Software for Exact Integration of Polynomials over Polyhedra: Appendix

J. A. De Loera[a], B. Dutra[a,*], M. Köppe[a], S. Moreinis[a,1], G. Pinto[a], J. Wu[a,2]

[a]*Department of Mathematics, UC Davis, One Shields Avenue, Davis, CA 95616, USA*

List of Tables

List of Figures

*Corresponding author.

Email addresses: deloera@math.ucdavis.edu (J. A. De Loera), bedutra@ucdavis.edu (B. Dutra), mkoeppe@math.ucdavis.edu (M. Köppe), moreinis@stanford.edu (S. Moreinis), gpinto@ucdavis.edu (G. Pinto), jwu2@caltech.edu (J. Wu)

[1]Present address: Stanford University, Stanford, California 94305

[2]Present address: California Institute of Technology, Pasadena, California 91125

Appendix A. Integration

In Tables A.1, A.3, A.5, A.7, and A.9 we display the average running integration time (in seconds) for the triangulation and cone decomposition methods. The column called "Vert." denotes the number of vertices the primal polytope has. Each vertex-count row is divided into two rows: the top row is the average running time for the primal polytopes (simplicial), and the bottom row is the average running time for the dual polytopes (simple). For example, consider Table A.1 and locate the column corresponding to the triangulation method for monomials of degree 10 in the first row (row 8). There are two numbers 0.15 and 2.18. This means that the average integration time using the triangulation method to integrate random degree 10 monomials over simplicial polytopes with exactly 8 vertices in dimension 3 took 0.15 seconds; and the average integration time using the triangulation method to integrate the same monomials over the dual polytopes (which are simple) in dimension 3 took 2.18 seconds. Tables A.2, A.4, A.6, A.8, and A.10 display the standard deviation in seconds.

Tables A.11 and A.12 display the average integration time when integrating over polytopes in Ziegler's database, while Tables A.13 and A.14 display the standard deviation.

Figures A.1 and A.2 display a histogram of relative running times for integrating over the random polytopes and their duals. Instead of displaying total running times for both integration methods individually, we plot the time for the triangulation method minus the time for the cone decomposition method all divided by the time of the triangulation method. Thus, a positive relative time difference means that the cone decomposition method was faster than the triangulation method.

Table A.1: Average time of 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 3

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
5	0.04	**0.02**	0.04	**0.02**	0.04	**0.02**	0.05	**0.02**	0.08	**0.05**	0.16	**0.13**	0.31	**0.31**	**0.58**	0.72
	0.03	0.04	**0.03**	0.04	**0.03**	0.04	**0.03**	0.04	**0.06**	0.11	**0.13**	0.28	**0.30**	0.75	**0.62**	1.74
6	0.07	**0.03**	0.07	**0.03**	0.07	**0.03**	0.07	**0.03**	0.12	**0.07**	0.21	**0.21**	**0.44**	0.54	**0.77**	1.21
	0.04	0.06	**0.04**	0.06	**0.04**	0.06	**0.05**	0.07	**0.08**	0.19	**0.20**	0.59	**0.44**	1.67	**0.87**	3.88
7	0.09	**0.03**	0.09	**0.04**	0.09	**0.04**	0.09	**0.04**	0.14	**0.10**	**0.27**	0.30	**0.55**	0.74	**0.99**	1.62
	0.07	0.09	**0.07**	0.09	**0.07**	0.10	**0.08**	0.11	**0.13**	0.31	**0.26**	1.01	**0.57**	2.83	**1.26**	6.64
8	0.09	**0.04**	0.10	**0.04**	0.09	**0.04**	0.10	**0.05**	0.17	**0.12**	**0.35**	0.36	**0.66**	0.97	**1.20**	2.28
	0.10	0.12	**0.10**	0.12	**0.10**	0.14	**0.11**	0.16	**0.17**	0.45	**0.34**	1.51	**0.80**	4.39	**1.51**	10.70
13	0.18	**0.09**	0.19	**0.08**	0.19	**0.09**	0.21	**0.11**	0.33	**0.28**	**0.70**	0.84	**1.30**	2.26	**2.37**	4.93
	0.41	0.49	**0.41**	0.51	**0.41**	0.51	**0.43**	0.61	**0.55**	1.73	**0.96**	6.07	**2.00**	18.32	**4.05**	45.34
18	0.34	**0.17**	0.30	**0.17**	0.36	**0.18**	0.39	**0.20**	0.56	**0.51**	**0.99**	1.46	**2.10**	3.70	**3.65**	8.23
	0.98	1.21	**0.99**	1.17	**0.99**	1.23	**1.08**	1.45	**1.25**	4.09	**2.05**	15.22	**4.21**	47.48	**8.66**	122.18
23	0.46	**0.29**	0.47	**0.30**	0.47	**0.29**	0.51	**0.34**	0.77	**0.72**	**1.41**	1.99	**2.70**	5.06	**4.96**	11.22
	1.76	2.13	**1.76**	2.12	**1.81**	2.21	**1.82**	2.60	**2.16**	7.46	**3.53**	28.72	**7.32**	90.03	**14.73**	235.44
28	0.64	**0.44**	0.61	**0.46**	0.64	**0.45**	0.68	**0.51**	1.00	**1.00**	**1.86**	2.66	**3.46**	6.65	**6.24**	14.77
	2.75	3.41	**2.82**	3.40	**2.76**	3.48	**2.89**	4.14	**3.47**	12.13	**5.52**	47.70	**11.12**	153.37	**22.93**	410.15

Table A.2: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 3

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Law.	Tri.
5	0.01	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.03	0.03	0.06	0.04	0.16
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.04	0.03	0.09	0.05	0.21
6	0.03	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.03	0.01	0.03	0.04	0.04	0.10	0.04	0.19
	0.01	0.02	0.01	0.02	0.01	0.01	0.01	0.02	0.01	0.03	0.02	0.08	0.05	0.29	0.12	0.55
7	0.03	0.01	0.03	0.01	0.02	0.01	0.02	0.01	0.02	0.02	0.03	0.04	0.05	0.12	0.05	0.26
	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.04	0.03	0.16	0.07	0.56	0.16	0.98
8	0.02	0.01	0.02	0.01	0.02	0.01	0.03	0.01	0.03	0.02	0.04	0.06	0.07	0.17	0.06	0.34
	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.06	0.04	0.21	0.13	0.66	0.13	1.32
13	0.03	0.01	0.03	0.01	0.04	0.01	0.03	0.01	0.04	0.04	0.11	0.12	0.11	0.37	0.11	0.59
	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.12	0.11	0.26	0.13	0.95	0.23	2.93	0.41	6.68
18	0.05	0.01	0.05	0.01	0.06	0.01	0.05	0.01	0.06	0.06	0.08	0.18	0.20	0.45	0.19	0.90
	0.21	0.22	0.21	0.21	0.22	0.21	0.24	0.22	0.22	0.57	0.25	2.09	0.46	6.57	0.73	16.30
23	0.04	0.01	0.05	0.01	0.05	0.01	0.06	0.02	0.07	0.07	0.10	0.21	0.21	0.68	0.22	1.10
	0.38	0.38	0.38	0.38	0.38	0.39	0.39	0.42	0.38	1.08	0.47	4.50	0.86	14.56	1.43	32.44
28	0.06	0.02	0.06	0.02	0.05	0.02	0.06	0.02	0.06	0.09	0.11	0.28	0.27	0.87	0.28	1.60
	0.68	0.70	0.69	0.69	0.68	0.69	0.70	0.79	0.71	1.96	0.83	8.00	1.52	27.06	2.90	75.80

Table A.3: Average time of 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 4

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
6	0.07	**0.04**	0.08	**0.04**	0.08	**0.03**	0.10	**0.04**	0.29	**0.16**	0.98	**0.69**	2.68	**2.36**	6.62	**6.29**
	0.11	0.13	**0.11**	0.13	**0.12**	0.14	**0.13**	0.18	**0.36**	0.80	**1.29**	4.02	**4.03**	14.76	**10.68**	43.32
7	0.12	**0.04**	0.13	**0.05**	0.13	**0.05**	0.15	**0.06**	0.46	**0.28**	1.73	**1.31**	4.61	**4.39**	**10.99**	11.60
	0.34	0.38	**0.35**	0.38	**0.35**	0.40	**0.37**	0.53	**0.73**	2.60	**2.36**	13.66	**6.98**	51.11	**18.65**	158.32
8	0.13	**0.05**	0.14	**0.05**	0.14	**0.06**	0.19	**0.08**	0.69	**0.40**	2.38	**1.98**	6.76	**6.49**	**15.89**	18.57
	0.70	0.80	**0.71**	0.79	**0.71**	0.82	**0.75**	1.09	**1.27**	5.80	**3.65**	31.49	**10.88**	124.72	**29.27**	394.53
9	0.19	**0.07**	0.16	**0.07**	0.17	**0.07**	0.22	**0.11**	0.92	**0.61**	3.28	**2.81**	**9.23**	9.73	**21.68**	27.33
	1.47	1.66	**1.46**	1.66	**1.49**	1.70	**1.56**	2.26	**2.28**	12.20	**5.85**	68.57	**17.05**	268.34	**45.80**	–
14	0.38	**0.18**	0.34	**0.18**	0.39	**0.19**	0.56	**0.30**	2.13	**1.69**	8.10	**8.04**	**23.11**	28.07	**55.94**	78.46
	9.61	10.97	**9.59**	11.03	**9.66**	11.39	**9.81**	15.53	**12.41**	95.81	**25.96**	–	**70.93**	–	**198.48**	–
19	0.57	**0.35**	0.58	**0.36**	0.64	**0.38**	0.86	**0.56**	3.64	**3.00**	14.59	**14.31**	**40.58**	49.09	**95.83**	134.31
	74.55	78.63	**74.10**	78.81	**74.71**	79.86	**74.78**	93.95	**81.77**	387.66	**116.60**	–	**234.77**	–	–	–
24	0.83	**0.60**	0.84	**0.60**	0.91	**0.63**	1.27	**0.93**	5.47	**4.70**	**20.62**	21.58	**59.77**	73.75	147.65	**0.00**
	191.14	200.77	**190.86**	200.56	**190.93**	205.28	**191.08**	239.63	**201.51**	–	**271.25**	–	**504.48**	–	–	–
29	1.19	**0.91**	1.15	**0.94**	1.22	**0.99**	1.70	**1.33**	7.42	**6.23**	**29.06**	29.25	**82.89**	97.86	**204.66**	269.14
	389.49	411.85	**388.78**	409.79	**390.25**	412.63	**390.91**	485.34	**408.69**	–	–	–	–	–	–	–

Table A.4: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 4

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
6	0.02	0.01	0.03	0.01	0.02	0.01	0.03	0.01	0.05	0.03	0.16	0.16	0.44	0.52	0.76	1.32
	0.03	0.03	0.03	0.04	0.03	0.04	0.04	0.04	0.06	0.21	0.21	1.05	0.66	3.98	1.34	10.29
7	0.02	0.01	0.03	0.01	0.02	0.01	0.03	0.01	0.08	0.06	0.28	0.29	0.66	0.96	1.46	2.38
	0.10	0.11	0.10	0.10	0.10	0.11	0.10	0.13	0.14	0.81	0.42	4.54	1.15	14.54	3.46	49.45
8	0.03	0.01	0.03	0.01	0.03	0.01	0.04	0.02	0.11	0.10	0.40	0.43	1.04	1.63	1.80	3.03
	0.23	0.24	0.23	0.24	0.23	0.24	0.23	0.28	0.29	1.46	0.64	8.31	1.97	32.45	4.55	105.30
9	0.04	0.01	0.04	0.01	0.04	0.02	0.04	0.02	0.14	0.13	0.56	0.64	1.22	1.72	2.45	5.49
	0.48	0.50	0.48	0.51	0.49	0.51	0.49	0.56	0.51	2.97	1.12	17.93	2.43	52.08	6.93	–
14	0.05	0.01	0.04	0.01	0.05	0.02	0.07	0.04	0.29	0.31	1.03	1.47	3.21	6.14	4.90	13.50
	4.93	4.97	4.91	4.97	4.92	5.00	4.93	5.70	5.04	22.33	6.16	–	13.72	–	25.28	–
19	0.05	0.01	0.05	0.01	0.06	0.02	0.11	0.06	0.55	0.53	2.04	2.81	5.11	9.79	9.58	21.71
	13.71	13.94	13.61	13.95	13.75	14.17	13.79	16.38	14.44	71.10	17.72	–	32.32	–	–	–
24	0.05	0.02	0.05	0.02	0.06	0.02	0.13	0.10	0.84	0.91	3.01	3.95	8.38	11.51	18.00	–
	26.76	27.64	26.57	27.54	26.72	28.44	26.73	34.93	28.04	–	37.61	–	66.73	–	–	–
29	0.06	0.02	0.06	0.02	0.07	0.04	0.17	0.12	1.13	1.09	4.06	5.10	11.70	16.21	22.31	38.30
	54.92	58.10	55.22	57.59	55.48	57.98	54.98	64.07	57.42	–	–	–	–	–	–	–

Table A.5: Average time of 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 5

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
7	0.05	**0.01**	0.12	**0.06**	0.11	**0.06**	0.18	**0.08**	1.03	**0.47**	5.05	**2.87**	20.78	**13.69**	61.27	**42.78**
	0.01	0.05	**0.02**	0.05	**0.02**	0.08	**0.11**	0.39	**2.39**	9.30	**20.37**	84.93	**109.08**	469.74	**419.73**	–
8	0.15	**0.08**	0.15	**0.08**	0.18	**0.08**	0.27	**0.13**	2.00	**1.00**	10.56	**6.56**	40.66	**30.40**	119.09	**94.22**
	0.02	0.25	**0.02**	0.27	**0.03**	0.41	**0.20**	2.21	**4.95**	59.68	**43.92**	–	**243.10**	–	–	–
9	0.19	**0.10**	0.19	**0.10**	0.24	**0.11**	0.38	**0.17**	3.09	**1.53**	18.17	**10.95**	66.55	**52.42**	197.45	**159.99**
	0.02	0.89	**0.02**	0.93	**0.04**	1.37	**0.33**	7.26	**8.45**	195.12	**77.70**	–	**439.60**	–	–	–
10	0.26	**0.12**	0.24	**0.12**	0.29	**0.13**	0.54	**0.25**	4.52	**2.45**	26.39	**16.80**	98.00	**74.71**	287.56	**244.13**
	0.02	2.36	**0.03**	2.45	**0.06**	3.60	**0.50**	18.50	**13.76**	–	**130.29**	–	–	–	–	–
15	0.70	**0.33**	0.70	**0.34**	0.82	**0.38**	1.72	**0.77**	16.33	**8.71**	93.47	**62.02**	384.07	**266.21**	–	–
	0.04	52.77	**0.06**	54.29	**0.21**	76.42	**2.70**	389.61	**79.04**	–	–	–	–	–	–	–
20	1.36	**0.74**	1.38	**0.76**	1.61	**0.82**	3.70	**1.58**	36.48	**16.73**	197.10	**119.15**	–	–	–	–
	0.08	284.72	**0.13**	290.09	**0.58**	398.24	**8.33**	–	**247.13**	–	–	–	–	–	–	–
25	2.18	**1.33**	2.21	**1.32**	2.64	**1.47**	5.51	**2.73**	54.61	**28.62**	308.82	**197.47**	–	–	–	–
	0.13	–	**0.23**	–	**1.20**	–	**18.77**	–	–	–	–	–	–	–	–	–
30	3.34	**2.15**	3.38	**2.17**	3.85	**2.38**	7.96	**4.09**	78.68	**42.18**	**453.28**	–	–	–	–	–
	0.20	–	**0.38**	–	**2.19**	–	**35.63**	–	–	–	–	–	–	–	–	–

Table A.6: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 5

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
7	0.00	0.00	0.02	0.01	0.02	0.01	0.03	0.01	0.26	0.15	1.11	0.81	3.62	3.69	9.57	9.96
	0.01	0.02	0.01	0.02	0.01	0.02	0.04	0.16	0.65	3.42	5.04	29.64	24.76	156.59	88.58	–
8	0.02	0.01	0.02	0.01	0.03	0.01	0.05	0.03	0.52	0.30	2.60	2.11	6.73	7.92	20.02	22.51
	0.01	0.07	0.01	0.07	0.01	0.10	0.06	0.77	1.42	22.48	10.73	–	48.05	–	–	–
9	0.04	0.01	0.03	0.02	0.03	0.01	0.07	0.03	0.84	0.49	4.35	3.22	13.49	15.16	30.70	33.57
	0.01	0.23	0.01	0.25	0.01	0.40	0.11	2.63	2.54	73.71	19.37	–	93.59	–	–	–
10	0.03	0.01	0.03	0.01	0.04	0.01	0.10	0.05	1.21	0.81	6.08	4.97	18.65	19.81	47.07	55.73
	0.01	0.58	0.01	0.59	0.02	0.92	0.17	7.21	4.62	–	36.97	–	–	–	–	–
15	0.05	0.02	0.06	0.02	0.08	0.02	0.40	0.17	4.26	2.55	24.25	16.11	75.35	61.92	–	–
	0.01	12.28	0.01	12.79	0.06	21.05	1.08	162.90	25.19	–	–	–	–	–	–	–
20	0.10	0.03	0.08	0.04	0.12	0.05	0.78	0.29	9.73	4.85	44.80	33.29	–	–	–	–
	0.01	57.61	0.02	59.66	0.15	75.42	2.89	–	74.46	–	–	–	–	–	–	–
25	0.11	0.05	0.12	0.06	0.19	0.07	1.08	0.50	13.97	8.34	71.04	50.42	–	–	–	–
	0.01	–	0.04	–	0.34	–	6.35	–	–	–	–	–	–	–	–	–
30	0.15	0.08	0.16	0.08	0.25	0.10	1.50	0.66	19.91	11.78	106.13	–	–	–	–	–
	0.02	–	0.08	–	0.65	–	11.67	–	–	–	–	–	–	–	–	–

Table A.7: Average time of 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 6

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
8	0.19	**0.10**	0.18	**0.10**	0.19	**0.10**	0.31	**0.15**	2.89	**1.25**	23.37	**11.11**	107.58	**60.10**	383.18	**240.80**
	0.02	0.15	**0.02**	0.16	**0.04**	0.27	**0.30**	2.18	**9.77**	77.23	**118.28**	–	–	–	–	–
9	0.23	**0.13**	0.23	**0.13**	0.26	**0.15**	0.55	**0.24**	6.72	**2.98**	52.82	**26.33**	243.92	**140.31**	–	–
	0.02	1.67	**0.02**	1.78	**0.07**	2.84	**0.67**	20.69	**24.31**	–	**306.31**	–	–	–	–	–
10	0.35	**0.17**	0.34	**0.18**	0.40	**0.18**	0.92	**0.39**	12.82	**5.47**	99.10	**48.93**	454.53	**271.30**	–	–
	0.03	9.45	**0.04**	9.91	**0.12**	15.26	**1.29**	107.74	**50.84**	–	–	–	–	–	–	–
11	0.40	**0.21**	0.39	**0.22**	0.49	**0.24**	1.40	**0.56**	20.04	**8.84**	166.73	**84.00**	–	–	–	–
	0.03	36.63	**0.05**	38.91	**0.19**	57.41	**2.38**	399.21	**101.11**	–	–	–	–	–	–	–
16	1.76	**0.63**	1.83	**0.64**	2.22	**0.78**	6.56	**2.15**	101.65	**39.95**	–	–	–	–	–	–
	0.15	–	**0.24**	–	**1.78**	–	**22.65**	–	–	–	–	–	–	–	–	–
21	4.48	**1.58**	4.59	**1.60**	5.47	**1.84**	15.14	**4.91**	232.43	**89.89**	–	–	–	–	–	–
	0.41	–	**0.73**	–	**7.21**	–	**91.92**	–	–	–	–	–	–	–	–	–
26	8.84	**3.25**	9.11	**3.31**	10.58	**3.76**	27.37	**9.21**	–	–	–	–	–	–	–	–
	0.97	–	**1.87**	–	**20.86**	–	**261.13**	–	–	–	–	–	–	–	–	–
31	14.57	**5.83**	15.05	**5.90**	17.27	**6.71**	43.07	**14.31**	–	–	–	–	–	–	–	–
	1.89	–	**3.84**	–	**45.75**	–	–	–	–	–	–	–	–	–	–	–

Table A.8: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 6

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Law.	Tri.	Law.	Tri.	Law.	Tri.	Law.	Tri.	Law.	Tri.	Law.	Tri.	Law.	Tri.	Law.	Tri.
8	0.04	0.01	0.02	0.01	0.02	0.01	0.06	0.02	0.85	0.41	6.07	3.25	22.13	15.84	53.30	54.69
	0.01	0.03	0.01	0.03	0.01	0.07	0.10	0.82	3.11	29.38	29.75	–	–	–	–	–
9	0.03	0.01	0.03	0.01	0.03	0.01	0.12	0.04	2.02	0.88	13.35	8.48	52.93	42.35	–	–
	0.01	0.49	0.01	0.53	0.02	0.95	0.22	8.88	8.43	–	82.34	–	–	–	–	–
10	0.03	0.02	0.04	0.02	0.05	0.02	0.20	0.09	3.97	2.04	24.19	15.84	90.54	80.24	–	–
	0.01	2.42	0.01	2.61	0.03	4.43	0.45	45.86	18.46	–	–	–	–	–	–	–
11	0.04	0.01	0.04	0.02	0.06	0.02	0.35	0.15	6.39	3.03	45.21	30.28	–	–	–	–
	0.01	8.60	0.01	9.02	0.06	14.68	0.76	148.63	35.40	–	–	–	–	–	–	–
16	0.14	0.03	0.14	0.03	0.23	0.05	1.74	0.58	31.65	13.27	–	–	–	–	–	–
	0.02	–	0.05	–	0.62	–	8.62	–	–	–	–	–	–	–	–	–
21	0.39	0.07	0.39	0.08	0.58	0.14	3.55	1.13	65.29	30.35	–	–	–	–	–	–
	0.06	–	0.19	–	2.67	–	35.45	–	–	–	–	–	–	–	–	–
26	0.76	0.18	0.79	0.17	1.02	0.29	6.30	2.25	–	–	–	–	–	–	–	–
	0.13	–	0.51	–	7.67	–	91.11	–	–	–	–	–	–	–	–	–
31	1.05	0.32	1.11	0.38	1.45	0.49	9.73	2.98	–	–	–	–	–	–	–	–
	0.22	–	1.03	–	16.43	–	–	–	–	–	–	–	–	–	–	–

Table A.9: Average time of 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 7

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
9	0.27	**0.17**	0.27	**0.18**	0.30	**0.19**	0.57	**0.26**	8.81	**2.79**	93.17	**32.27**	–	–	–	–
	0.02	0.54	**0.03**	0.58	**0.07**	1.11	**0.74**	10.80	**34.18**	–	–	–	–	–	–	–
10	0.41	**0.22**	0.41	**0.21**	0.46	**0.24**	1.19	**0.43**	22.67	**7.41**	244.00	**91.76**	–	–	–	–
	0.04	11.66	**0.05**	12.43	**0.16**	21.47	**2.06**	190.98	**106.43**	–	–	–	–	–	–	–
11	0.57	**0.29**	0.57	**0.28**	0.71	**0.31**	2.21	**0.71**	49.52	**14.52**	–	–	–	–	–	–
	0.05	100.04	**0.08**	105.87	**0.33**	172.43	**5.05**	–	**282.41**	–	–	–	–	–	–	–
12	0.82	**0.37**	0.84	**0.37**	1.05	**0.41**	3.94	**1.11**	85.45	**25.33**	–	–	–	–	–	–
	0.09	–	**0.14**	–	**0.71**	–	**12.08**	–	–	–	–	–	–	–	–	–
17	4.55	**1.21**	4.65	**1.25**	5.90	**1.55**	20.88	**5.47**	–	–	–	–	–	–	–	–
	0.59	–	**1.55**	–	**11.41**	–	**239.05**	–	–	–	–	–	–	–	–	–
22	15.57	**2.77**	15.92	**2.78**	19.20	**3.68**	59.77	**13.33**	–	–	–	–	–	–	–	–
	2.80	–	**8.85**	–	**69.71**	–	–	–	–	–	–	–	–	–	–	–
27	36.38	**10.59**	37.29	**12.15**	43.45	**11.66**	124.57	**33.48**	–	–	–	–	–	–	–	–
	8.93	–	**30.61**	–	**243.60**	–	–	–	–	–	–	–	–	–	–	–
32	71.84	**40.74**	73.22	**40.92**	83.16	**42.47**	212.32	**74.26**	–	–	–	–	–	–	–	–
	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–

Table A.10: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when integrating over random simplicial polytopes and their duals in dimension 7

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Vert.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
9	0.02	0.01	0.02	0.01	0.03	0.02	0.12	0.04	3.27	1.06	30.54	11.66	–	–	–	–
	0.01	0.13	0.01	0.14	0.02	0.33	0.32	5.40	13.06	–	–	–	–	–	–	–
10	0.03	0.01	0.03	0.01	0.05	0.02	0.35	0.10	8.90	2.92	78.30	31.76	–	–	–	–
	0.01	3.51	0.01	3.66	0.05	7.71	0.91	101.07	42.87	–	–	–	–	–	–	–
11	0.04	0.02	0.05	0.02	0.07	0.02	0.66	0.22	19.16	5.82	–	–	–	–	–	–
	0.01	28.45	0.01	30.27	0.10	53.92	2.07	–	106.40	–	–	–	–	–	–	–
12	0.05	0.02	0.04	0.02	0.12	0.03	1.30	0.36	30.26	9.60	–	–	–	–	–	–
	0.02	–	0.03	–	0.27	–	5.23	–	–	–	–	–	–	–	–	–
17	0.29	0.06	0.24	0.06	0.66	0.15	6.88	1.88	–	–	–	–	–	–	–	–
	0.10	–	0.42	–	4.44	–	107.21	–	–	–	–	–	–	–	–	–
22	1.04	0.22	1.14	0.25	2.15	0.48	18.10	4.75	–	–	–	–	–	–	–	–
	0.33	–	2.37	–	27.79	–	–	–	–	–	–	–	–	–	–	–
27	2.82	2.32	2.97	2.54	3.77	2.38	36.56	10.00	–	–	–	–	–	–	–	–
	1.30	–	8.83	–	83.61	–	–	–	–	–	–	–	–	–	–	–
32	5.82	8.73	6.31	9.48	8.66	10.25	54.87	19.02	–	–	–	–	–	–	–	–
	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–

Table A.11: Average time of 50 random trials using the triangulation and cone decomposition method when integrating over polytopes in Ziegler's Database and their duals: Part I

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Polyt.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
CUT3_3-4	0.01	**0.01**	0.01	**0.01**	0.01	**0.01**	0.01	**0.01**	0.03	**0.02**	0.07	**0.05**	0.16	**0.12**	0.27	**0.25**
	0.01	0.01	0.01	**0.01**	0.01	**0.01**	0.01	**0.01**	0.03	**0.02**	0.09	**0.06**	0.20	**0.13**	0.33	**0.28**
HC_3-4	0.01	**0.01**	**0.01**	0.01	0.01	**0.01**	0.01	**0.01**	0.03	**0.02**	0.06	**0.05**	0.14	**0.12**	0.27	**0.26**
	0.01	0.01	0.01	0.01	**0.01**	0.01	**0.01**	0.01	0.03	**0.02**	0.07	**0.06**	0.16	**0.13**	0.33	**0.29**
CRO_3-6	0.06	**0.02**	0.06	**0.03**	0.07	**0.03**	0.08	**0.03**	0.14	**0.08**	0.26	**0.20**	**0.49**	0.51	**0.86**	1.07
	0.01	0.02	**0.01**	0.02	**0.01**	0.02	**0.01**	0.03	**0.04**	0.08	**0.10**	0.24	**0.22**	0.64	**0.44**	1.44
Sharir	**0.03**	0.04	**0.03**	0.04	**0.03**	0.04	**0.04**	0.05	**0.06**	0.12	**0.13**	0.37	**0.29**	0.93	**0.56**	2.11
	0.09	**0.02**	0.09	**0.02**	0.07	**0.02**	0.11	**0.04**	**0.23**	0.24	**0.75**	1.06	**2.09**	3.35	**4.98**	8.74
CF_4-5	0.01	**0.01**	**0.02**	0.02	0.02	**0.02**	0.03	**0.02**	0.12	**0.07**	0.43	**0.26**	1.14	**0.80**	2.74	**2.17**
	0.01	**0.01**	0.01	**0.01**	0.01	**0.01**	0.02	**0.02**	0.12	**0.07**	0.46	**0.29**	1.32	**0.92**	3.40	**2.56**
BIR3_4-6	0.06	**0.03**	0.07	**0.03**	0.08	**0.03**	0.13	**0.04**	0.43	**0.22**	1.45	**0.85**	3.86	**2.65**	8.79	**6.78**
	0.01	0.02	**0.01**	0.02	**0.01**	0.02	**0.03**	0.04	**0.18**	0.32	**0.72**	1.39	**1.98**	4.57	**4.64**	12.68
cyclic_4_8	0.13	**0.05**	0.14	**0.05**	0.14	**0.05**	0.19	**0.08**	0.69	**0.46**	2.47	**2.14**	**7.11**	7.84	**16.93**	20.61
	0.01	0.08	**0.01**	0.09	**0.02**	0.10	**0.04**	0.22	**0.34**	1.84	**1.52**	9.92	**4.42**	32.93	**10.97**	93.66
neig._4_8	0.13	**0.05**	0.12	**0.05**	0.15	**0.05**	0.21	**0.09**	0.75	**0.49**	2.74	**2.11**	7.18	**6.95**	**16.37**	18.46
	0.02	0.07	**0.01**	0.07	**0.02**	0.09	**0.04**	0.21	**0.32**	1.86	**1.39**	9.07	**4.16**	32.62	**10.54**	87.88
CRO_4-8	0.12	**0.04**	0.12	**0.05**	0.12	**0.05**	0.20	**0.09**	0.79	**0.48**	2.65	**1.92**	6.71	**6.27**	**14.93**	15.93
	0.01	0.04	**0.01**	0.04	**0.01**	0.05	**0.05**	0.15	**0.33**	1.19	**1.31**	5.56	**3.69**	18.29	**8.68**	49.37
HC_4-8	0.10	**0.04**	0.12	**0.05**	0.12	**0.05**	0.19	**0.09**	0.76	**0.47**	2.52	**1.95**	6.65	**6.18**	**14.86**	16.05
	0.01	0.04	**0.01**	0.04	**0.01**	0.05	**0.04**	0.15	**0.33**	1.19	**1.30**	5.50	**3.68**	17.48	**8.64**	46.90
CF_5-6	**0.03**	0.03	**0.03**	0.03	0.03	**0.03**	0.05	**0.04**	0.36	**0.18**	1.83	**0.96**	6.68	**3.86**	18.78	**12.63**
	0.01	0.01	**0.01**	0.01	0.02	**0.01**	0.04	**0.02**	0.43	**0.20**	2.38	**1.14**	8.84	**4.85**	24.65	**15.04**
CNG_5-6a	0.03	**0.03**	0.03	**0.03**	0.04	**0.03**	0.07	**0.04**	0.50	**0.22**	2.46	**1.13**	8.89	**4.62**	24.99	**14.11**
	0.01	**0.01**	**0.01**	0.01	0.01	**0.01**	0.04	**0.02**	0.41	**0.19**	2.23	**1.08**	8.32	**4.64**	23.19	**14.29**
CNG_5-6b	0.03	**0.03**	0.03	**0.03**	0.04	**0.03**	0.08	**0.04**	0.56	**0.25**	2.77	**1.30**	10.04	**5.29**	26.89	**15.95**
	0.01	**0.01**	0.01	**0.01**	0.01	**0.01**	0.04	**0.02**	0.40	**0.19**	2.15	**1.06**	8.16	**4.50**	22.67	**14.28**
EQU_5-7a	0.09	**0.04**	0.09	**0.05**	0.09	**0.05**	0.15	**0.07**	1.17	**0.47**	5.78	**2.91**	20.70	**13.20**	60.10	**36.75**
	0.01	0.02	**0.01**	0.03	**0.02**	0.03	0.08	**0.08**	**0.97**	0.99	**5.06**	6.07	**19.08**	24.79	**51.44**	78.54
EQU_5-7b	0.07	**0.05**	0.07	**0.04**	0.09	**0.05**	0.17	**0.08**	1.31	**0.54**	6.28	**3.42**	22.97	**14.56**	61.79	**40.46**
	0.01	0.02	**0.01**	0.02	**0.02**	0.03	**0.08**	0.09	**0.92**	1.00	**4.78**	5.99	**18.05**	24.75	**48.18**	76.85
CYC_5-8	0.12	**0.06**	0.12	**0.06**	0.16	**0.07**	0.36	**0.15**	2.87	**1.33**	14.46	**7.44**	48.30	**31.94**	131.47	**92.46**
	0.01	0.09	**0.01**	0.09	**0.02**	0.12	**0.08**	0.36	**1.06**	5.19	**6.33**	35.46	**23.34**	160.99	**67.77**	518.99
OA_5-10	0.20	**0.09**	0.21	**0.10**	0.28	**0.12**	0.78	**0.31**	6.80	**3.35**	32.05	**18.63**	123.58	**75.14**	305.59	**233.52**
	0.12	**0.12**	0.13	**0.12**	**0.17**	0.19	**0.55**	0.73	**5.41**	10.30	**27.98**	60.28	**96.47**	257.16	**265.99**	–
CRO_5-10	0.18	**0.08**	0.20	**0.09**	0.26	**0.11**	0.62	**0.27**	5.67	**2.83**	29.40	**16.53**	103.78	**65.95**	281.22	**210.45**
	0.02	0.34	**0.02**	0.36	**0.03**	0.46	**0.13**	1.40	**1.70**	17.82	**9.53**	110.11	**35.86**	492.79	**101.28**	–
EG_5-12	0.28	**0.14**	0.28	**0.15**	0.41	**0.19**	1.19	**0.54**	10.74	**6.36**	53.37	**34.49**	182.85	**143.71**	485.90	**453.55**
	0.13	0.84	**0.11**	0.86	**0.15**	1.03	**0.43**	2.63	**4.62**	30.66	**24.76**	195.17	**92.42**	–	**259.01**	–
HC_5-16	0.40	**0.07**	0.44	**0.07**	0.56	**0.13**	1.58	**0.69**	14.05	**9.59**	66.97	**54.12**	240.55	**218.57**	–	–
	0.32	**0.27**	0.32	**0.28**	0.41	**0.40**	**1.21**	1.39	**13.97**	19.05	**75.62**	117.91	**302.22**	–	–	–

Table A.12: Average time of 50 random trials using the triangulation and cone decomposition method when integrating over polytopes in Ziegler's Database and their duals: Part II

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Polyt.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
OA_5-18	0.48	**0.30**	0.49	**0.33**	0.59	**0.38**	1.33	**1.04**	**10.34**	11.01	**48.89**	59.80	**170.60**	245.08	**450.50**	–
	0.25	**0.11**	0.26	**0.11**	0.29	**0.15**	0.61	**0.56**	**5.82**	8.94	**32.62**	62.08	**129.58**	283.92	**373.38**	–
OA_5-24	0.73	**0.59**	0.74	**0.60**	0.88	**0.73**	1.89	**1.76**	**15.01**	18.75	**66.89**	104.28	**234.41**	430.15	–	–
	0.25	**0.12**	0.24	**0.12**	0.30	**0.17**	0.86	**0.71**	10.39	**9.62**	**54.98**	65.51	**213.49**	276.15	–	–
CF_6-7	**0.05**	0.05	**0.05**	0.05	0.05	**0.05**	0.11	**0.06**	1.13	**0.45**	7.98	**3.61**	34.39	**16.44**	116.42	**63.29**
	0.01	**0.01**	0.01	**0.01**	0.02	**0.01**	0.07	**0.03**	1.19	**0.47**	9.18	**3.95**	41.39	**19.73**	142.85	**73.92**
CUT4_6-8	0.15	**0.07**	0.15	**0.07**	0.18	**0.08**	0.48	**0.17**	5.60	**2.18**	38.65	**16.72**	159.99	**74.45**	541.50	**280.34**
	0.01	0.05	**0.02**	0.05	**0.03**	0.08	**0.19**	0.46	**3.16**	8.98	**23.13**	72.48	**99.75**	375.33	**338.40**	–
OA_6-13	0.57	**0.22**	0.62	**0.22**	0.98	**0.30**	4.68	**1.36**	68.07	**25.07**	460.63	**196.98**	–	–	–	–
	0.54	5.54	**0.56**	5.66	**0.63**	6.20	**2.00**	13.88	**34.60**	221.25	**280.42**	–	–	–	–	–
AS_6-18	1.17	**0.54**	1.28	**0.54**	2.06	**0.77**	9.81	**3.75**	153.49	**72.17**	–	–	–	–	–	–
	1.28	80.36	**1.30**	78.79	**1.58**	86.04	**5.33**	121.87	**97.82**	–	–	–	–	–	–	–
HC_6-32	2.30	**2.02**	2.41	**2.08**	3.60	**2.63**	18.27	**10.12**	219.43	**176.50**	–	–	–	–	–	–
	3.20	3.75	**3.26**	3.79	**3.63**	4.38	**10.86**	13.50	**190.81**	247.44	–	–	–	–	–	–
3simp.	**5.53**	6.16	**5.52**	6.15	**5.96**	7.03	**13.22**	15.62	**181.39**	248.60	–	–	–	–	–	–
	1.23	1.55	**1.43**	1.61	2.93	**2.19**	17.30	**9.56**	210.92	**174.20**	–	–	–	–	–	–
CF_7-8	0.09	**0.09**	0.09	**0.09**	0.10	**0.09**	0.26	**0.12**	3.92	**1.28**	36.10	**12.79**	181.61	**72.83**	–	**322.01**
	0.02	**0.01**	0.02	**0.01**	0.02	**0.02**	0.10	**0.04**	2.41	**0.88**	24.69	**10.19**	133.56	**64.09**	–	**290.68**
OA_7-18	3.35	**0.88**	3.56	**0.89**	6.44	**1.41**	39.70	**9.30**	–	**266.91**	–	–	–	–	–	–
	14.18	–	**14.33**	–	**15.49**	–	**33.84**	–	–	–	–	–	–	–	–	–
HC_7-64	**13.86**	67.19	**14.59**	62.60	**23.50**	67.31	**125.24**	145.07	–	–	–	–	–	–	–	–
	61.64	–	**61.92**	318.97	**63.37**	–	**96.22**	–	–	–	–	–	–	–	–	–
CF_8-9	**0.14**	0.14	0.15	**0.14**	0.16	**0.15**	0.48	**0.21**	10.54	**3.00**	113.44	**35.48**	–	**240.04**	–	–
	0.01	**0.01**	0.01	**0.01**	0.03	**0.01**	0.18	**0.06**	5.15	**1.84**	62.70	**24.40**	404.53	**176.99**	–	–
HAM_8-16	1.57	**0.60**	1.71	**0.64**	4.42	**1.03**	41.22	**8.16**	–	**338.43**	–	–	–	–	–	–
	0.12	–	**0.16**	–	**0.56**	–	**6.70**	–	**255.09**	–	–	–	–	–	–	–
OA_8-25	39.62	**10.16**	41.19	**10.57**	67.33	**13.57**	428.97	**84.51**	–	–	–	–	–	–	–	–
	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
HC_8-128	**88.28**	–	**93.99**	–	**142.61**	–	–	–	–	–	–	–	–	–	–	–
	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
CF_9-10	0.21	**0.21**	**0.21**	0.21	0.24	**0.21**	0.67	**0.30**	17.41	**4.38**	248.64	**71.24**	–	–	–	–
	0.02	**0.01**	0.02	**0.01**	0.03	**0.01**	0.24	**0.07**	9.89	**3.07**	161.97	**54.30**	–	–	–	–
BIR4_9-24	6.13	**2.23**	6.34	**2.26**	10.58	**3.04**	104.00	**23.12**	–	–	–	–	–	–	–	–
	1.55	**0.17**	1.60	**0.17**	2.36	**0.37**	21.30	**5.10**	–	**309.35**	–	–	–	–	–	–
CF_10-11	0.34	**0.34**	0.35	**0.34**	0.43	**0.35**	1.07	**0.44**	32.64	**7.28**	–	**145.63**	–	–	–	–
	0.02	**0.02**	0.02	**0.02**	0.04	**0.02**	0.33	**0.08**	17.70	**4.30**	338.93	**93.45**	–	–	–	–
CUT5_10-16	3.02	**0.94**	3.66	**0.95**	10.22	**1.57**	106.94	**14.23**	–	–	–	–	–	–	–	–
	37.84	–	**39.37**	–	**61.50**	–	**468.14**	–	–	–	–	–	–	–	–	–
MJ_16-17	2.68	**2.50**	2.46	**2.45**	2.74	**2.52**	8.36	**2.86**	–	**65.65**	–	–	–	–	–	–
	0.07	**0.02**	0.08	**0.03**	0.14	**0.04**	1.65	**0.25**	241.75	**37.75**	–	–	–	–	–	–
MJ_32-33	87.99	**79.35**	83.26	**80.07**	89.65	**78.56**	241.00	**84.81**	–	–	–	–	–	–	–	–
	1.43	**0.14**	1.66	**0.14**	4.58	**0.21**	88.90	**2.11**	–	–	–	–	–	–	–	–

Table A.13: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when integrating over polytopes in Ziegler's Database and their duals: Part I

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Polyt.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
CUT3_3-4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.02	0.01
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.04	0.02
HC_3-4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.02	0.01
CRO_3-6	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.03	0.01	0.03	0.02	0.04	0.04	0.04	0.05
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.07	0.02	0.09
SharirCube	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.03	0.10	0.03	0.17
	0.03	0.01	0.02	0.01	0.03	0.01	0.03	0.01	0.03	0.04	0.06	0.11	0.19	0.45	0.22	0.98
CF_4-5	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.07	0.04	0.17	0.10	0.27	0.19
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.07	0.04	0.18	0.11	0.31	0.18
BIR3_4-6	0.02	0.01	0.02	0.01	0.02	0.01	0.03	0.01	0.07	0.04	0.20	0.12	0.52	0.32	0.95	0.56
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.06	0.11	0.20	0.28	0.60	0.42	1.12
cyclic_4_8	0.02	0.01	0.03	0.01	0.03	0.01	0.03	0.01	0.10	0.07	0.37	0.30	1.07	0.98	1.95	1.76
	0.01	0.01	0.01	0.02	0.01	0.02	0.01	0.04	0.05	0.34	0.21	1.39	0.55	4.47	0.89	9.98
neighborly_4_8	0.03	0.01	0.02	0.01	0.03	0.01	0.03	0.02	0.13	0.08	0.43	0.30	1.16	0.89	1.98	1.55
	0.01	0.01	0.01	0.02	0.01	0.02	0.01	0.03	0.05	0.30	0.20	1.32	0.52	4.39	0.91	9.23
CRO_4-8	0.03	0.01	0.02	0.01	0.03	0.01	0.04	0.02	0.12	0.08	0.39	0.28	0.93	0.86	1.45	1.66
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.04	0.07	0.23	0.23	0.89	0.56	2.53	0.90	5.96
HC_4-8	0.02	0.01	0.03	0.01	0.03	0.01	0.04	0.02	0.12	0.08	0.37	0.28	0.88	0.85	1.40	1.55
	0.01	0.02	0.01	0.02	0.01	0.01	0.01	0.03	0.07	0.22	0.23	0.88	0.56	2.47	0.89	4.82
CF_5-6	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.10	0.04	0.40	0.21	1.07	0.59	2.51	1.50
	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.12	0.05	0.53	0.25	1.48	0.75	3.41	1.65
CNG_5-6a	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.13	0.05	0.55	0.26	1.44	0.73	3.63	1.83
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.11	0.05	0.49	0.24	1.30	0.70	3.06	1.59
CNG_5-6b	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.14	0.06	0.59	0.28	1.56	0.82	3.67	1.93
	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.10	0.04	0.47	0.23	1.34	0.67	3.22	1.53
EQU_5-7a	0.02	0.01	0.02	0.01	0.02	0.01	0.04	0.02	0.39	0.21	1.61	1.22	4.15	5.31	11.53	15.63
	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.02	0.29	0.28	1.19	1.39	3.24	4.02	7.35	10.50
EQU_5-7b	0.02	0.01	0.02	0.01	0.02	0.01	0.04	0.02	0.38	0.24	1.66	1.58	4.84	5.63	10.17	15.35
	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.02	0.26	0.27	1.09	1.33	3.08	3.98	6.91	10.15
CYC_5-8	0.02	0.01	0.02	0.01	0.02	0.01	0.09	0.04	0.75	0.37	3.11	1.88	7.59	7.00	21.22	17.43
	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.09	0.29	1.42	1.42	8.32	3.71	29.55	8.97	87.05
CRO_5-10	0.03	0.02	0.03	0.01	0.03	0.02	0.18	0.07	1.58	0.76	7.08	3.78	18.14	10.77	40.88	28.49
	0.01	0.02	0.01	0.03	0.01	0.05	0.04	0.37	0.50	4.88	2.29	25.56	6.33	88.98	14.16	–
OA_5-10	0.03	0.01	0.04	0.01	0.04	0.02	0.19	0.07	1.83	0.85	7.06	4.06	20.08	12.02	42.53	31.61
	0.03	0.01	0.03	0.01	0.03	0.03	0.14	0.23	1.41	3.23	6.36	15.15	15.49	49.81	36.49	–
EG_5-12	0.03	0.01	0.04	0.01	0.05	0.02	0.31	0.15	2.73	1.89	11.47	8.35	30.14	28.61	65.35	82.11
	0.03	0.04	0.03	0.05	0.03	0.10	0.14	0.71	1.42	8.72	6.21	43.64	17.40	–	38.79	–
HC_5-16	0.04	0.01	0.04	0.01	0.07	0.02	0.40	0.21	3.63	2.55	14.76	12.04	39.96	34.68	–	–
	0.03	0.02	0.03	0.02	0.05	0.06	0.41	0.53	4.29	6.41	18.86	46.34	55.01	–	–	–

Table A.14: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when integrating over polytopes in Ziegler's Database and their duals: Part II

	Monomial Degree															
	1		2		5		10		20		30		40		50	
Polyt.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
OA_5-18	0.04	0.01	0.05	0.02	0.05	0.03	0.27	0.24	2.44	2.97	10.00	13.66	27.39	43.23	55.52	–
	0.03	0.02	0.04	0.02	0.04	0.02	0.17	0.17	1.74	2.70	8.09	16.30	23.42	58.11	54.35	–
OA_5-24	0.04	0.02	0.04	0.02	0.06	0.04	0.36	0.40	3.47	4.89	12.91	21.89	34.79	71.33	–	–
	0.03	0.01	0.04	0.02	0.04	0.04	0.30	0.32	3.27	3.68	14.14	27.75	38.73	100.59	–	–
CF_6-7	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.01	0.35	0.13	1.98	0.89	7.28	3.26	15.35	8.28
	0.01	0.01	0.01	0.01	0.01	0.01	0.03	0.01	0.39	0.15	2.42	0.99	8.78	3.74	21.06	9.63
CUT4_6-8	0.03	0.01	0.03	0.01	0.03	0.01	0.13	0.03	1.83	0.68	10.19	4.25	36.25	15.47	96.52	37.03
	0.01	0.01	0.01	0.01	0.01	0.02	0.07	0.14	1.08	2.78	6.30	17.77	22.96	78.43	50.39	–
OA_6-13	0.03	0.01	0.04	0.01	0.16	0.04	1.54	0.41	22.69	7.56	121.28	49.35	–	–	–	–
	0.04	0.36	0.05	0.33	0.06	0.43	0.60	3.35	11.81	73.60	77.74	–	–	–	–	–
AS_6-18	0.04	0.02	0.06	0.02	0.33	0.09	3.15	1.19	47.65	24.01	–	–	–	–	–	–
	0.05	10.75	0.04	10.08	0.16	11.34	1.96	19.56	35.15	–	–	–	–	–	–	–
HC_6-32	0.09	0.06	0.09	0.05	0.49	0.22	5.39	2.75	64.81	54.29	–	–	–	–	–	–
	0.08	0.33	0.10	0.30	0.25	0.50	3.74	4.70	70.68	125.24	–	–	–	–	–	–
3simp3simp	0.12	0.38	0.14	0.33	0.28	0.43	3.83	4.75	64.72	119.50	–	–	–	–	–	–
	0.08	0.06	0.10	0.05	0.50	0.20	4.80	2.59	58.49	51.17	–	–	–	–	–	–
CF_7-8	0.01	0.01	0.01	0.01	0.01	0.01	0.08	0.02	1.47	0.45	11.93	4.02	49.86	19.13	–	58.79
	0.01	0.01	0.01	0.01	0.01	0.01	0.04	0.01	0.87	0.31	7.56	2.99	34.95	16.48	–	48.40
OA_7-18	0.09	0.03	0.17	0.03	1.26	0.19	16.26	3.89	–	105.21	–	–	–	–	–	–
	0.47	–	0.59	–	0.70	–	9.35	–	–	–	–	–	–	–	–	–
HC_7-64	0.28	12.59	0.53	7.79	3.46	7.61	45.72	39.44	–	–	–	–	–	–	–	–
	1.09	–	1.00	109.17	1.67	–	22.81	–	–	–	–	–	–	–	–	–
CF_8-9	0.01	0.01	0.01	0.01	0.01	0.01	0.14	0.03	4.66	1.22	37.51	11.47	–	67.08	–	–
	0.01	0.01	0.01	0.01	0.01	0.01	0.06	0.02	1.95	0.68	19.03	7.13	110.59	48.06	–	–
HAM_8-16	0.09	0.02	0.14	0.02	1.12	0.14	16.02	2.71	–	143.89	–	–	–	–	–	–
	0.01	–	0.02	–	0.19	–	2.66	–	120.94	–	–	–	–	–	–	–
OA_8-25	3.62	1.95	2.38	1.34	10.71	1.73	150.44	26.36	–	–	–	–	–	–	–	–
	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
HC_8-128	1.05	–	2.32	–	18.55	–	–	–	–	–	–	–	–	–	–	–
	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
CF_9-10	0.01	0.01	0.01	0.01	0.02	0.01	0.24	0.04	7.78	1.83	86.82	24.36	–	–	–	–
	0.01	0.01	0.01	0.01	0.01	0.01	0.10	0.03	4.28	1.28	54.41	17.58	–	–	–	–
BIR4_9-24	0.11	0.03	0.17	0.04	2.15	0.32	54.28	9.95	–	–	–	–	–	–	–	–
	0.06	0.01	0.05	0.01	0.50	0.10	13.22	2.82	–	151.23	–	–	–	–	–	–
CF_10-11	0.01	0.01	0.01	0.01	0.02	0.01	0.26	0.04	15.97	3.25	–	58.13	–	–	–	–
	0.01	0.01	0.01	0.01	0.01	0.01	0.11	0.03	8.30	1.97	127.19	34.44	–	–	–	–
CUT5_10-16	0.08	0.02	0.22	0.03	2.40	0.19	39.16	4.43	–	–	–	–	–	–	–	–
	0.56	–	0.98	–	9.69	–	178.06	–	–	–	–	–	–	–	–	–
OA_10-44	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
MJ_16-17	0.02	0.02	0.06	0.06	0.15	0.02	2.24	0.13	–	28.89	–	–	–	–	–	–
	0.01	0.01	0.01	0.01	0.02	0.01	0.55	0.08	116.23	19.61	–	–	–	–	–	–
MJ_32-33	0.42	2.22	0.92	1.43	3.12	1.71	49.34	0.78	–	–	–	–	–	–	–	–
	0.02	0.01	0.06	0.01	0.59	0.02	24.31	0.54	–	–	–	–	–	–	–	–

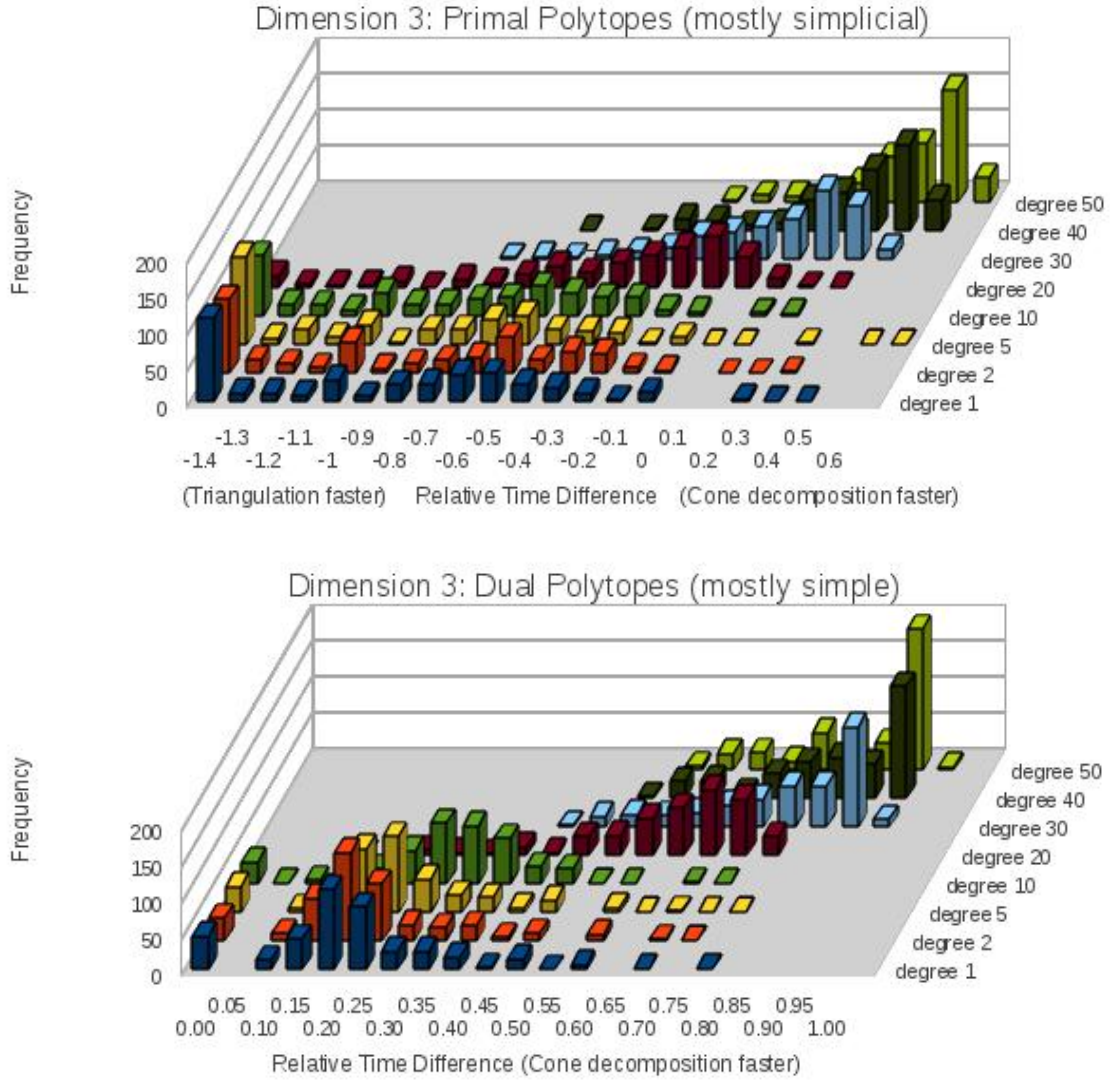

Figure A.1: Integration over random polytopes: Dimension 3

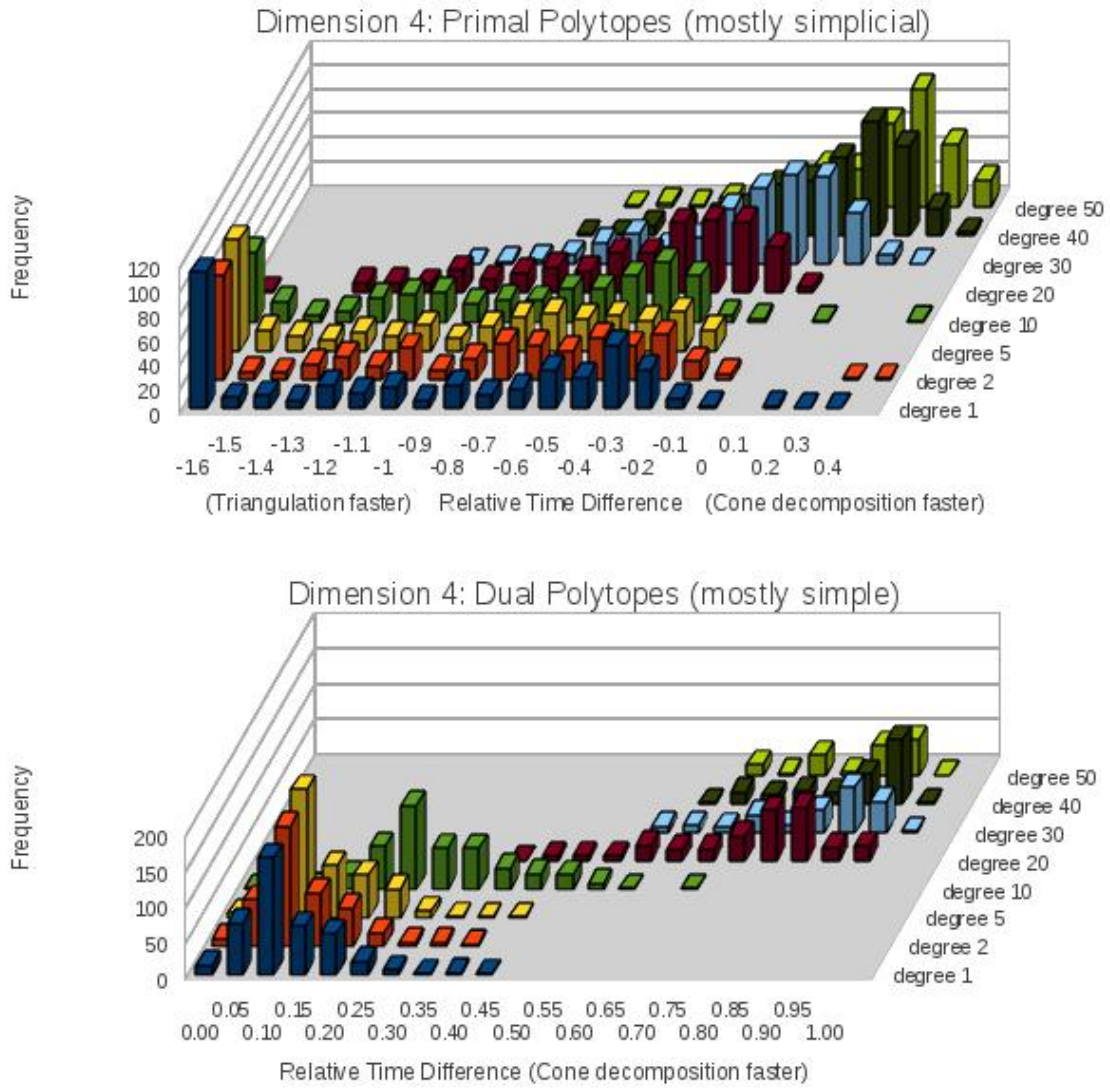

Figure A.2: Integration over random polytopes: Dimension 4

Appendix B. Volume

Tables B.1, and B.2 display the average and standard deviation time when computing the volume of the same random polytopes we used for integrating.

Table B.1: Average time of 50 random trials using the triangulation and cone decomposition method when computing volumes of random simplicial polytopes and their duals

	Primal Vertex Count															
	$d+2$		$d+3$		$d+4$		$d+5$		$d+10$		$d+15$		$d+20$		$d+25$	
d	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
3	0.04	**0.02**	0.06	**0.03**	0.08	**0.03**	0.09	**0.04**	0.19	**0.09**	0.31	**0.17**	0.47	**0.29**	0.61	**0.44**
	0.03	0.04	**0.04**	0.06	**0.07**	0.09	**0.11**	0.12	**0.42**	0.49	**1.05**	1.17	**1.79**	2.12	**2.81**	3.34
4	0.08	**0.04**	0.10	**0.05**	0.14	**0.06**	0.15	**0.07**	0.36	**0.17**	0.55	**0.35**	0.84	**0.60**	1.15	**0.93**
	0.11	0.13	**0.35**	0.39	**0.71**	0.79	**1.53**	1.65	**9.73**	11.05	**75.24**	78.72	**195.12**	202.11	**402.06**	407.94
5	0.12	**0.06**	0.16	**0.07**	0.21	**0.10**	0.29	**0.12**	0.76	**0.33**	1.44	**0.72**	2.36	**1.30**	3.47	**2.13**
	0.01	0.03	**0.01**	0.14	**0.02**	0.47	**0.03**	1.21	**0.05**	21.27	**0.07**	94.77	**0.12**	276.83	**0.17**	–
6	0.21	**0.10**	0.28	**0.13**	0.38	**0.16**	0.51	**0.22**	1.85	**0.64**	4.61	**1.58**	9.31	**3.23**	15.26	**5.78**
	0.02	0.09	**0.02**	0.82	**0.03**	4.47	**0.04**	16.13	**0.12**	–	**0.26**	–	**0.63**	–	**1.08**	–
7	0.32	**0.18**	0.42	**0.22**	0.62	**0.29**	0.87	**0.36**	4.75	**1.24**	15.85	**3.66**	36.62	**11.50**	71.68	**46.35**
	0.02	0.25	**0.04**	5.40	**0.06**	43.20	**0.09**	230.40	**0.40**	–	**1.57**	–	**4.22**	–	–	–
8	0.48	**0.28**	0.64	**0.36**	0.96	**0.45**	1.52	**0.58**	11.40	**2.34**	50.03	**11.52**	170.16	**105.26**	–	–
	0.04	0.82	**0.06**	33.20	**0.11**	499.47	**0.20**	–	**1.86**	–	**11.59**	–	**55.11**	–	**188.05**	–
10	0.97	**0.69**	1.46	**0.84**	2.33	**1.07**	4.22	**1.40**	65.07	**10.52**	–	–	–	–	–	–
	0.07	8.75	**0.18**	–	**0.37**	–	**0.87**	–	**50.48**	–	–	–	–	–	–	–
15	5.76	**4.56**	9.20	**5.30**	18.88	**6.55**	46.12	**9.01**	–	–	–	–	–	–	–	–
	0.47	–	**2.07**	–	**11.46**	–	**80.01**	–	–	–	–	–	–	–	–	–

Table B.2: Standard deviation of the time for 50 random trials using the triangulation and cone decomposition method when computing volumes of random simplicial polytopes and their duals

	Vertices of the primal polytope															
	$d+2$		$d+3$		$d+4$		$d+5$		$d+10$		$d+15$		$d+20$		$d+25$	
d	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.	Cone.	Tri.
3	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.03	0.01	0.03	0.01	0.04	0.01	0.09	0.01
	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.11	0.11	0.23	0.21	0.38	0.38	0.69	0.68
4	0.02	0.01	0.02	0.01	0.03	0.01	0.03	0.01	0.06	0.01	0.04	0.02	0.07	0.02	0.08	0.02
	0.03	0.03	0.11	0.11	0.23	0.24	0.53	0.51	5.00	4.98	13.88	14.01	27.03	27.87	56.88	58.39
5	0.03	0.01	0.03	0.01	0.03	0.01	0.03	0.01	0.06	0.01	0.09	0.03	0.11	0.05	0.19	0.10
	0.01	0.01	0.01	0.04	0.01	0.11	0.01	0.26	0.01	4.54	0.01	16.12	0.01	48.51	0.02	–
6	0.03	0.01	0.03	0.01	0.03	0.02	0.04	0.02	0.13	0.03	0.38	0.07	0.76	0.15	1.06	0.38
	0.01	0.02	0.01	0.24	0.01	1.12	0.01	3.58	0.02	–	0.03	–	0.08	–	0.12	–
7	0.03	0.01	0.04	0.01	0.05	0.02	0.06	0.02	0.32	0.06	1.21	0.23	2.92	1.79	5.85	9.87
	0.01	0.06	0.01	1.47	0.01	12.15	0.02	80.53	0.06	–	0.18	–	0.60	–	–	–
8	0.03	0.01	0.05	0.01	0.05	0.01	0.07	0.02	0.66	0.12	3.57	1.74	21.00	20.00	–	–
	0.01	0.24	0.01	8.33	0.01	126.97	0.02	–	0.28	–	1.99	–	10.25	–	34.85	–
10	0.04	0.02	0.04	0.02	0.06	0.02	0.11	0.05	3.40	1.09	–	–	–	–	–	–
	0.01	2.47	0.02	–	0.03	–	0.09	–	9.82	–	–	–	–	–	–	–
15	0.07	0.04	0.10	0.07	0.26	0.09	0.98	0.19	–	–	–	–	–	–	–	–
	0.02	–	0.16	–	1.12	–	10.38	–	–	–	–	–	–	–	–	–