theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}} \left\| \frac{f(t+h) - f(t)}{h} \right\| dt &\leq \frac{1}{h} \int_{\mathbb{R}} \left(\int_t^{t+h} \|g(s)\| \, ds \right) dt \\
&\leq \frac{1}{h} \int_{\mathbb{R}} \left(\int_0^h \|g(s+t)\| \, ds \right) dt \\
&\leq \frac{1}{h} \int_0^h \left(\int_{\mathbb{R}} \|g(s+t)\| \, dt \right) ds \\
&\leq \int_{\mathbb{R}} \|g(t)\| \, dt.
\end{aligned}$$

If $\phi \in C_c^\infty(0,T)$, this estimate justifies the use of the dominated convergence theorem and the previous result on the pointwise *a.e.* convergence of f_t to get

$$\begin{aligned}
\int_0^T \phi'(t) f(t)\, dt &= \lim_{h\to 0} \int_0^T \left[\frac{\phi(t+h) - \phi(t)}{h} \right] f(t)\, dt \\
&= -\lim_{h\to 0} \int_0^T \phi(t) \left[\frac{f(t) - f(t-h)}{h} \right] dt \\
&= -\int_0^T \phi(t) g(t)\, dt,
\end{aligned}$$

which shows that g is the weak derivative of f.

Conversely, if $f_t = g \in L^1(0,T)$ in the sense of weak derivatives, let

$$\tilde{f}(t) = \int_0^t g(s)\, ds.$$

Then the previous argument implies that $\tilde{f}_t = g$, so the weak derivative $(f - \tilde{f})_t$ is zero. Proposition 6.34 then implies that $f - \tilde{f}$ is constant pointwise *a.e.*, which gives (6.43). □

We can also characterize the weak derivative of a vector-valued function in terms of weak derivatives of the real-valued functions obtained by duality.

PROPOSITION 6.36. *Let X be a Banach space with dual X'. If $f, g \in L^1(0,T;X)$, then f is weakly differentiable with $f_t = g$ if and only if for every $\omega \in X'$*

$$\frac{d}{dt}\langle \omega, f\rangle = \langle \omega, g\rangle \qquad \text{as a real-valued weak derivative in } (0,T). \tag{6.44}$$

PROOF. If $f_t = g$, then

$$\int_0^T \phi' f\, dt = -\int_0^T \phi g\, dt \qquad \text{for all } \phi \in C_c^\infty(0,T).$$

Acting on this equation by $\omega \in X'$ and using the continuity of the integral, we get

$$\int_0^T \phi' \langle \omega, f\rangle\, dt = -\int_0^T \phi \langle \omega, g\rangle\, dt \qquad \text{for all } \phi \in C_c^\infty(0,T)$$

which is (6.44). Conversely, if (6.44) holds, then

$$\left\langle \omega, \int_0^T (\phi' f + \phi g)\, dt \right\rangle = 0 \qquad \text{for all } \omega \in X',$$

which implies that

$$\int_0^T (\phi' f + \phi g)\, dt = 0.$$

Therefore f is weakly differentiable with $f_t = g$. □

A consequence of these results is that any of the natural ways of defining what one means for an abstract evolution equation to hold in a weak sense leads to the same notion of a solution. To be more explicit, suppose that $X \hookrightarrow Y$ are Banach spaces with X continuously and densely embedded in Y and $F : X \times (0,T) \to Y$. Then a function $u \in L^1(0,T;X)$ is a weak solution of the equation

$$u_t = F(u,t)$$

if it has a weak derivative $u_t \in L^1(0,T;Y)$ and $u_t = F(u,t)$ for t pointwise *a.e.* in $(0,T)$. Equivalent ways of stating this property are that

$$u(t) = u_0 + \int_0^t F(u(s),s)\, ds \qquad \text{for } t \text{ pointwise } a.e. \text{ in } (0,T);$$

or that

$$\frac{d}{dt}\langle \omega, u(t)\rangle = \langle \omega, F(u(t),t)\rangle \qquad \text{for every } \omega \in Y'$$

in the sense of real-valued weak derivatives. Moreover, by approximating arbitrary smooth functions $w : (0,T) \to Y'$ by linear combinations of functions of the form $w(t) = \phi(t)\omega$, we see that this is equivalent to the statement that

$$-\int_0^T \langle w_t(t), u(t)\rangle\, dt = \int_0^T \langle w(t), F(u(t),t)\rangle\, dt \qquad \text{for every } w \in C_c^\infty(0,T;Y').$$

We define Sobolev spaces of vector-valued functions in the same way as for scalar-valued functions, and they have similar properties.

Definition 6.37. Suppose that X is a Banach space, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The Banach space $W^{k,p}(0,T;X)$ consists of all (equivalence classes of) measurable functions $u : (0,T) \to X$ whose weak derivatives of order $0 \le j \le k$ belong to $L^p(0,T;X)$. If $1 \le p < \infty$, then the $W^{k,p}$-norm is defined by

$$\|u\|_{W^{k,p}(0,T;X)} = \left(\sum_{j=1}^{k} \left\|\partial_t^j u\right\|_X^p\, dt\right)^{1/p};$$

if $p = \infty$, then

$$\|u\|_{W^{k,p}(0,T;X)} = \sup_{1\le j\le k} \left\|\partial_t^j u\right\|_X.$$

If $p = 2$, and $X = \mathcal{H}$ is a Hilbert space, then $W^{k,2}(0,T;\mathcal{H}) = H^k(0,T;\mathcal{H})$ is the Hilbert space with inner product

$$(u,v)_{H^k(0,T;\mathcal{H})} = \int_0^T (u(t),v(t))_{\mathcal{H}}\, dt.$$

The Sobolev embedding theorem for scalar-valued functions of a single variable carries over to the vector-valued case.

Theorem 6.38. *If $1 \le p \le \infty$ and $u \in W^{1,p}(0,T;X)$, then $u \in C([0,T];X)$. Moreover, there exists a constant $C = C(p,T)$ such that*

$$\|u\|_{L^\infty(0,T;X)} \le C\, \|u\|_{W^{1,p}(0,T;X)}.$$

PROOF. From Theorem 6.35, we have

$$\|u(t) - u(s)\| \leq \int_s^t \|u_t(r)\| \, dr.$$

Since $\|u_t\| \in L^1(0,T)$, its integral is absolutely continuous, so u is uniformly continuous on $(0,T)$ and extends to a continuous function on $[0,T]$.

If $h : (0,T) \to \mathbb{R}$ is defined by $h = \|u\|$, then

$$|h(t) - h(s)| \leq \|u(t) - u(s)\| \leq \int_s^t \|u_t(r)\| \, dr.$$

It follows that h is absolutely continuous and $|h_t| \leq \|u_t\|$ pointwise a.e. on $(0,T)$. Therefore, by the Sobolev embedding theorem for real valued functions,

$$\|u\|_{L^\infty(0,T;X)} = \|h\|_{L^\infty(0,T)} \leq C \|h\|_{W^{1,p}(0,T)} \leq C \|u\|_{W^{1,p}(0,T;X)}.$$

$\square$

6.A.4. The Radon-Nikodym property. Although we do not use this discussion elsewhere, it is interesting to consider the relationship between weak differentiability and absolute continuity in the vector-valued case.

The definition of absolute continuity of vector-valued functions is a natural generalization of the real-valued definition. We say that $f : [0,T] \to X$ is absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{n=1}^{N} \|f(t_n) - f(t_{n-1})\| < \epsilon$$

for every collection $\{[t_0, t_1], [t_2, t_3], \dots, [t_{N-1}, t_N]\}$ of non-overlapping subintervals of $[0,T]$ such that

$$\sum_{n=1}^{N} |t_n - t_{n-1}| < \delta.$$

Similarly, $f : [0,T] \to X$ is Lipschitz continuous on $[0,T]$ if there exists a constant $M \geq 0$ such that

$$\|f(s) - f(t)\| \leq M|s - t| \qquad \text{for all } s, t \in [0,T].$$

It follows immediately that a Lipschitz continuous function is absolutely continuous (with $\delta = \epsilon/M$).

A real-valued function is weakly differentiable with integrable derivative if and only if it is absolutely continuous *c.f.* Theorem 3.60. This is one of the few properties of real-valued integrals that does not carry over to Bochner integrals in arbitrary Banach spaces. It follows from the integral representation in Theorem 6.35 that every weakly differentiable function with integrable derivative is absolutely continuous, but it can happen that an absolutely continuous vector-valued function is not weakly differentiable.

EXAMPLE 6.39. Define $f : (0,1) \to L^1(0,1)$ by

$$f(t) = t\chi_{[0,t]}.$$

Then f is Lipschitz continuous, and therefore absolutely continuous. Nevertheless, the derivative $f'(t)$ does not exist for any $t \in (0,1)$ since the limit as $h \to 0$ of the difference quotient

$$\frac{f(t+h)-f(t)}{h}$$

does not converge in $L^1(0,1)$, so by Theorem 6.35 f is not weakly differentiable.

A Banach space for which every absolutely continuous function has an integrable weak derivative is said to have the Radon-Nikodym property. Any reflexive Banach space has this property but, as the previous example shows, the space $L^1(0,1)$ does not. One can use the Radon-Nikodym property to study the geometric structure of Banach spaces, but this question is not relevant for our purposes. Most of the spaces we use are reflexive, and even if they are not, we do not need an explicit characterization of the weakly differentiable functions.

6.B. Hilbert triples

Hilbert triples provide a useful framework for the study of weak and variational solutions of PDEs. We consider real Hilbert spaces for simplicity. For complex Hilbert spaces, one has to replace duals by antiduals, as appropriate.

Definition 6.40. A Hilbert triple consists of three separable Hilbert spaces

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$$

such that $\mathcal{V}$ is densely embedded in $\mathcal{H}$, $\mathcal{H}$ is densely embedded in $\mathcal{V}'$, and

$$\langle f, v\rangle = (f, v)_{\mathcal{H}} \qquad \text{for every } f \in \mathcal{H} \text{ and } v \in \mathcal{V}.$$

Hilbert triples are also referred to as Gelfand triples, variational triples, or rigged Hilbert spaces. In this definition, $\langle\cdot,\cdot\rangle : \mathcal{V}' \times \mathcal{V} \to \mathbb{R}$ denotes the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$, and $(\cdot,\cdot)_{\mathcal{H}} : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ denotes the inner product on $\mathcal{H}$. Thus, we identify: (a) the space $\mathcal{V}$ with a dense subspace of $\mathcal{H}$ through the embedding; (b) the dual of the 'pivot' space $\mathcal{H}$ with itself through its own inner product, as usual for a Hilbert space; (c) the space $\mathcal{H}$ with a subspace of the dual space $\mathcal{V}'$, where $\mathcal{H}$ acts on $\mathcal{V}$ through the $\mathcal{H}$-inner product, *not* the $\mathcal{V}$-inner product.

In the elliptic and parabolic problems considered above involving a uniformly elliptic, second-order operator, we have

$$\mathcal{V} = H_0^1(\Omega), \qquad \mathcal{H} = L^2(\Omega), \qquad \mathcal{V}' = H^{-1}(\Omega),$$
$$(f,g)_{\mathcal{H}} = \int_\Omega fg\,dx, \quad (f,g)_{\mathcal{V}} = \int_\Omega Df\cdot Dg\,dx,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set. Nothing will be lost by thinking about this case. The embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is inclusion. The embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is defined by the identification of an L^2-function with its corresponding regular distribution, and the action of $f \in L^2(\Omega)$ on a test function $v \in H_0^1(\Omega)$ is given by

$$\langle f, v\rangle = \int_\Omega fv\,dx.$$

The isomorphism between $\mathcal{V}$ and its dual space $\mathcal{V}'$ is then given by

$$-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega).$$

Thus, a Hilbert triple allows us to represent a 'concrete' operator, such as $-\Delta$, as an isomorphism between a Hilbert space and its dual.

As suggested by this example, in studying evolution equations such as the heat equation $u_t = \Delta u$, we are interested in functions u that take values in $\mathcal{V}$ whose weak time-derivatives u_t takes values in $\mathcal{V}'$. The basic facts about such functions are given in the next theorem, which states roughly that the natural identities for time derivatives hold provided that the duality pairings they involve make sense.

THEOREM 6.41. *Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$ be a Hilbert triple. If $u \in L^2(0,T;\mathcal{V})$ and $u_t \in L^2(0,T;\mathcal{V}')$, then $u \in C([0,T];\mathcal{H})$. Moreover:*

(1) *for any $v \in \mathcal{V}$, the real-valued function $t \mapsto (u(t),v)_{\mathcal{H}}$ is weakly differentiable in $(0,T)$ and*

$$\frac{d}{dt}(u(t),v)_{\mathcal{H}} = \langle u_t(t), v\rangle; \tag{6.45}$$

(2) *the real-valued function $t \mapsto \|u(t)\|_{\mathcal{H}}^2$ is weakly differentiable in $(0,T)$ and*

$$\frac{d}{dt}\|u\|_{\mathcal{H}}^2 = 2\langle u_t, u\rangle; \tag{6.46}$$

(3) *there is a constant $C = C(T)$ such that*

$$\|u\|_{L^\infty(0,T;\mathcal{H})} \le C\left(\|u\|_{L^2(0,T;\mathcal{V})} + \|u_t\|_{L^2(0,T;\mathcal{V}')}\right). \tag{6.47}$$

PROOF. We extend u to a compactly supported map $\tilde{u} : (-\infty,\infty) \to \mathcal{V}$ with $\tilde{u}_t \in L^2(\mathbb{R};\mathcal{V}')$. For example, we can do this by reflection of u in the endpoints of the interval [**4**]: Write $u = \phi u + \psi u$ on $[0,T]$ where $\phi, \psi \in C_c^\infty(\mathbb{R})$ are nonnegative test functions such that $\phi + \psi = 1$ on $[0,T]$ and $\operatorname{supp}\phi \subset [-T/4, 3T/4]$, $\operatorname{supp}\phi \subset [T/4, 5T/4]$; then extend ϕu, ψu to compactly supported, weakly differentiable functions $v, w : (-\infty,\infty) \to \mathcal{V}$ defined by

$$v(t) = \begin{cases} \phi(t)u(t) & \text{if } 0 \le t \le T, \\ \phi(-t)u(-t) & \text{if } -T \le t < 0, \\ 0 & \text{if } |t| > T, \end{cases}$$

$$w(t) = \begin{cases} \psi(t)u(t) & \text{if } 0 \le t \le T, \\ \psi(2T-t)u(2T-t) & \text{if } T < t \le 2T, \\ 0 & \text{if } |t-T| > T, \end{cases}$$

and finally define $\tilde{u} = v + w$. Next, we mollify the extension $\tilde{u}$ with the standard mollifier $\eta^\epsilon : \mathbb{R} \to \mathbb{R}$ to obtain a smooth approximation

$$u^\epsilon = \eta^\epsilon * \tilde{u} \in C_c^\infty(\mathbb{R};\mathcal{V}), \qquad u^\epsilon(t) = \int_{-\infty}^{\infty} \eta^\epsilon(t-s)\tilde{u}(s)\,ds.$$

The same results that apply to mollifiers of real-valued functions apply to these vector-valued functions. As $\epsilon \to 0^+$, we have: $u^\epsilon \to u$ in $L^2(0,T;\mathcal{V})$, $u_t^\epsilon = \eta^\epsilon * u_t \to u_t$ in $L^2(0,T;\mathcal{V}')$, and $u^\epsilon(t) \to u(t)$ in $\mathcal{V}$ for t pointwise *a.e.* in $(0,T)$. Moreover, as a consequence of the boundedness of the extension operator and the fact that mollification does not increase the norm of a function, there exists a constant $0 < C < 1$ such that for all $0 < \epsilon \le 1$, say,

$$C\|u^\epsilon\|_{L^2(\mathbb{R};\mathcal{V})} \le \|u\|_{L^2(0,T;\mathcal{V})} \le \|u^\epsilon\|_{L^2(\mathbb{R};\mathcal{V})}. \tag{6.48}$$

Since u^ϵ is a smooth $\mathcal{V}$-valued function and $\mathcal{V} \hookrightarrow \mathcal{H}$, we have

$$(u^\epsilon(t), u^\epsilon(t))_{\mathcal{H}} = \int_{-\infty}^{t} \frac{d}{ds}(u^\epsilon(s), u^\epsilon(s))_{\mathcal{H}}\,ds = 2\int_{-\infty}^{t}(u_s^\epsilon(s), u^\epsilon(s))_{\mathcal{H}}\,ds. \tag{6.49}$$

Using the analogous formula for $u^\epsilon - u^\delta$, the duality estimate and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\left\|u^\epsilon(t) - u^\delta(t)\right\|_{\mathcal{H}}^2 &\le 2\int_{-\infty}^{\infty} \left\|u_s^\epsilon(s) - u_s^\delta(s)\right\|_{\mathcal{V}'} \left\|u^\epsilon(s) - u^\delta(s)\right\|_{\mathcal{V}}\, ds \\ &\le 2\left\|u_t^\epsilon - u_t^\delta\right\|_{L^2(\mathbb{R};\mathcal{V}')} \left\|u^\epsilon - u^\delta\right\|_{L^2(\mathbb{R};\mathcal{V})}.\end{aligned}$$

Since $\{u^\epsilon\}$ is Cauchy in $L^2(\mathbb{R};\mathcal{V})$ and $\{u_t^\epsilon\}$ is Cauchy in $L^2(\mathbb{R};\mathcal{V}')$, it follows that $\{u^\epsilon\}$ is Cauchy in $C_c(\mathbb{R};\mathcal{H})$, and therefore converges uniformly on $[0,T]$ to a function $v \in C([0,T];\mathcal{H})$. Since u^ϵ converges pointwise *a.e.* to u, it follows that u is equivalent to v, so $u \in C([0,T];\mathcal{H})$ after being redefined, if necessary, on a set of measure zero.

Taking the limit of (6.49) as $\epsilon \to 0^+$, we find that for $t \in [0,T]$

$$\|u(t)\|_{\mathcal{H}}^2 = \|u(0)\|_{\mathcal{H}}^2 + 2\int_0^t \langle u_s(s), u(s)\rangle\, ds,$$

which implies that $\|u\|_{\mathcal{H}}^2 : [0,T] \to \mathbb{R}$ is absolutely continuous and (6.46) holds. Moreover, (6.47) follows from (6.48), (6.49), and the Cauchy-Schwartz inequality.

Finally, if $\phi \in C_c^\infty(0,T)$ is a test function $\phi : (0,T) \to \mathbb{R}$ and $v \in \mathcal{V}$, then $\phi v \in C_c^\infty(0,T;\mathcal{V})$. Therefore, since $u_t^\epsilon \to u_t$ in $L^2(0,T;\mathcal{V}')$,

$$\int_0^T \langle u_t^\epsilon, \phi v\rangle\, dt \to \int_0^T \langle u_t, \phi v\rangle\, dt.$$

Also, since u^ϵ is a smooth $\mathcal{V}$-valued function,

$$\int_0^T \langle u_t^\epsilon, \phi v\rangle\, dt = -\int_0^T \phi' \langle u^\epsilon, v\rangle\, dt \to -\int_0^T \phi' \langle u, v\rangle\, dt$$

We conclude that for every $\phi \in C_c^\infty(0,T)$ and $v \in \mathcal{V}$

$$\int_0^T \phi \langle u_t, v\rangle\, dt = -\int_0^T \phi_t \langle u, v\rangle\, dt$$

which is the weak form of (6.45). □

We further have the following integration by parts formula.

THEOREM 6.42. *Suppose that* $u, v \in L^2(0,T;\mathcal{V})$ *and* $u_t, v_t \in L^2(0,T;\mathcal{V}')$. *Then*

$$\int_0^T \langle u_t, v\rangle\, dt = (u(T), v(T))_{\mathcal{H}} - (u(0), v(0))_{\mathcal{H}} - \int_0^T \langle u, v_t\rangle\, dt.$$

PROOF. This result holds for smooth functions $u, v \in C^\infty([0,T];\mathcal{V})$. Therefore by density and Theorem 6.41 it holds for all functions $u, v \in L^2(0,T;\mathcal{V})$ with $u_t, v_t \in L^2(0,T;\mathcal{V}')$. □

CHAPTER 7

Hyperbolic Equations

Hyperbolic PDEs arise in physical applications as models of waves, such as acoustic, elastic, electromagnetic, or gravitational waves. The qualitative properties of hyperbolic PDEs differ sharply from those of parabolic PDEs. For example, they have finite domains of influence and dependence, and singularities in solutions propagate without being smoothed.

7.1. The wave equation

The prototypical example of a hyperbolic PDE is the wave equation

$$u_{tt} = \Delta u. \tag{7.1}$$

To begin with, consider the one-dimensional wave equation on $\mathbb{R}$,

$$u_{tt} = u_{xx}.$$

The general solution is the d'Alembert solution

$$u(x,t) = f(x-t) + g(x+t)$$

where f, g are arbitrary functions, as one may verify directly. This solution describes a superposition of two traveling waves with arbitrary profiles, one propagating with speed one to the right, the other with speed one to the left.

Let us compare this solution with the general solution of the one-dimensional heat equation

$$u_t = u_{xx},$$

which is given for $t > 0$ by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} f(y)\, dy.$$

Some of the qualitative properties of the wave equation that differ from those of the heat equation, which are evident from these solutions, are:

1. (1) the wave equation has finite propagation speed and domains of influence;
2. (2) the wave equation is reversible in time;
3. (3) solutions of the wave equation do not become smoother in time;
4. (4) the wave equation does not satisfy a maximum principle.

A suitable IBVP for the wave equation with Dirichlet BCs on a bounded open set $\Omega \subset \mathbb{R}^n$ for $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} &u_{tt} = \Delta u && \text{for } x \in \Omega \text{ and } t \in \mathbb{R},\\ &u(x,t) = 0 && \text{for } x \in \partial\Omega \text{ and } t \in \mathbb{R},\\ &u(x,0) = g(x), \quad u_t(x,0) = h(x) && \text{for } x \in \Omega. \end{aligned} \tag{7.2}$$

We require two initial conditions since the wave equation is second-order in time. For example, in two space dimensions, this IBVP would describe the small vibrations of an elastic membrane, with displacement $z = u(x, y, t)$, such as a drum. The membrane is fixed at its edge $\partial\Omega$, and has initial displacement g and initial velocity h. We could also add a nonhomogeneous term to the PDE, which would describe an external force, but we omit it for simplicity.

7.1.1. Energy estimate. To obtain the basic energy estimate for the wave equation, we multiple (7.1) by u_t and write

$$\begin{aligned} u_t u_{tt} &= \left(\frac{1}{2}u_t\right)_t, \\ u_t \Delta u &= \operatorname{div}(u_t Du) - Du \cdot Du_t = \operatorname{div}(u_t Du) - \left(\frac{1}{2}|Du|^2\right)_t \end{aligned}$$

to get

$$\left(\frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2\right) - \operatorname{div}(u_t Du) = 0. \tag{7.3}$$

This is the differential form of conservation of energy. The quantity $\frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2$ is the energy density (kinetic plus potential energy) and $-u_t Du$ is the energy flux.

If u is a solution of (7.2), then integration of (7.3) over Ω, use of the divergence theorem, and the BC $u = 0$ on $\partial\Omega$ (which implies that $u_t = 0$) gives

$$\frac{dE}{dt} = 0$$

where $E(t)$ is the total energy

$$E(t) = \int_\Omega \left(\frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2\right) dx.$$

Thus, the total energy remains constant. This result provides an L^2-energy estimate for solutions of the wave equation.

We will use this estimate to construct weak solutions of a general wave equation by a Galerkin method. Despite the qualitative difference in the properties of parabolic and hyperbolic PDEs, the proof is similar to the proof in Chapter 6 for the existence of weak solutions of parabolic PDEs. Some of the details are, however, more delicate; the lack of smoothing of hyperbolic PDEs is reflected analytically by weaker estimates for their solutions. For additional discussion see [**35**].

7.2. Definition of weak solutions

We consider a uniformly elliptic, second-order operator of the form (6.5). For simplicity, we assume that $b^i = 0$. In that case,

$$Lu = -\sum_{i,j=1}^n \partial_i \left(a^{ij}(x,t)\partial_j u\right) + c(x,t)u, \tag{7.4}$$

and L is formally self-adjoint. The first-order spatial derivative terms would be straightforward to include at the expense of complicating the energy estimates. We could also include appropriate first-order time derivatives in the equation proportional to u_t.

Generalizing (7.2), we consider the following IBVP for a second-order hyperbolic PDE

$$
\begin{aligned}
&u_{tt} + Lu = f && \text{in } \Omega \times (0,T),\\
&u = 0 && \text{on } \partial\Omega \times (0,T),\\
&u = g, \quad u_t = h && \text{on } t = 0.
\end{aligned}
\tag{7.5}
$$

To formulate a definition of a weak solution of (7.5), let $a(u,v;t) = (Lu,v)_{L^2}$ be the bilinear form associated with L in (7.4),

$$
a(u,v;t) = \sum_{i,j=1}^{n} \int_\Omega a^{ij}(x,t)\partial_i u(x)\partial_j u(x)\,dx + \int_\Omega c(x,t)u(x)v(x)\,dx. \tag{7.6}
$$

We make the following assumptions.

ASSUMPTION 7.1. *The set $\Omega \subset \mathbb{R}^n$ is bounded and open, $T > 0$, and:*

(1) *the coefficients of a in (7.6) satisfy*

$$
a^{ij}, c \in L^\infty(\Omega \times (0,T)), \qquad a^{ij}_t, c_t \in L^\infty(\Omega \times (0,T));
$$

(2) *$a^{ij} = a^{ji}$ for $1 \le i,j \le n$ and the uniform ellipticity condition (6.6) holds for some constant $\theta > 0$;*

(3) *$f \in L^2\left(0,T;L^2(\Omega)\right)$, $g \in H^1_0(\Omega)$, and $h \in L^2(\Omega)$.*

Then $a(u,v;t) = a(v,u;t)$ is a symmetric bilinear form on $H^1_0(\Omega)$ Moreover, there exist constants $C > 0$, $\beta > 0$, and $\gamma \in \mathbb{R}$ such that for every $u, v \in H^1_0(\Omega)$

$$
\begin{aligned}
\beta \|u\|^2_{H^1_0} &\le a(u,u;t) + \gamma\|u\|^2_{L^2},\\
|a(u,v;t)| &\le C\, \|u\|_{H^1_0} \|v\|_{H^1_0}.\\
|a_t(u,v;t)| &\le C\, \|u\|_{H^1_0} \|v\|_{H^1_0}.
\end{aligned}
\tag{7.7}
$$

We define weak solutions of (7.5) as follows.

DEFINITION 7.2. A function $u : [0,T] \to H^1_0(\Omega)$ is a weak solution of (7.5) if:

(1) u has weak derivatives u_t and u_{tt} and

$$
u \in C\left([0,T];H^1_0(\Omega)\right), \quad u_t \in C\left([0,T];L^2(\Omega)\right), \quad u_{tt} \in L^2\left(0,T;H^{-1}(\Omega)\right);
$$

(2) For every $v \in H^1_0(\Omega)$,

$$
\langle u_{tt}(t), v\rangle + a\left(u(t), v; t\right) = \left(f(t), v\right)_{L^2} \tag{7.8}
$$

for t pointwise a.e. in $[0,T]$ where a is defined in (7.6);

(3) $u(0) = g$ and $u_t(0) = h$.

We then have the following existence result.

THEOREM 7.3. *Suppose that the conditions in Assumption 7.1 are satisfied. Then for every $f \in L^2\left(0,T;L^2(\Omega)\right)$, $g \in H^1_0(\Omega)$, and $h \in L^2(\Omega)$, there is a unique weak solution of (7.5), in the sense of Definition 7.2. Moreover, there is a constant C, depending only on Ω, T, and the coefficients of L, such that*

$$
\begin{aligned}
&\|u\|_{L^\infty(0,T;H^1_0)} + \|u_t\|_{L^\infty(0,T;L^2)} + \|u_{tt}\|_{L^2(0,T;H^{-1})}\\
&\qquad\qquad \le C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{H^1_0} + \|h\|_{L^2}\right).
\end{aligned}
$$

7.3. Existence of weak solutions

We prove an existence result in this section. The continuity and uniqueness of weak solutions is proved in the next sections.

7.3.1. Construction of approximate solutions. As for the Galerkin approximation of the heat equation, let E_N be the N-dimensional subspace of $H_0^1(\Omega)$ given in (6.15)–(6.16) and P_N the orthogonal projection onto E_N given by (6.17).

DEFINITION 7.4. A function $u_N : [0,T] \to E_N$ is an approximate solution of (7.5) if:

(1) $u_N \in L^2(0,T;E_N)$, $u_{Nt} \in L^2(0,T;E_N)$, and $u_{Ntt} \in L^2(0,T;E_N)$;
(2) for every $v \in E_N$

$$(u_{Ntt}(t), v)_{L^2} + a\left(u_N(t), v; t\right) = (f(t), v)_{L^2} \tag{7.9}$$

pointwise *a.e.* in $t \in (0,T)$;
(3) $u_N(0) = P_N g$, and $u_{Nt}(0) = P_N h$.

Since $u_N \in H^2(0,T;E_N)$, it follows from the Sobolev embedding theorem for functions of a single variable t that $u_N \in C^1([0,T];E_N)$, so the initial condition (3) makes sense. Equation (7.9) is equivalent to an $N\times N$ linear system of second-order ODEs with coefficients that are L^∞ functions of t. By standard ODE theory, it has a solution $u_N \in H^2(0,T;E_N)$; if $a(w_j,w_k;t)$ and $(f(t),w_j)_{L^2}$ are continuous functions of time, then $u_N \in C^2(0,T;E_N)$. Thus, we have the following existence result.

PROPOSITION 7.5. *For every $N \in \mathbb{N}$, there exists a unique approximate solution $u_N : [0,T] \to E_N$ of (7.5) with*

$$u_N \in C^1\left([0,T];E_N\right), \qquad u_{Ntt} \in L^2\left(0,T;E_N\right).$$

7.3.2. Energy estimates for approximate solutions. The derivation of energy estimates for the approximate solutions follows the derivation of the *a priori* energy estimates for the wave equation.

PROPOSITION 7.6. *There exists a constant C, depending only on T, Ω, and the coefficient functions a^{ij}, c, such that for every $N \in \mathbb{N}$ the approximate solution u_N given by Proposition 7.5 satisfies*

$$\begin{aligned} &\|u_N\|_{L^\infty(0,T;H_0^1)} + \|u_{Nt}\|_{L^\infty(0,T;L^2)} + \|u_{Ntt}\|_{L^2(0,T;H^{-1})} \\ &\qquad\qquad \le C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{H_0^1} + \|h\|_{L^2}\right). \end{aligned} \tag{7.10}$$

PROOF. Taking $v = u_{Nt}(t) \in E_N$ in (7.9), we find that

$$(u_{Ntt}(t), u_{Nt}(t))_{L^2} + a\left(u_N(t), u_{Nt}(t); t\right) = (f(t), u_{Nt}(t))_{L^2}$$

pointwise *a.e.* in $(0,T)$. Since a is symmetric, it follows that

$$\frac{1}{2}\frac{d}{dt}\left[\|u_{Nt}\|_{L^2}^2 + a\left(u_N, u_N; t\right)\right] = (f, u_{Nt})_{L^2} + a_t\left(u_N, u_N; t\right).$$

Integrating this equation with respect to t, we get

$$\begin{aligned}
&\|u_{Nt}\|_{L^2}^2 + a\left(u_N, u_N; t\right) \\
&\qquad = 2\int_0^t \left[(f, u_{Ns})_{L^2} + a_s\left(u_N, u_N; s\right)\right] ds + a\left(P_N g, P_N g; 0\right) + \|P_N h\|_{L^2}^2 \\
&\qquad \le \int_0^t \left(\|u_{Ns}\|_{L^2}^2 + C\|u_N\|_{H_0^1}^2\right) ds + \|f\|_{L^2(0,T;L^2)}^2 + C\|g\|_{H_0^1}^2 + \|h\|_{L^2}^2,
\end{aligned}$$

where we have used (7.7), the fact that $\|P_N h\|_{L^2} \le \|h\|$, $\|P_N g\|_{H_0^1} \le \|g\|_{H_0^1}$, and the inequality

$$\begin{aligned}
2\int_0^t (f, u_{Ns})_{L^2} &\le 2\left(\int_0^t \|f\|_{L^2}^2\, ds\right)^{1/2}\left(\int_0^t \|u_{Ns}\|_{L^2}^2\, ds\right)^{1/2} \\
&\le \int_0^t \|u_{Ns}\|_{L^2}^2\, ds + \int_0^T \|f\|_{L^2}^2\, ds.
\end{aligned}$$

Using the uniform ellipticity condition in (7.7) to estimate $\|u_N\|_{H_0^1}^2$ in terms of $a(u_N, u_N; t)$ and a lower L^2-norm of u_N, we get for $0 \le t \le T$ that

$$\begin{aligned}
\|u_{Nt}\|_{L^2}^2 + \beta\|u_N\|_{H_0^1}^2 \le \int_0^t \left(\|u_{Ns}\|_{L^2}^2 + C\|u_N\|_{H_0^1}^2\right) ds + \gamma\|u_N\|_{L^2}^2 \\
+ \|f\|_{L^2(0,T;L^2)}^2 + C\|g\|_{H_0^1}^2 + \|h\|_{L^2}^2.
\end{aligned} \tag{7.11}$$

We estimate the L^2-norm of u_N by

$$\begin{aligned}
\|u_N\|_{L^2}^2 &= 2\int_0^t (u_N, u_N)_{L^2}\, ds + \|P_N g\|_{L^2}^2 \\
&\le 2\left(\int_0^t \|u_N\|_{L^2}^2\, ds\right)^{1/2}\left(\int_0^t \|u_{Ns}\|_{L^2}^2\, ds\right)^{1/2} + \|g\|_{L^2}^2 \\
&\le \int_0^t \left(\|u_N\|_{L^2}^2 + \|u_{Ns}\|_{L^2}^2\right) ds + \|g\|_{L^2}^2 \\
&\le \int_0^t \left(\|u_{Ns}\|_{L^2}^2 + C\|u_N\|_{H_0^1}^2\right) ds + C\|g\|_{H_0^1}^2.
\end{aligned}$$

Using this result in (7.11), we find that

$$\begin{aligned}
\|u_{Nt}\|_{L^2}^2 + \|u_N\|_{H_0^1}^2 \le C_1\int_0^t \left(\|u_{Ns}\|_{L^2}^2 + \|u_N\|_{H_0^1}^2\right) ds \\
+ C_2\left(\|f\|_{L^2(0,T;L^2)}^2 + \|g\|_{H_0^1}^2 + \|h\|_{L^2}^2\right)
\end{aligned} \tag{7.12}$$

for some constants $C_1, C_2 > 0$. Thus, defining $E : [0, T] \to \mathbb{R}$ by

$$E = \|u_{Nt}\|_{L^2}^2 + \|u_N\|_{H_0^1}^2,$$

we have

$$E(t) \le C_1\int_0^t E(s)\, ds + C_2\left(\|f\|_{L^2(0,T;L^2)}^2 + \|h\|_{L^2}^2 + \|g\|_{H_0^1}^2\right).$$

Gronwall's inequality (Lemma 1.47) implies that

$$E(t) \le C_2\left(\|f\|_{L^2(0,T;L^2)}^2 + \|h\|_{L^2}^2 + \|g\|_{H_0^1}^2\right) e^{C_1 t},$$

and we conclude that there is a constant C such that

$$\sup_{[0,T]}\left(\|u_{Nt}\|_{L^2}^2+\|u_N\|_{H_0^1}^2\right)\le C\left(\|f\|_{L^2(0,T;L^2)}^2+\|h\|_{L^2}^2+\|g\|_{H_0^1}^2\right). \tag{7.13}$$

Finally, from the Galerkin equation (7.9), we have for every $v\in E_N$ that

$$(u_{Ntt},v)_{L^2}=(f,v)_{L^2}-a\left(u_N,v;t\right)$$

pointwise *a.e.* in t. Since $u_{Ntt}\in E_N$, it follows that

$$\|u_{Ntt}\|_{H^{-1}}=\sup_{v\in E_N\setminus\{0\}}\frac{(u_{Ntt},v)_{L^2}}{\|v\|_{H_0^1}}\le C\left(\|f\|_{L^2}+\|u_N\|_{H_0^1}\right).$$

Squaring this inequality, integrating with respect to t, and using (7.10) we get

$$\begin{aligned}\int_0^T\|u_{Ntt}\|_{H^{-1}}^2\,dt&\le C\int_0^T\left(\|f\|_{L^2}^2+\|u_N\|_{H_0^1}^2\right)dt\\&\le C\left(\|f\|_{L^2(0,T;L^2)}^2+\|h\|_{L^2}^2+\|g\|_{H_0^1}^2\right).\end{aligned} \tag{7.14}$$

Combining (7.13)–(7.14), we get (7.10). □

7.3.3. Convergence of approximate solutions. The uniform estimates for the approximate solutions allows us to obtain a weak solution as the limit of a subsequence of approximate solutions in an appropriate weak-star topology. We use a weak-star topology because the estimates are L^∞ in time, and L^∞ is not reflexive. From Theorem 6.30, if X is reflexive Banach space, such as a Hilbert space, then

$$u_N\overset{*}{\rightharpoonup}u\qquad\text{in }L^\infty\left(0,T;X\right)$$

if and only if

$$\int_0^T\langle u_N(t),w(t)\rangle\,dt\to\int_0^T\langle u(t),w(t)\rangle\,dt\qquad\text{for every }w\in L^1\left(0,T;X'\right).$$

Theorem 1.19 then gives us weak-star compactness of the approximations and convergence of a subsequence as stated in the following proposition.

Proposition 7.7. *There is a subsequence $\{u_N\}$ of approximate solutions and a function u with such that*

$$\begin{aligned}&u_N\overset{*}{\rightharpoonup}u&&\text{as }N\to\infty\text{ in }L^\infty\left(0,T;H_0^1\right),\\&u_{Nt}\overset{*}{\rightharpoonup}u_t&&\text{as }N\to\infty\text{ in }L^\infty\left(0,T;L^2\right),\\&u_{Ntt}\rightharpoonup u_{tt}&&\text{as }N\to\infty\text{ in }L^2\left(0,T;H^{-1}\right),\end{aligned}$$

where u satisfies (7.8).

Proof. By Proposition 7.6, the approximate solutions $\{u_N\}$ are uniformly bounded in $L^\infty(0,T;H_0^1)$, and their time-derivatives are uniformly bounded in $L^\infty(0,T;L^2)$. It follows from the Banach-Alaoglu theorem, and the usual argument that a weak limit of derivatives is the derivative of the weak limit, that there is a subsequence of approximate solutions, which we still denote by $\{u_N\}$, such that

$$u_N\overset{*}{\rightharpoonup}u\quad\text{in }L^\infty(0,T;H_0^1),\qquad u_{Nt}\overset{*}{\rightharpoonup}u_t\quad\text{in }L^\infty(0,T;L^2).$$

Moreover, since $\{u_{Ntt}\}$ is uniformly bounded in $L^2(0,T;H^{-1})$, we can choose the subsequence so that

$$u_{Ntt}\rightharpoonup u_{tt}\qquad\text{in }L^2(0,T;H^{-1}).$$

Thus, the weak-star limit u satisfies

$$u \in L^\infty(0,T;H_0^1), \quad u_t \in L^\infty(0,T;L^2), \quad u_{tt} \in L^2(0,T;H^{-1}). \tag{7.15}$$

Passing to the limit $N \to \infty$ in the Galerkin equations(7.9), we find that u satisfies (7.8) for every $v \in H_0^1(\Omega)$. In detail, consider time-dependent test functions of the form $w(t) = \phi(t)v$ where $\phi \in C_c^\infty(0,T)$ and $v \in E_M$, as for the parabolic equation. Multiplying (7.9) by $\phi(t)$ and integrating the result with respect to t, we find that for $N \geq M$

$$\int_0^T (u_{Ntt}, w)_{L^2}\ dt + \int_0^T a\left(u_N, w; t\right)\ dt = \int_0^T (f, w)_{L^2}\ dt.$$

Taking the limit of this equation as $N \to \infty$, we get

$$\int_0^T (u_{tt}, w)_{L^2}\ dt + \int_0^T a\left(u, w; t\right)\ dt = \int_0^T (f, w)_{L^2}\ dt.$$

By density, this equation holds for $w(t) = \phi(t)v$ where $v \in H_0^1(\Omega)$, and then since $\phi \in C_c^\infty(0,T)$ is arbitrary, we get(7.9). □

7.4. Continuity of weak solutions

In this section, we show that the weak solutions obtained above satisfy the continuity requirement (1) in Definition 7.2. To do this, we show that u and u_t are weakly continuous with values in H_0^1, and L^2 respectively, then use the energy estimate to show that the 'energy' $E : (0,T) \to \mathbb{R}$ defined by

$$E = \|u_t\|_{L^2} + a(u,u;t) \tag{7.16}$$

is a continuous function of time. This gives continuity in norm, which together with weak continuity implies strong continuity. The argument is essentially the same as the proof that if a sequence $\{x_n\}$ converges weakly to x in a Hilbert space $\mathcal{H}$ and the norms also converge, then the sequence converges strongly:

$$(x - x_n, x - x_n) = \|x\|^2 - 2(x, x_n) + \|x_n\|^2 \to \|x\|^2 - 2(x,x) + \|x\|^2 = 0.$$

See (7.23) below for the analogous formula in this argument.

We begin by proving the weak continuity, which follows from the next lemma.

Lemma 7.8. *Suppose that $\mathcal{V}$, $\mathcal{H}$ are Hilbert spaces and $\mathcal{V} \hookrightarrow \mathcal{H}$ is densely and continuously embedded in $\mathcal{H}$. If*

$$u \in L^\infty\left(0,T;\mathcal{V}\right), \qquad u_t \in L^2\left(0,T;\mathcal{H}\right),$$

then $u \in C_w\left([0,T];\mathcal{V}\right)$ is weakly continuous.

Proof. We have $u \in H^1\left(0,T;\mathcal{H}\right)$ and the Sobolev embedding theorem, Theorem 6.38, implies that $u \in C\left([0,T];\mathcal{H}\right)$. Let $\omega \in \mathcal{V}'$, and choose $\omega_n \in \mathcal{H}$ such that $\omega_n \to \omega$ in $\mathcal{V}'$. Then

$$|\langle \omega_n, u(t)\rangle - \langle \omega, u(t)\rangle| = |\langle \omega_n - \omega, u(t)\rangle| \leq \|\omega_n - \omega\|_{\mathcal{V}'} \|u(t)\|_{\mathcal{V}}.$$

Thus,

$$\sup_{[0,T]} |\langle \omega_n, u\rangle - \langle \omega, u\rangle| \leq \|\omega_n - \omega\|_{\mathcal{V}'} \|u\|_{L^\infty(0,T;\mathcal{V})} \to 0 \qquad \text{as } n \to \infty,$$

so $\langle \omega_n, u\rangle$ converges uniformly to $\langle \omega, u\rangle$. Since $\langle \omega_n, u\rangle \in C\left([0,T];\mathcal{V}\right)$ it follows that $\langle \omega, u\rangle \in C\left([0,T];\mathcal{V}\right)$, meaning that u is weakly continuous into $\mathcal{V}$. □

LEMMA 7.9. *Let u be a weak solution constructed in Proposition 7.7. Then*

$$u \in C_w([0,T];H_0^1), \qquad u_t \in C_w([0,T];L^2) \tag{7.17}$$

PROOF. This follows at once from Lemma 7.9 and the fact that

$$u \in L^\infty\left(0,T;H_0^1\right), \quad u_t \in L^\infty\left(0,T;H^{-1}\right), \quad u_{tt} \in L^2\left(0,T;H^{-1}\right),$$

where $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. □

Next, we prove that the energy is continuous. In doing this, we have to be careful not to assume more regularity in time that we know.

LEMMA 7.10. *Suppose that L is given by (7.4) and a by (7.6), where the coefficients satisfy the conditions in Assumption 7.1. If*

$$u \in L^2\left(0,T;H_0^1(\Omega)\right), \quad u_t \in L^2\left(0,T;L^2(\Omega)\right), \quad u_{tt} \in L^2\left(0,T;H^{-1}(\Omega)\right),$$

and

$$u_{tt} + Lu \in L^2\left(0,T;L^2(\Omega)\right), \tag{7.18}$$

then

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_{L^2}^2 + a(u,u;t)\right) = (u_{tt}+Lu,u_t)_{L^2} + \frac{1}{2}a_t(u,u;t). \tag{7.19}$$

and $E:(0,T)\to\mathbb{R}$ defined in (7.16) is an absolutely continuous function.

PROOF. We show first that (7.19) holds in the sense of (real-valued) distributions on $(0,T)$. The relation would be immediate if u was sufficiently smooth to allow us to expand the derivatives with respect to t. We prove it for general u by mollification.

It is sufficient to show that (7.19) holds in the distributional sense on compact subsets of $(0,T)$. Let $\zeta \in C_c^\infty(\mathbb{R})$ be a cut-off function that is equal to one on some subinterval $I \Subset (0,T)$ and zero on $\mathbb{R}\setminus(0,T)$. Extend u to a compactly supported function $\zeta u : \mathbb{R} \to H_0^1(\Omega)$, and mollify this function with the standard mollifier $\eta^\epsilon : \mathbb{R}\to\mathbb{R}$ to obtain

$$u^\epsilon = \eta^\epsilon * (\zeta u) \in C_c^\infty\left(\mathbb{R};H_0^1\right).$$

Mollifying (7.18), we also have that

$$u_{tt}^\epsilon + Lu^\epsilon \in L^2\left(\mathbb{R};L^2\right). \tag{7.20}$$

Without (7.18), we would only have $Lu^\epsilon \in L^2\left(\mathbb{R};H^{-1}\right)$.

Since u^ϵ is a smooth, H_0^1-valued function and a is symmetric, we have that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|u_t^\epsilon\|_{L^2}^2 + a\left(u^\epsilon,u^\epsilon;t\right)\right) &= \langle u_{tt}^\epsilon, u_t^\epsilon\rangle + a\left(u^\epsilon,u_t^\epsilon;t\right) + \frac{1}{2}a_t\left(u^\epsilon,u^\epsilon;t\right)\\
&= \langle u_{tt}^\epsilon, u_t^\epsilon\rangle + \langle Lu^\epsilon, u_t^\epsilon\rangle + \frac{1}{2}a_t\left(u^\epsilon,u^\epsilon;t\right)\\
&= \langle u_{tt}^\epsilon + Lu^\epsilon, u_t^\epsilon\rangle + \frac{1}{2}a_t\left(u^\epsilon,u^\epsilon;t\right)\\
&= \left(u_{tt}^\epsilon + Lu^\epsilon, u_t^\epsilon\right)_{L^2} + \frac{1}{2}a_t\left(u^\epsilon,u^\epsilon;t\right).
\end{aligned} \tag{7.21}$$

Here, we have used (7.20) and the identity

$$a(u,v;t) = \langle L(t)u,v\rangle \qquad \text{for } u,v \in H_0^1.$$

Note that we cannot use this identity to rewrite $a(u,u_t;t)$ if u is the unmollified function, since we know only that $u_t \in L^2$. Taking the limit of (7.21) as $\epsilon \to 0^+$, we get the same equation for ζu, and hence (7.19) holds on every compact subinterval of $(0,T)$, which proves the equation.

The right-hand side of (7.19) belongs to $L^1(0,T)$ since

$$\begin{aligned}\int_0^T (u_{tt}+Lu,u_t)_{L^2}\,dt &\le \int_0^T \|u_{tt}+Lu\|_{L^2}\|u_t\|_{L^2}\,dt\\ &\le \|u_{tt}+Lu\|_{L^2(0,T;L^2)}\|u_t\|_{L^2(0,T;L^2)},\\ \int_0^T a_t(u,u;t)\,dt &\le \int_0^T C\|u\|_{H_0^1}^2\,dt\\ &\le C\|u\|^2_{L^2(0,T;H_0^1)}.\end{aligned}$$

Thus, E in (7.16) is the integral of an L^1-function, so it is absolutely continuous. □

PROPOSITION 7.11. *Let u be a weak solution constructed in Proposition 7.7. Then*

$$u\in C\left([0,T];H_0^1(\Omega)\right),\qquad u_t\in C\left([0,T];L^2(\Omega)\right). \tag{7.22}$$

PROOF. Using the weak continuity of u, u_t from Lemma 7.9, the continuity of E from Lemma 7.10, energy, and the continuity of a_t on H_0^1, we find that as $t\to t_0$,

$$\begin{aligned}&\|u_t(t)-u_t(t_0)\|^2_{L^2}+a\left(u(t)-u(t_0),u(t)-u(t_0);t_0\right)\\ &\quad=\|u_t(t)\|^2_{L^2}-2\left(u_t(t),u_t(t_0)\right)_{L^2}+\|u_t(t_0)\|^2_{L^2}\\ &\quad\quad+a\left(u(t),u(t);t_0\right)-2a\left(u(t),u(t_0);t_0\right)+a\left(u(t_0),u(t_0);t_0\right)\\ &\quad=\|u_t(t)\|_{L^2}+a\left(u(t),u(t);t\right)+\|u_t(t_0)\|_{L^2}+a\left(u(t_0),u(t_0);t_0\right)\\ &\quad\quad+a\left(u(t),u(t);t_0\right)-a\left(u(t),u(t);t\right)\\ &\quad\quad-2\left(u_t(t),u_t(t_0)\right)_{L^2}-2a\left(u(t),u(t_0);t_0\right)\\ &\quad=E(t)+E(t_0)+a\left(u(t),u(t);t_0\right)-a\left(u(t),u(t);t\right)\\ &\quad\quad-2\left\{\left(u_t(t),u_t(t_0)\right)_{L^2}+a\left(u(t),u(t_0);t_0\right)\right\}\\ &\quad\to E(t_0)+E(t_0)-2\left\{\|u_t(t_0)\|_{L^2}+a\left(u(t_0),u(t_0);t_0\right)\right\}=0.\end{aligned} \tag{7.23}$$

Finally, using this result, the coercivity estimate

$$\theta\|u(t)-u(t_0)\|^2_{H_0^1}\le a\left(u(t)-u(t_0),u(t)-u(t_0);t_0\right)+\gamma\|\,\|u(t)-u(t_0)\|^2_{L^2}$$

and the fact that $u\in C\left(0,T;L^2\right)$ by Sobolev embedding, we conclude that

$$\lim_{t\to t_0}\|u_t(t)-u_t(t_0)\|_{L^2}=0,\qquad \lim_{t\to t_0}\|u(t)-u(t_0)\|_{H_0^1}=0,$$

which proves (7.22). □

This completes the proof of the existence of a weak solution in the sense of Definition 7.2.

7.5. Uniqueness of weak solutions

The proof of uniqueness of weak solutions of the IBVP (7.5) for the second-order hyperbolic PDE requires a more careful argument than for the corresponding parabolic IBVP. To get an energy estimate in the parabolic case, we use the test function $v=u(t)$; this is permissible since $u(t)\in H_0^1(\Omega)$. To get an estimate in the hyperbolic case, we would like to take $v=u_t(t)$, but we cannot do this directly, since we know only that $u_t(t) \in L^2(\Omega)$. Instead we fix $t_0 \in (0,T)$ and use as a test function

$$
\begin{aligned}
v(t) &= \int_{t_0}^{t} u(s)\, ds \qquad && \text{for } 0 < t \leq t_0, \\
v(t) &= 0 && \text{for } t_0 < t < T.
\end{aligned} \tag{7.24}
$$

To motivate this choice, consider an *a priori* estimate for the wave equation. Suppose that

$$u_{tt} = \Delta u, \qquad u(0) = u_t(0) = 0.$$

Multiplying the PDE by v in (7.24), and using the fact that $v_t = u$ we get for $0 < t < t_0$ that

$$\left(v u_t - \frac{1}{2} u^2 + \frac{1}{2}|Dv|^2 \right)_t - \operatorname{div}(v Du) = 0.$$

We integrate this equation over Ω to get

$$\frac{d}{dt} \int_\Omega \left(v u_t - \frac{1}{2} u^2 + \frac{1}{2}|Dv|^2 \right)\, dx = 0$$

The boundary terms $vDu{\cdot}\nu$ vanish since $u = 0$ on $\partial\Omega$ implies that $v = 0$. Integrating this equation with respect to t over $(0, t_0)$, and using the fact that $u = u_t = 0$ at $t = 0$ and $v = 0$ at $t = t_0$, we find that

$$\|u\|_{L^2}^2 (t_0) + \|v\|_{H_0^1}^2 (0) = 0.$$

Since this holds for every $t_0 \in (0,T)$, we conclude that $u = 0$.

The proof of the next proposition is the same calculation for weak solutions.

Proposition 7.12. *A weak solution of (7.5) in the sense of Definition 7.2 is unique.*

Proof. Since the equation is linear, to show uniqueness it is sufficient to show that the only solution u of (7.5) with zero data ($f = 0$, $g = 0$, $h = 0$) is $u = 0$.

Let $v \in C\left([0,T]; H_0^1\right)$ be given by (7.24). Using $v(t)$ in (7.8), we get for $0 < t < t_0$ that

$$\langle u_{tt}(t), v(t) \rangle + a\left(u(t), v(t); t\right) = 0.$$

Since $u = v_t$ and a is a symmetric bilinear form on H_0^1, it follows that

$$\frac{d}{dt} \left[(u_t, v)_{L^2} - \frac{1}{2}\left(u, u\right)_{L^2} + \frac{1}{2} a\left(v, v; t\right) \right] = a_t(v, v; t).$$

Integrating this equation from 0 to t_0, and using the fact that

$$u(0) = 0, \qquad u_t(0) = 0, \qquad v(t_0) = 0,$$

we get

$$\|u(t_0)\|_{L^2} + a\left(v(0), v(0); 0\right) = -2 \int_0^{t_0} a(v, v; t)\, dt.$$

Using the coercivity and boundedness estimates for a in (7.7), we find that

$$\|u(t_0)\|_{L^2}^2 + \beta \|v(0)\|_{H_0^1}^2 \leq C \int_0^{t_0} \|v(t)\|_{H_0^1}^2 \, dt + \gamma \|v(0)\|_{L^2}^2 . \tag{7.25}$$

Writing $w(t) = -v(t_0 - t)$ for $0 < t < t_0$, we have from (7.24) that

$$w(t) = -\int_{t_0}^{t_0 - t} u(s)\, ds = \int_0^t u(t_0 - s)\, ds$$

and

$$v(0) = -w(t_0) = -\int_0^{t_0} u(t_0 - s)\,ds = \int_0^{t_0} u(t)\,dt,$$

$$\int_0^{t_0} \|v(t)\|_{H_0^1}^2\,dt = \int_0^{t_0} \|w(t_0 - t)\|_{H_0^1}^2\,dt = \int_0^{t_0} \|w(t)\|_{H_0^1}^2\,dt.$$

Using these expressions in (7.25), we get an estimate of the form

$$\|u(t_0)\|_{L^2}^2 + \|w(t_0)\|_{H_0^1}^2 \leq C \int_0^{t_0} \left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{H_0^1}^2\right)\,dt$$

for every $0 < t_0 < T$. Since $u(0) = 0$ and $w(0) = 0$, Gronwall's inequality implies that u, w are zero on $[0, T]$, which proves the uniqueness of weak solutions. □

This proposition completes the proof of Theorem 7.3. For the regularity theory of these weak solutions see §7.2.3 of [**9**].

CHAPTER 8

Friedrich symmetric systems

In this chapter, we describe a theory due to Friedrich [**13**] for positive symmetric systems, which gives the existence and uniqueness of weak solutions of boundary value problems under appropriate positivity conditions on the PDE and the boundary conditions. No assumptions about the type of the PDE are required, and the theory applies equally well to hyperbolic, elliptic, and mixed-type systems.

8.1. A BVP for symmetric systems

Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Consider a BVP for an $m \times m$ system of PDEs for $u : \Omega \to \mathbb{R}^m$ of the form

$$\begin{aligned} &A^i \partial_i u + Cu = f && \text{in } \Omega, \\ &B_- u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{8.1}$$

where A^i, C, B_- are $m \times m$ coefficient matrices, $f : \Omega \to \mathbb{R}^m$, and we use the summation convention. We assume throughout that A^i is symmetric.

We define a boundary matrix on $\partial\Omega$ by

$$B = \nu_i A^i \tag{8.2}$$

where ν is the outward unit normal to $\partial\Omega$. We assume that the boundary is noncharacteristic and that (8.1) satisfies the following smoothness conditions.

Definition 8.1. The BVP (8.1) is smooth if:

(1) The domain Ω is bounded and has C^2-boundary.
(2) The symmetric matrices $A^i : \overline{\Omega} \to \mathbb{R}^{m\times m}$ are continuously differentiable on the closure $\overline{\Omega}$, and $C : \overline{\Omega} \to \mathbb{R}^{m\times m}$ is continuous on $\overline{\Omega}$.
(3) The boundary matrix $B_- : \partial\Omega \to \mathbb{R}^{m\times m}$ is continuous on $\partial\Omega$.

These assumptions can be relaxed, but our goal is to describe the theory in its basic form with a minimum of technicalities.

Let L denote the operator in (8.1) and L^* its formal adjoint,

$$L = A^i \partial_i + C, \qquad L^* = -A^i \partial_i + C^T - \partial_i A^i. \tag{8.3}$$

For brevity, we write spaces of continuously differentiable and square integrable vector-valued functions as

$$C^1(\overline{\Omega}) = C^1(\overline{\Omega}; \mathbb{R}^m), \qquad L^2(\Omega) = L^2(\Omega; \mathbb{R}^m),$$

with a similar notation for matrix-valued functions.

Proposition 8.2 (Green's identity). *If the smoothness assumptions in Definition 8.1 are satisfied and $u, v \in C^1(\overline{\Omega})$, then*

$$\int_\Omega v^T L u \, dx - \int_\Omega u^T L^* v \, dx = \int_{\partial\Omega} v^T B u \, dS, \tag{8.4}$$

where B is defined in (8.2).

PROOF. Using the symmetry of A^i, we have

$$v^T Lu = u^T L^* v + \partial_i \left(v^T A^i u\right).$$

The result follows by integration and the use of Green's theorem. □

The smoothness assumptions are sufficient to ensure that Green's theorem applies, although it also holds under weaker assumptions.

PROPOSITION 8.3 (Energy identity). *If the smoothness assumptions in Definition 8.1 are satisfied and $u \in C^1(\overline{\Omega})$, then*

$$\int_\Omega u^T \left(C + C^T - \partial_i A^i\right) u \, dx + \int_{\partial\Omega} u^T B u \, dS = 2 \int_\Omega f^T u \, dx \tag{8.5}$$

where B is defined in (8.2) and $Lu = f$.

PROOF. Taking the inner product of the equation $Lu = f$ with u, adding the transposed equation, and combining the derivatives of u, we get

$$\partial_i \left(u^T A^i u\right) + u^T \left(C + C^T - \partial_i A^i\right) u = 2 f^T u.$$

The result follows by integration and the use of Green's theorem. □

To get energy estimates, we want to ensure that the volume integral in (8.5) is positive, which leads to the following definition.

DEFINITION 8.4. The system in (8.1) is a positive symmetric system if the matrices A^i are symmetric and there exists a constant $c > 0$ such that

$$C + C^T - \partial_i A^i \geq 2cI. \tag{8.6}$$

8.2. Boundary conditions

We assume that the domain has non-characteristic boundary, meaning that the boundary matrix $B = \nu_i A^i$ is nonsingular on $\partial\Omega$. The analysis extends to characteristic boundaries with constant multiplicity, meaning that the rank of B is constant on $\partial\Omega$ [**25, 34**].

To get estimates, we need the boundary terms in (8.5) to be positive for all u such that $B_- u = 0$. Furthermore, to get estimates for the adjoint problem, we need the adjoint boundary terms to be negative. This is the case if the boundary conditions are maximally positive in the following sense [**13**].

DEFINITION 8.5. Let $B = \nu_i A^i$ be a nonsingular, symmetric boundary matrix. A boundary condition $B_- u = 0$ on $\partial\Omega$ is maximally positive if there is a (not necessarily symmetric) matrix function $M : \partial\Omega \to \mathbb{R}^{m\times m}$ such that:

(1) $B = B_+ + B_-$ where $B_+ = B + M$, and $B_- = B - M$;
(2) $M + M^T \geq 0$ (positivity);
(3) $\mathbb{R}^m = \ker B_+ \oplus \ker B_-$ (maximality).

The adjoint boundary condition to $B_- u = 0$ is $B_+^T v = 0$, as can be seen from the decomposition

$$v^T B u = u^T B_+^T v + v^T B_- u.$$

If $B_- u = 0$ on $\partial\Omega$, then

$$u^T B u = u^T \left(B_+ - B_-\right) u = u^T \left(M + M^T\right) u \geq 0, \tag{8.7}$$

while if $B_+^T v = 0$ on $\partial\Omega$, then

$$v^T B v = v^T \left(-B_+ + B_-\right) v = -v^T \left(M + M^T\right) v \leq 0. \tag{8.8}$$

The boundary condition $B_- u = 0$ can also be formulated as: $u \in \mathcal{N}_+$ where $\mathcal{N}_+ = \ker B_-$ is a family of subspaces defined on $\partial\Omega$. An equivalent way to state Definition 8.5 is that the subspace $\mathcal{N}_+$ is a maximally positive subspace for B, meaning that B is positive (≥ 0) on $\mathcal{N}_+$ and not positive on any strictly larger subspace of $\mathbb{R}^m$ that contains $\mathcal{N}_+$.

The adjoint boundary condition $B_+^T v = 0$ may be written as $v \in \mathcal{N}_-$ where $\mathcal{N}_- = \ker B_+^T$ is a maximally negative subspace that complements $\mathcal{N}_+$, and

$$\mathbb{R}^m = \mathcal{N}_+ \oplus \mathcal{N}_-, \qquad \mathcal{N}_+ = \left(B\mathcal{N}_-\right)^\perp, \qquad \mathcal{N}_- = \left(B\mathcal{N}_+\right)^\perp.$$

We may consider $\mathbb{R}^m$ as a vector space with an indefinite inner product given by $\langle u, v\rangle = u^T B v$. It follows from standard results about indefinite inner product spaces that if $\mathbb{R}^m = \mathcal{N}_+ \oplus \mathcal{N}_-$ where $\mathcal{N}_+$ is a maximally positive subspace for $\langle\cdot,\cdot\rangle$, then $\mathcal{N}_-$ is a maximally negative subspace. Moreover, the dimension of $\mathcal{N}_+$ is equal to the number of positive eigenvalues of B, and the dimension of $\mathcal{N}_-$ is equal to the number of negative eigenvalues of B. In particular, the dimensions of $\mathcal{N}_\pm$ are constant on each connected component of $\partial\Omega$ if B is continuous and non-singular.

8.3. Uniqueness of smooth solutions

Under the above positivity assumptions, we can estimate a smooth solution u of (8.1) by the right-hand side f. A similar result holds for the adjoint problem. Let

$$\|u\| = \left(\int_\Omega |u|^2\, dx\right)^{1/2}, \qquad (u, v) = \int_\Omega u^T v\, dx \tag{8.9}$$

denote the standard L^2-norm and inner product, where $|u|$ denotes the Euclidean norm of $u \in \mathbb{R}^m$.

THEOREM 8.6. *Let L, L^* denote the operators in (8.3), and suppose that the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied. If $u \in C^1(\overline{\Omega})$ and $B_- u = 0$ on $\partial\Omega$, then $c\|u\| \leq \|Lu\|$. If $v \in C^1(\overline{\Omega})$ and $B_+^T v = 0$ on $\partial\Omega$, then $c\|v\| \leq \|L^* v\|$.*

PROOF. If $B_- u = 0$, then the energy identity (8.5), the positivity conditions (8.6)–(8.7), and the Cauchy-Schwartz inequality imply that

$$\begin{aligned} 2c\|u\|^2 &\leq \int_\Omega u^T \left(C + C^T - \partial_i A^i\right) u\, dx + \int_{\partial\Omega} u^T B u\, dS \\ &\leq 2\int_\Omega u^T L u\, dx \\ &\leq 2\|u\|\, \|Lu\| \end{aligned}$$

so $c\|u\| \leq \|Lu\|$. Similarly, if $B_+^T v = 0$, then Green's formula and (8.8) imply that

$$2c\|v\|^2 \leq 2\int_\Omega v^T L^* v\, dx - \int_{\partial\Omega} v^T B v\, dS = 2\int_\Omega v^T L^* v\, dx \leq 2\|v\|\, \|L^* v\|,$$

which proves the result for L^*. $\square$

COROLLARY 8.7. *If the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied, then a smooth solution* $u \in C^1(\overline{\Omega})$ *of (8.1) is unique.*

PROOF. If u_1, u_2 are two solutions and $u = u_1 - u_2$, then $Lu = 0$ and $B_- u = 0$, so Theorem 8.6 implies that $u = 0$. $\square$

8.4. Existence of weak solutions

We define weak solutions of (8.1) as follows.

DEFINITION 8.8. Let $f \in L^2(\Omega)$. A function $u \in L^2(\Omega)$ is a weak solution of (8.1) if

$$\int_\Omega u^T L^* v\, dx = \int_\Omega f^T v\, dx \qquad \text{for all } v \in D^*,$$

where L^* is the operator defined in (8.3), the space of test functions v is

$$D^* = \left\{ v \in C^1(\overline{\Omega}) : B_+^T v = 0 \text{ on } \partial\Omega \right\}, \tag{8.10}$$

and B_+ is the boundary matrix in Definition 8.5.

It follows from Green's theorem that a smooth function $u \in C^1(\overline{\Omega})$ is a weak solution of (8.1) if and only if it is a classical solution i.e., it satisfies (8.1) pointwise. In general, a weak solution u is a distributional solution of $Lu = f$ in Ω with $u, Lu \in L^2(\Omega)$. The boundary condition $B_- u = 0$ is enforced weakly by the use of test functions v that are not compactly supported in Ω and satisfy the adjoint boundary condition $B_+^T v = 0$.

In particular, functions $u, v \in H^1(\Omega)$ satisfy the integration by parts formula

$$\int_\Omega v^T \partial_i u\, dx = -\int_\Omega u^T \partial_i v\, dx + \int_{\partial\Omega} \nu_i (\gamma v)^T (\gamma u)\, dx'$$

where the trace map

$$\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega) \tag{8.11}$$

is defined by the pointwise evaluation of smooth functions on $\partial\Omega$ extended by density and boundedness to $H^1(\Omega)$.The trace map is not, however, well-defined for general $u \in L^2(\Omega)$.

It follows that if $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, satisfying Definition 8.8, then

$$\int_{\mathbb{R}^{n-1}} v^T \gamma B u\, dx' = 0 \qquad \text{for all } v \in D^*,$$

which implies that $\gamma B_- u = 0$. A similar result holds in a distributional sense if $u, Lu \in L^2(\Omega)$, in which case $\gamma B u \in H^{-1/2}(\partial\Omega)$.

The existence of weak solutions follows immediately from the the Riesz representation theorem and the estimate for the adjoint L^* in Theorem 8.6.

THEOREM 8.9. *If the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied, then there is a weak solution* $u \in L^2(\Omega)$ *of (8.1) for every* $f \in L^2(\Omega)$.

PROOF. We write $H = L^2(\Omega)$, where H is equipped with its standard norm and inner product given in (8.9). Let

$$L^* : D^* \subset H \to H$$

where the domain D^* of L^* is given by (8.10), and denote the range of L^* by $W = L^*(D^*) \subset H$. From Theorem 8.6,

$$c\|v\| \leq \|L^*v\| \qquad \text{for all } v \in D^*, \tag{8.12}$$

which implies, in particular, that $L^* : D^* \to W$ is one-to-one.

Define a linear functional $\ell : W \to \mathbb{R}$ by

$$\ell(w) = (f, v) \text{ where } L^*v = w.$$

This functional is well-defined since L^* is one-to-one. Furthermore, ℓ is bounded on W since (8.12) implies that

$$|\ell(w)| \leq \|f\|\,\|v\| \leq \frac{1}{c}\|f\|\,\|w\|.$$

By the Riesz representation theorem, there exists $u \in \overline{W} \subset H$ such that $(u, w) = \ell(w)$ for all $w \in W$, which implies that

$$(u, L^*v) = (f, v) \qquad \text{for all } v \in D^*.$$

This identity it just the statement that u is a weak solution of (8.1). □

8.5. Weak equals strong

A weak solution of (8.1) does not satisfy the same boundary condition as a test function in Definition 8.8. As a result, we cannot derive an energy equation analogous to (8.5) directly from the weak formulation and use it to prove the uniqueness of a weak solution.

To close the gap between the existence of weak solutions and the uniqueness of smooth solutions, we use the fact that weak solutions are strong solutions, meaning that they can be obtained as a limit of smooth solutions.

Definition 8.10. Let $f \in L^2(\Omega)$. A function $u \in L^2(\Omega)$ is a strong solution of (8.1) there exists a sequence of functions $u_n \in C^1(\overline{\Omega})$ such that $B_- u_n = 0$ on $\partial\Omega$ and $u_n \to u$, $Lu_n \to f$ in $L^2(\Omega)$ as $n \to \infty$.

In operator-theoretic terms, this definition says that u is a strong solution of (8.1) if the pair $(u, f) \in L^2(\Omega) \times L^2(\Omega)$ belongs to the closure of the graph of the operator

$$\begin{aligned} &L : D \subset L^2(\Omega) \to L^2(\Omega), \\ &D = \left\{u \in C^1(\overline{\Omega}) : B_- u = 0 \text{ on } \partial\Omega\right\}. \end{aligned}$$

If $\overline{D}$ is the domain of the closure, then

$$\overline{D} \supset \{u \in H^1(\Omega) : \gamma B_- u = 0\},$$

but, in general, it is difficult to give an explicit description of $\overline{D}$.

We will prove that a weak solution is a strong solution by mollifying the weak solution. In fact, Friedrichs [**12**] introduced mollifiers for exactly this purpose. The proof depends on the following lemma regarding the commutator of the differential operator L with a smoothing operator.

Let

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$$

denote the standard mollifier (η is a compactly supported, non-negative, radially symmetric C^∞-function with unit integral), and let

$$J_\epsilon : L^2(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \qquad J_\epsilon u = \eta_\epsilon * u \tag{8.13}$$

denote the associated smoothing operator.

LEMMA 8.11 (Friedrich). *Define $J_\epsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by (8.13) and $L : C_c^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by (8.3), where $A^i \in C_c^1(\mathbb{R}^n)$ and $C \in C_c(\mathbb{R}^n)$. Then the commutator*

$$[J_\epsilon, L] = J_\epsilon L - L J_\epsilon, \qquad [J_\epsilon, L] : C_c^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

extends to a bounded linear operator $\overline{[J_\epsilon, L]} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ whose norm is uniformly bounded in ϵ. Furthermore, for every $u \in L^2(\mathbb{R}^n)$

$$\overline{[J_\epsilon, L]}\, u \to 0 \qquad \text{in } L^2(\mathbb{R}^n) \text{ as } \epsilon \to 0^+.$$

PROOF. For $u \in C_c^1$, we have

$$\begin{aligned}
[J_\epsilon, L]\, u &= \eta_\epsilon * \left(A^i \partial_i u + Cu\right) - A^i \partial_i \left(\eta_\epsilon * u\right) - C\left(\eta_\epsilon * u\right) \\
&= \eta_\epsilon * \left(A^i \partial_i u\right) - A^i \left(\eta_\epsilon * \partial_i u\right) + \eta_\epsilon * (Cu) - C\left(\eta_\epsilon * u\right).
\end{aligned}$$

By standard properties of mollifiers, if $f \in L^2$ then $\eta_\epsilon * f \to f$ in L^2 as $\epsilon \to 0^+$, so $[J_\epsilon, L]\, u \to 0$ in L^2 when $u \in C_c^1$.

We may write the previous equation as

$$\begin{aligned}
[J_\epsilon, L]\, u(x) = \int \eta_\epsilon(x-y) \Big\{ &\left[A^i(y) - A^i(x)\right] \partial_i u(y) \\
&+ \left[C(y) - C(x)\right] u(y) \Big\}\, dy,
\end{aligned}$$

and an integration by parts gives

$$\begin{aligned}
[J_\epsilon, L]\, u(x) &= \int \partial_i \eta_\epsilon(x-y) \left[A^i(y) - A^i(x)\right] u(y)\, dy \\
&\quad + \int \eta_\epsilon(x-y) \left[C(y) - C(x) - \partial_i A^i(y)\right] u(y)\, dy.
\end{aligned} \tag{8.14}$$

The first term on the right-hand side of (8.14) is bounded uniformly in ϵ because the large factor $\partial_i \eta_\epsilon(x-y)$ is balanced by the factor $A^i(y) - A^i(x)$, which is small on the support of $\eta_\epsilon(x-y)$. To estimate this term, we use the Lipschitz continuity of A^i — with Lipschitz constant K^i, say — to get

$$\begin{aligned}
&\left| \int \partial_i \eta_\epsilon(x-y) \left[A^i(y) - A^i(x)\right] u(y)\, dy \right| \\
&\qquad \le K^i \int \left|\partial_i \eta_\epsilon(x-y)\right| |x-y|\, |u(y)|\, dy \\
&\qquad \le K^i \Big[\left(|x \partial_i \eta_\epsilon|\right) * |u| \Big](x).
\end{aligned}$$

Young's inequality implies that

$$\left\| \left(|x \partial_i \eta_\epsilon|\right) * |u| \right\|_{L^2} \le \left\| x \partial_i \eta_\epsilon \right\|_{L^1} \|u\|_{L^2},$$

and the L^1-norm

$$E_i = \left\| x \partial_i \eta_\epsilon \right\|_{L^1} = \frac{1}{\epsilon^{n+1}} \int |x| \left| \partial_i \eta\left(\frac{x}{\epsilon}\right) \right| dx = \int |x|\, \left|\partial_i \eta(x)\right|\, dx$$

is independent of ϵ. It follows that

$$\left\| \int \partial_i \eta_\epsilon(x-y) \left[A^i(y) - A^i(x)\right] u(y)\, dy \right\|_{L^2} \leq K \left\|u\right\|_{L^2}$$

where $K = E_i K^i$.

The second term on the right-hand side of (8.14) is straightforward to estimate:

$$\begin{aligned}\left| \int \eta_\epsilon(x-y) \left[C(y) - C(x) - \partial_i A^i(y)\right] u(y)\, dy \right| \\ &\leq M \int \eta_\epsilon(x-y) \left|u(y)\right|\, dy \\ &\leq M \big(\eta_\epsilon * |u|\big)(x)\end{aligned}$$

where $M = \sup\left\{2|C| + |\partial_i A^i|\right\}$ is a bound for the coefficient matrices (with $|\cdot|$ denoting the L^2-matrix norm). Young's inequality and the fact that $\|\eta_\epsilon\|_{L_1} = 1$ imply that

$$\begin{aligned}\left\| \int \eta_\epsilon(x-y) \left[C(y) - C(x) - \partial_i A^i(y)\right] u(y)\, dy \right\|_{L^2} \\ &\leq M \left\|\eta_\epsilon * |u|\right\|_{L^2} \\ &\leq M \|u\|_{L^2}.\end{aligned}$$

Thus, $[J_\epsilon, L]$ is bounded on the dense subset C_c^1 of L^2, so it extends uniquely to a linear operator on L^2 whose norm is bounded by $K + M$ independently of ϵ. Furthermore, since $[J_\epsilon, L]u \to 0$ as $\epsilon \to 0^+$ for all u in a dense subset of L^2, it follows that $\overline{[J_\epsilon, L]}u \to 0$ for all $u \in L^2$. □

Next, we prove the "weak equals strong" theorem.

THEOREM 8.12. *Suppose that the smoothness assumptions in Definition 8.1 are satisfied, $B = \nu_i A^i$ is nonsingular on $\partial\Omega$, and $f \in L^2(\Omega)$. Then a function $u \in L^2(\Omega)$ is a weak solution of (8.1) if and only if it is a strong solution.*

PROOF. Suppose u is a strong solution of (8.1), meaning that there is a sequence (u_n) of smooth solutions such that $u_n \to u$ and $Lu_n \to f$ in $L^2(\Omega)$ as $n \to \infty$. These solutions satisfy $(u_n, L^* v) = (Lu_n, v)$ for all $v \in D^*$, and taking the limit of this equation as $n \to \infty$, we get that $(u, L^* v) = (f, v)$ for all $v \in D^*$. This means that u is a weak solution.

To prove that a weak solution is a strong solution, we use a partition of unity to localize the problem and mollifiers to smooth the weak solution. In the interior of the domain, we use a standard mollifier. On the boundary, we make a change of coordinates to "flatten" the boundary and mollify only in the tangential directions to preserve the boundary condition. The smoothness of the mollified solution in the normal direction then follows from the PDE, since we can express the normal derivative of a solution in terms of the tangential derivatives if the boundary is non-characteristic.

In more detail, suppose that $u \in L^2(\Omega)$ is a weak solution of (8.1), meaning that $(u, L^* v) = (f, v)$ for all $v \in D^*$, where $(\cdot, \cdot)$ denotes the standard inner product on $L^2(\Omega)$ and D^* is defined in (8.10).

Let $\{U_j\}$ be a finite open cover of $\overline{\Omega}$ by interior or boundary coordinate patches U_j. An interior patch is compactly contained in Ω and diffeomorphic to a ball

$\{|x| < 1\}$; a boundary patch intersects Ω in a region that is diffeomorphic to a half-ball $\{x_1 > 0,\ |x| < 1\}$. Introduce a subordinate partition of unity $\{\phi_j\}$ with $\operatorname{supp}\phi_j \subset U_j$ and $\sum_j \phi_j = 1$ on $\overline{\Omega}$, and let

$$u = \sum_j u_j, \qquad u_j = \phi_j u.$$

We claim that $u \in L^2(\Omega)$ is a weak solution of $Lu = f$, with the boundary condition $B_- u = 0$, if and only if each $u_j \in L^2(\Omega)$ is a weak solution of

$$Lu_j = \phi_j f + [\partial_i \phi_j] A^i u, \tag{8.15}$$

with the same boundary condition. The right-hand side of (8.15) depends on u, but it belongs to $L^2(\Omega)$ since it involves no derivatives of u.

To verify this claim, suppose that $Lu = f$. Then, by use of the equations $(u, \phi v) = (\phi u, v)$ and $(u, L^* v) = (f, u)$, we get for all $v \in D^*$ that

$$\begin{aligned}(u_j, L^* v) &= (u, \phi_j L^* v)\\ &= (u, L^*[\phi_j v]) + \big(u, [\partial_i \phi_j] A^i v\big)\\ &= (f, \phi_j v) + \big(u, [\partial_i \phi_j] A^i v\big)\\ &= \big(\phi_j f + [\partial_i \phi_j] A^i u, v\big)\,,\end{aligned} \tag{8.16}$$

which shows that u_j is a weak solution of (8.15). Conversely, suppose that u_j a weak solution of (8.15). Then by summing (8.16) over j and using the equation $\sum_j [\partial_i \phi_j] = 0$, we find that $u = \sum_j u_j$ is a weak solution of $Lu = f$. Thus, to prove that a weak solution u is a strong solution it suffices to prove that each u_j is a strong solution. We may therefore assume without loss of generality that u is supported in an interior or boundary patch.

First, suppose that u is supported in an interior patch. Since $u \in L^2(\Omega)$ is compactly supported in Ω, we may extend u by zero on Ω^c and extend other functions to compactly supported functions on $\mathbb{R}^n$. Then $u_\epsilon = J_\epsilon u \in C_c^\infty$ is well-defined and, by standard properties of mollifiers, $u_\epsilon \to u$ in L^2 as $\epsilon \to 0^+$. We will show that $Lu_\epsilon \to Lu$ in L^2, which proves that u is a strong solution.

Using the self-adjointness of J_ϵ, we have for all $v \in D^*$ that

$$\begin{aligned}(u_\epsilon, L^* v) &= (u, J_\epsilon L^* v)\\ &= (u, L^* J_\epsilon v) + (u, [J_\epsilon, L^*] v)\\ &= (f, J_\epsilon v) + (u, [J_\epsilon, L^*] v)\\ &= (J_\epsilon f, v) + (u, [J_\epsilon, L^*] v)\,.\end{aligned}$$

Lemma 8.11, applied to L^*, implies that $[J_\epsilon, L^*]$ is bounded on L^2. Moreover, a density argument shows that its Hilbert-space adjoint is

$$[J_\epsilon, L^*]^* = -\overline{[J_\epsilon, L]}.$$

Thus,

$$(u_\epsilon, L^* v) = \Big(J_\epsilon f - \overline{[J_\epsilon, L]} u, v\Big) \qquad \text{for all } v \in D^*,$$

which means that u_ϵ is a weak solution of

$$Lu_\epsilon = f_\epsilon, \qquad f_\epsilon = J_\epsilon f - \overline{[J_\epsilon, L]} u.$$

Since u_ϵ is smooth, it is a classical solution that satisfies the boundary condition $B_- u_\epsilon = 0$ pointwise.. Lemma 8.11 and the properties of mollifiers imply that $f_\epsilon \to f$ in L^2 as $\epsilon \to 0^+$, which proves that u is a strong solution.

Second, suppose that u is supported in a boundary patch $\overline{\Omega} \cap U_j$. In this case, we obtain a smooth approximation by mollifying u in the tangential directions. The PDE then implies that u is smooth in the normal direction.

By making a C^2-change of the independent variable, we may assume without loss of generality that Ω is a half-space

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x^1 > 0\},$$

and u is compactly supported in $\overline{\mathbb{R}}^n_+$. We write

$$x = (x^1, x'), \qquad x' = (x^2, \dots, x^n) \in \mathbb{R}^{n-1}.$$

Since we assume that the boundary is non-characteristic, A^1 is nonsingular on $x^1 = 0$, in which case it is non-singular in a neighborhood of the boundary by continuity. Restricting the support of u appropriately, we may assume that A^1 is nonsingular everywhere, and multiplication of the PDE by the inverse matrix puts the equation $Lu = f$ in the form

$$\partial_1 u + L'u = f, \qquad L' = A^{i'} \partial_{i'} + C \qquad \text{in } x^1 > 0, \tag{8.17}$$

where the sum is taken over $2 \le i' \le n$, and the matrices $A^{i'}(x^1, x')$ need not be symmetric. The weak form of the equation transforms correspondingly under a smooth change of independent variable.

We may regard $u \in L^2(\mathbb{R}^n_+)$ equivalently as a vector-valued function of the normal variable $u \in L^2\left(\mathbb{R}_+; L^2\right)$ where $u : x^1 \mapsto u(x^1, \cdot)$, and we abbreviate the range space $L^2(\mathbb{R}^{n-1})$ of functions of the tangential variable x' to L^2. If $(\cdot,\cdot)'$ denotes the L^2-inner product with respect to $x' \in \mathbb{R}^{n-1}$, then the inner product on this space is the same as the $L^2(\mathbb{R}^n)$-inner product:

$$(u, v)_{L^2(\mathbb{R}_+; L^2)} = \int_{\mathbb{R}_+} (u, v)'\, dx^1 = (u, v).$$

We denote other spaces similarly. For example, $L^2\left(\mathbb{R}_+; H^1\right)$ consists of functions with square-integrable tangential derivatives, with inner product

$$(u, v)_{L^2(\mathbb{R}_+; H^1)} = \int_{\mathbb{R}_+} \left\{ (u, v)' + \sum_{i'=2}^{n} (\partial_{i'} u, \partial_{i'} v)' \right\} dx^1;$$

and $H^1\left(\mathbb{R}_+; L^2\right)$ consists of functions with square-integrable normal derivatives, with inner product

$$(u, v)_{H^1(\mathbb{R}_+; L^2)} = \int_{\mathbb{R}_+} \left\{ (u, v)' + (\partial_1 u, \partial_1 v)' \right\} dx^1.$$

In particular, $H^1(\mathbb{R}^n_+) = L^2\left(\mathbb{R}_+; H^1\right) \cap H^1\left(\mathbb{R}_+; L^2\right)$.

Let η'_ϵ be the standard mollifier with respect to $x' \in \mathbb{R}^{n-1}$, and define the associated tangential smoothing operator $J'_\epsilon : u \mapsto u_\epsilon$ by

$$u_\epsilon(x^1, x') = \int_{\mathbb{R}^{n-1}} \eta'_\epsilon(x' - y') u(x^1, y')\, dy'.$$

If $u \in L^2(\mathbb{R}^n_+)$, then $u_\epsilon \in L^2\left(\mathbb{R}_+; H^1\right)$. Fubini's theorem and standard properties of mollifiers imply that $u_\epsilon \to u$ in $L^2(\mathbb{R}^n_+)$ as $\epsilon \to 0^+$.

Mollifying the weak form of (8.17) in the tangential directions and using the fact that J'_ϵ commutes with ∂_1, we get — as in the interior case — that

$$\begin{aligned}(u_\epsilon, L^* v) &= \left(u_\epsilon, \left\{-\partial_1 + L'^*\right\} v\right)\\ &= \left(u, \left\{-\partial_1 + L'^*\right\} J'_\epsilon v\right) + \left(u, \left[J'_\epsilon, L'^*\right] v\right)\\ &= \left(J'_\epsilon f - \overline{[J'_\epsilon, L']} u, v\right),\end{aligned}$$

meaning that u_ϵ is a weak solution of

$$\partial_1 u_\epsilon + L' u_\epsilon = f_\epsilon, \qquad f_\epsilon = J'_\epsilon f - \overline{[J'_\epsilon, L']} u. \tag{8.18}$$

Lemma 8.11 applied to the tangential commutator implies that

$$\overline{[J'_\epsilon, L']} u \in L^2(\mathbb{R}_+; L^2)$$

and $f_\epsilon \to f$ in $L^2(\mathbb{R}_+; L^2)$. Moreover, (8.18) shows that $u_\epsilon \in H^1(\mathbb{R}_+; L^2)$. Thus, we have constructed $u_\epsilon \in H^1(\mathbb{R}^n_+)$ such that

$$u_\epsilon \to u, \quad L u_\epsilon \to L u \qquad \text{in } L^2(\mathbb{R}^n_+) \text{ as } \epsilon \to 0^+. \tag{8.19}$$

In view of (8.19), we just need to show that weak H^1-solutions are strong solutions.[1] By making a linear transformation of u, we can transform the boundary condition $B_- u = 0$ into

$$u_1 = u_2 = \cdots = u_r = 0,$$

where r is the dimension of $\ker B_-$. We decompose $u = u^+ + u^-$ where

$$u^+ = (u_1, \ldots, u_r, 0, \ldots, 0)^T, \qquad u^- = (0, \ldots, 0, u_{r+1}, \ldots, u_n)^T,$$

in which case the boundary condition is $u^+ = 0$ on $x^1 = 0$, with u^- arbitrary.

If $u \in H^1(\mathbb{R}^n_+)$ is a weak solution of $Lu = f$, then $\gamma u^+ = 0$, where γ is the trace map in (8.11). This condition implies that [**9**]

$$u^+ \in H^1_0(\mathbb{R}^n_+), \qquad H^1_0(\mathbb{R}^n_+) = \overline{C_c^\infty(\mathbb{R}^n_+)}.$$

Consequently, there exist $u^+_\epsilon \in C^1_c(\mathbb{R}^n_+)$ such that $u^+_\epsilon \to u^+$ in $H^1(\mathbb{R}^n_+)$ as $n \to \infty$. Since u^+_ϵ has compact support in $\mathbb{R}^n_+$, it satisfies the boundary condition pointwise. Furthermore, by density, there exist $u^-_\epsilon \in C^1_c(\overline{\mathbb{R}}^n_+)$ such that $u^-_\epsilon \to u^-$ in $H^1(\mathbb{R}^n_+)$. Let $u_\epsilon = u^+_\epsilon + u^-_\epsilon$. Then $u_\epsilon \in C^1_c(\overline{\mathbb{R}}^n_+)$, $B_- u_\epsilon = 0$, and $u_\epsilon \to u$ in $H^1(\mathbb{R}^n_+)$. Since $L : H^1(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+)$ is bounded, $u_\epsilon \to u$, $L u_\epsilon \to L u$ in $L^2(\mathbb{R}^n_+)$, which proves that u is a strong solution. □

If the boundary is not smooth, or the boundary matrix B is singular and the dimension of its null-space changes, then difficulties may arise with the tangential mollification near the boundary; in that case weak solutions might not be strong solutions e.g. see [**30**].

Note that Theorem 8.12 is based entirely on mollification and does not depend on any positivity or symmetry conditions

Corollary 8.13. *Let $f \in L^2(\Omega)$. If the smoothness conditions in Definition 8.1 and the positivity conditions in Definition 8.4, Definition 8.5 are satisfied, then a weak solution $u \in L^2(\Omega)$ of (8.1) is unique and* $c\|u\| \le \|f\|$.

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[1] If we had defined strong solutions equivalently as the limit of H^1-solutions instead of C^1-solutions, we wouldn't need this step.

PROOF. Let $u \in L^2(\Omega)$ be a weak solution of (8.1). By Theorem 8.12, there is a sequence (u_n) of smooth solutions $u_n \in C^1(\overline{\Omega})$ of (8.1) with $Lu_n = f_n$ such that $u_n \to u$ and $f_n \to f$ in L^2. Theorem 8.6 implies that $c\|u_n\| \leq \|f_n\|$ and, taking the limit of this inequality as $n \to \infty$, we get $c\|u\| \leq \|f\|$. In particular, $f = 0$ implies that $u = 0$, so a weak solution is unique. $\square$

A further issue is the regularity of weak solutions, which follows from energy estimates for their derivatives. As shown in Rauch [**33**] and the references cited there, if the boundary is non-characteristic, then the solution is as regular as the data allows: If A^i and $\partial\Omega$ are C^{k+1}, C is C^k, and $f \in H^k(\Omega)$, then $u \in H^k(\Omega)$.

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