LattE integrale 1.6

Tutorial and update

Matthias Köppe, UC Davis

V. Baldoni
N. Berline
J.A. De Loera
B. Dutra
S. Moreinis
G. Pinto
M. Vergne
J. Wu

Image credit: Wikipedia

Rational polytopes and counting problems

Rational polytope

$$P = \operatorname{conv}\{\mathbf{v}^1, \dots, \mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq \mathbf{b} \,\}$$

Integer dilates

Consider

$$nP = \operatorname{conv}\{n\mathbf{v}^1, \dots, n\mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq n\mathbf{b} \,\}$$

for $n \in \mathbf{N}$.

Ehrhart function

$$i_P : \mathbf{N} \to \mathbf{N}, \quad n \mapsto \#(nP \cap \mathbf{Z}^d)$$

Ehrhart series (generating function)

$$\mathrm{Ehr}_P(z) = \sum_{n=0}^{\infty} i_P(n) z^n$$

For lattice polytopes P, $\dim P = d$

i_P is a polynomial function of degree d, the Ehrhart polynomial of P

Goals

- Compute the exact counting function (polynomial, series, quasi-polynomial)
- … its asymptotics (highest coefficients)
- … or weighted versions (motivated by symmetry techniques)

Rational polytopes and counting problems

Rational polytope

$$P = \operatorname{conv}\{\mathbf{v}^1, \dots, \mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq \mathbf{b} \,\}$$

Integer dilates

Consider

$$nP = \operatorname{conv}\{n\mathbf{v}^1, \dots, n\mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq n\mathbf{b} \,\}$$

for $n \in \mathbf{N}$.

Ehrhart function

$$i_P : \mathbf{N} \to \mathbf{N}, \quad n \mapsto \#(nP \cap \mathbf{Z}^d)$$

Ehrhart series (generating function)

$$\mathrm{Ehr}_P(z) = \sum_{n=0}^{\infty} i_P(n) z^n$$

For lattice polytopes P, $\dim P = d$

i_P is a polynomial function of degree d, the Ehrhart polynomial of P

Goals

- Compute the exact counting function (polynomial, series, quasi-polynomial)
- ... its asymptotics (highest coefficients)
- ... or weighted versions (motivated by symmetry techniques)

Rational polytopes and counting problems

Rational polytope

$$P = \operatorname{conv}\{\mathbf{v}^1, \dots, \mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \le \mathbf{b} \,\}$$

Integer dilates

Consider

$$nP = \operatorname{conv}\{n\mathbf{v}^1, \dots, n\mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \le n\mathbf{b} \,\}$$

for $n \in \mathbf{N}$.

Ehrhart function

$$i_P \colon \mathbf{N} \to \mathbf{N}, \quad n \mapsto \#(nP \cap \mathbf{Z}^d)$$

Ehrhart series (generating function)

$$\mathrm{Ehr}_P(z) = \sum_{n=0}^{\infty} i_P(n) z^n$$

For lattice polytopes P, $\dim P = d$

i_P is a polynomial function of degree d, the Ehrhart polynomial of P

Goals

- Compute the exact counting function (polynomial, series, quasi-polynomial)
- ... its asymptotics (highest coefficients)
- ... or weighted versions (motivated by symmetry techniques)

Rational polytopes and counting problems

Rational polytope

$$P = \mathrm{conv}\{\mathbf{v}^1, \dots, \mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq \mathbf{b} \,\}$$

Integer dilates

Consider

$$nP = \mathrm{conv}\{n\mathbf{v}^1, \dots, n\mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq n\mathbf{b} \,\}$$

for $n \in \mathbf{N}$.

Ehrhart function

$$i_P \colon \mathbf{N} \to \mathbf{N}, \quad n \mapsto \#(nP \cap \mathbf{Z}^d)$$

Ehrhart series (generating function)

$$\mathrm{Ehr}_P(z) = \sum_{n=0}^{\infty} i_P(n) z^n$$

For lattice polytopes P, $\dim P = d$

i_P is a polynomial function of degree d, the Ehrhart polynomial of P

Goals

- Compute the exact counting function (polynomial, series, quasi-polynomial)
- . . . its asymptotics (highest coefficients)
- . . . or weighted versions (motivated by symmetry techniques)

Rational polytopes and counting problems

Rational polytope

$$P = \operatorname{conv}\{\mathbf{v}^1, \dots, \mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \le \mathbf{b} \,\}$$

Integer dilates

Consider

$$nP = \operatorname{conv}\{n\mathbf{v}^1, \dots, n\mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \le n\mathbf{b} \,\}$$

for $n \in \mathbf{N}$.

Ehrhart function

$$i_P \colon \mathbf{N} \to \mathbf{N}, \quad n \mapsto \#(nP \cap \mathbf{Z}^d)$$

Ehrhart series (generating function)

$$\mathrm{Ehr}_P(z) = \sum_{n=0}^{\infty} i_P(n) z^n$$

For lattice polytopes P, $\dim P = d$

i_P is a polynomial function of degree d, the Ehrhart polynomial of P

Goals

- Compute the exact counting function (polynomial, series, quasi-polynomial)
- . . . its asymptotics (highest coefficients)
- . . . or weighted versions (motivated by symmetry techniques)

Rational polytopes and counting problems

Rational polytope

$$P = \operatorname{conv}\{\mathbf{v}^1, \dots, \mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq \mathbf{b} \,\}$$

Integer dilates

Consider

$$nP = \operatorname{conv}\{n\mathbf{v}^1, \dots, n\mathbf{v}^k\} \subseteq \mathbf{R}^d$$
$$= \{\, \mathbf{x} \in \mathbf{R}^d : A\mathbf{x} \leq n\mathbf{b} \,\}$$

for $n \in \mathbf{N}$.

Ehrhart function

$$i_P \colon \mathbf{N} \to \mathbf{N}, \quad n \mapsto \#(nP \cap \mathbf{Z}^d)$$

Ehrhart series (generating function)

$$\mathrm{Ehr}_P(z) = \sum_{n=0}^{\infty} i_P(n) z^n$$

For lattice polytopes P, $\dim P = d$

i_P is a polynomial function of degree d, the Ehrhart polynomial of P

Goals

- Compute the exact counting function (polynomial, series, quasi-polynomial)
- ... its asymptotics (highest coefficients)
- ... or weighted versions (motivated by symmetry techniques)

Complexity results

Hardness results

- Detecting lattice points in polytopes (even simplices) is NP-hard
- Counting lattice points in polytopes is #P-hard
- Computing the volume of polytopes is #P-hard (Dyer–Frieze, 1988)
- Approximating the volume of polytopes is hard (Elekes, 1986)

Polynomiality results

- Detecting lattice points is polynomial time in fixed dimension (Lenstra, 1983)
- Counting lattice points is polynomial time in fixed dimension (Barvinok, 1994)
- Computing Ehrhart polynomials of integral polytopes is polynomial time in fixed dimension (Barvinok, 1994)
- Computing the first k (fixed) coefficients of Ehrhart quasi-polynomials (for a given coset) of rational simplices is polynomial time in varying dimension (Barvinok, 2005)
- Computing the first k (fixed) coefficients of **weighted** Ehrhart quasi-polynomials **(as closed formulas)** of rational simplices is polynomial time in varying dimension (Baldoni–Berline–De Loera–**Kö.**–Vergne, 2012)

Complexity results

Hardness results

- Detecting lattice points in polytopes (even simplices) is NP-hard
- Counting lattice points in polytopes is #P-hard
- Computing the volume of polytopes is #P-hard (Dyer–Frieze, 1988)
- Approximating the volume of polytopes is hard (Elekes, 1986)

Polynomiality results

- Detecting lattice points is polynomial time in fixed dimension (Lenstra, 1983)
- Counting lattice points is polynomial time in fixed dimension (Barvinok, 1994)
- Computing Ehrhart polynomials of integral polytopes is polynomial time in fixed dimension (Barvinok, 1994)
- Computing the first k (fixed) coefficients of Ehrhart quasi-polynomials (for a given coset) of rational simplices is polynomial time in varying dimension (Barvinok, 2005)
- Computing the first k (fixed) coefficients of **weighted** Ehrhart quasi-polynomials **(as closed formulas)** of rational simplices is polynomial time in varying dimension (Baldoni–Berline–De Loera–**Kö.**–Vergne, 2012)

Complexity results

Hardness results

- Detecting lattice points in polytopes (even simplices) is NP-hard
- Counting lattice points in polytopes is #P-hard
- Computing the volume of polytopes is #P-hard (Dyer–Frieze, 1988)
- Approximating the volume of polytopes is hard (Elekes, 1986)

Polynomiality results

- Detecting lattice points is polynomial time in fixed dimension (Lenstra, 1983)
- Counting lattice points is polynomial time in fixed dimension (Barvinok, 1994)
- Computing Ehrhart polynomials of integral polytopes is polynomial time in fixed dimension (Barvinok, 1994)
- Computing the first k (fixed) coefficients of Ehrhart quasi-polynomials (for a given coset) of rational simplices is polynomial time in varying dimension (Barvinok, 2005)
- Computing the first k (fixed) coefficients of **weighted** Ehrhart quasi-polynomials **(as closed formulas)** of rational simplices is polynomial time in varying dimension (Baldoni–Berline–De Loera–**Kö.**–Vergne, 2012)

Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed polyhedra

Thm (Brion, 1988)

$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$=$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),

Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$=$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),

Generating functions

Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

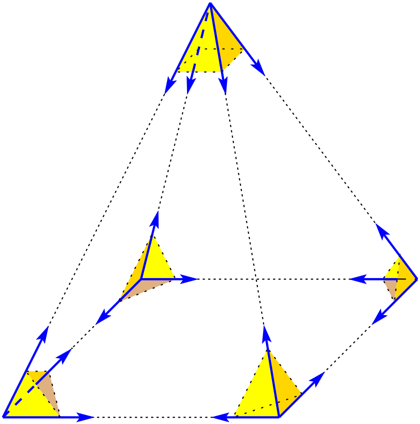
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$=$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

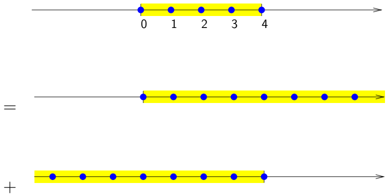
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$=$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

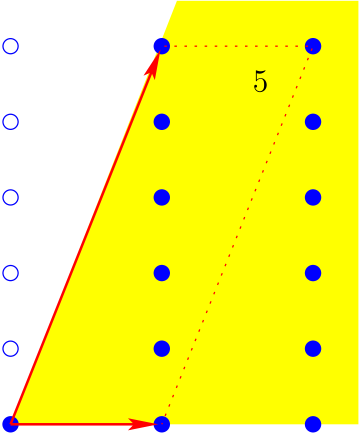
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$[P \cup Q] = [P] + [Q] - [P \cap Q]$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Generating functions

Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

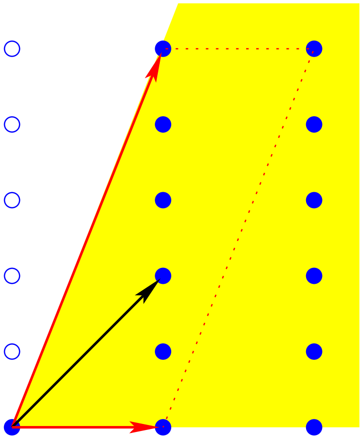
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$[P \cup Q] = [P] + [Q] - [P \cap Q]$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

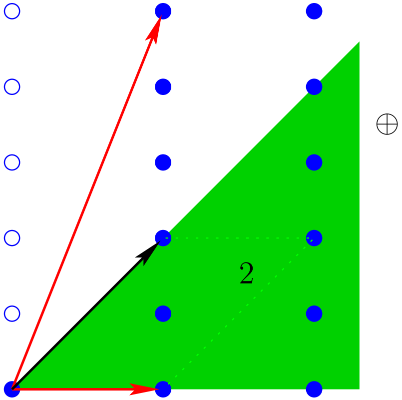
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$[P \cup Q] = [P] + [Q] - [P \cap Q]$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

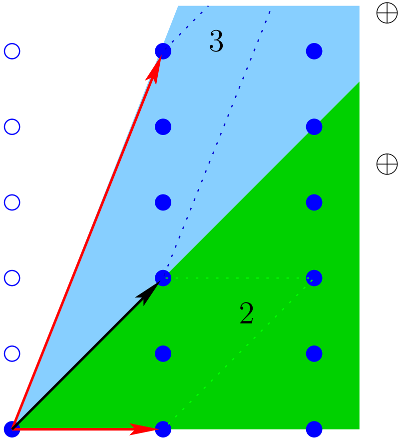
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$[P \cup Q] = [P] + [Q] - [P \cap Q]$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

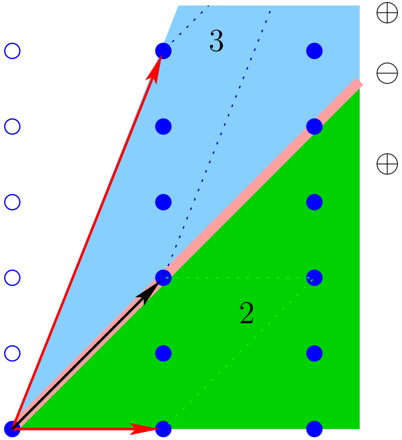
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$[P \cup Q] = [P] + [Q] - [P \cap Q]$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Generating functions

Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

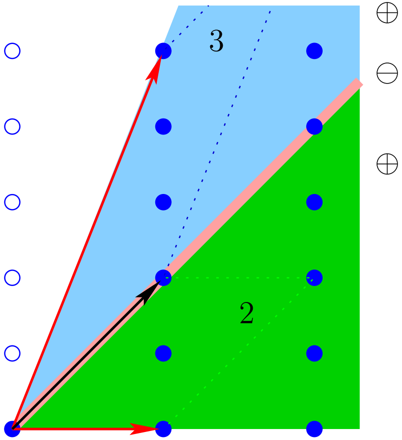
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$g_{P\cup Q}(\mathbf{z}) = g_P(\mathbf{z}) + g_Q(\mathbf{z}) - g_{P\cap Q}(\mathbf{z})$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),



Generating functions

Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

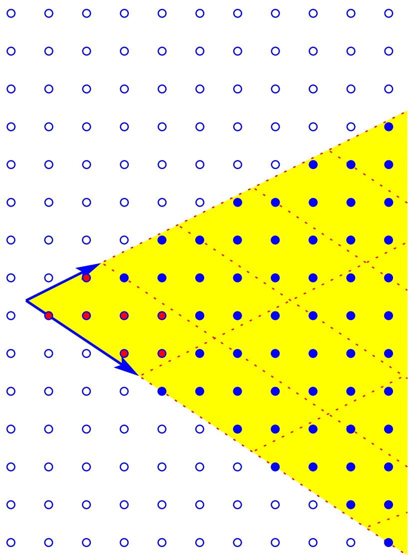
$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

Valuation property (linearity)

$$g_{P\cup Q}(\mathbf{z}) = g_P(\mathbf{z}) + g_Q(\mathbf{z}) - g_{P\cap Q}(\mathbf{z})$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \ldots, \mathbf{b}_d \in \mathbf{Z}^d$),



Generating functions

Thm (Khovanskii–Pukhlikov–Lawrence, 1990s)

Rational-function-valued valuation (linear map) $\mathcal{F}\colon [P] \mapsto g_P(\mathbf{z})$, agrees with $\sum_{\mathbf{a}\in P} \mathbf{z}^{\mathbf{a}}$ for pointed, zero on non-pointed polyhedra

Thm (Brion, 1988)

$$g_P(\mathbf{z}) = \sum_{C_i \text{ vertex cone}} g_{C_i}(\mathbf{z})$$

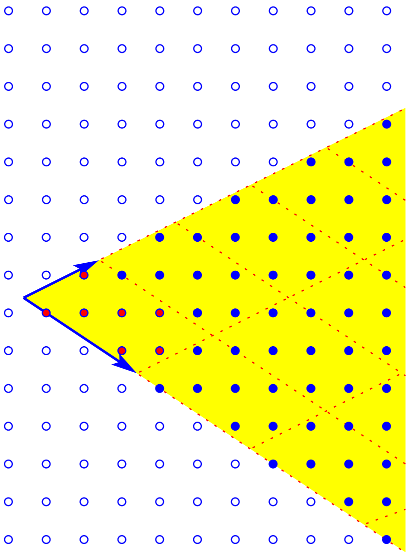
Valuation property (linearity)

$$g_{P\cup Q}(\mathbf{z}) = g_P(\mathbf{z}) + g_Q(\mathbf{z}) - g_{P\cap Q}(\mathbf{z})$$

For simplicial cones $C \subseteq \mathbf{R}^d$:

(generated by rays $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbf{Z}^d$),

$$g_C(\mathbf{z}) = \frac{\sum_{\mathbf{a}\in\Pi\cap\mathbf{Z}^n} \mathbf{z}^{\mathbf{a}}}{\prod_{j=1}^{d}(1-\mathbf{z}^{\mathbf{b}_j})}$$



Continuous generating functions: Brion's formula for integrals

M. Brion, Ann. Sci. École Norm. Sup. **21** (1988), 653–663.

Theorem (Brion)

Let Δ be the simplex that is the convex hull of $(d+1)$ affinely independent vertices $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{d+1}$ in $\mathbf{R}^n$.
Let ℓ be a linear form which is regular w.r.t. Δ, i.e.,

$$\langle \ell, \mathbf{s}_i \rangle \neq \langle \ell, \mathbf{s}_j \rangle \quad \text{for } i \neq j$$

Then:

$$\int_\Delta e^\ell \, \mathrm{d}m = d! \operatorname{vol}(\Delta, \mathrm{d}m) \sum_{i=1}^{d+1} \frac{e^{\langle \ell, \mathbf{s}_i \rangle}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle}.$$

By expanding the exponential as a Taylor series:

Corollary

$$\int_\Delta \ell^M \, \mathrm{d}m = d! \operatorname{vol}(\Delta, \mathrm{d}m) \frac{M!}{(M+d)!} \Big(\sum_{i=1}^{d+1} \frac{\langle \ell, \mathbf{s}_i \rangle^{M+d}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle} \Big).$$

M. Brion, Ann. Sci. École Norm. Sup. **21** (1988), 653–663.

Theorem (Brion)

Let Δ be the simplex that is the convex hull of $(d+1)$ affinely independent vertices $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{d+1}$ in $\mathbf{R}^n$.
Let ℓ be a linear form which is regular w.r.t. Δ, i.e.,

$$\langle \ell, \mathbf{s}_i \rangle \neq \langle \ell, \mathbf{s}_j \rangle \quad \text{for } i \neq j$$

Then:

$$\int_\Delta e^\ell \, \mathrm{d}m = d! \operatorname{vol}(\Delta, \mathrm{d}m) \sum_{i=1}^{d+1} \frac{e^{\langle \ell, \mathbf{s}_i \rangle}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle}.$$

By expanding the exponential as a Taylor series:

Corollary

$$\int_\Delta \ell^M \, \mathrm{d}m = d! \operatorname{vol}(\Delta, \mathrm{d}m) \frac{M!}{(M+d)!} \Big(\sum_{i=1}^{d+1} \frac{\langle \ell, \mathbf{s}_i \rangle^{M+d}}{\prod_{j \neq i} \langle \ell, \mathbf{s}_i - \mathbf{s}_j \rangle} \Big).$$

Powers of linear forms are enough: The polynomial Waring problem

J. Alexander and A. Hirschowitz, J. Algebraic Geom. **4** (1995), 201–222.

Theorem (Alexander–Hirschowitz, 1995)

A generic homogeneous polynomial of degree M in n variables is expressible as the sum of

$$r(M, n) = \left\lceil \frac{\binom{n+M-1}{M}}{n} \right\rceil$$

M-th powers of linear forms, with the exception of the cases $r(3, 5) = 8$, $r(4, 3) = 6$, $r(4, 4) = 10$, $r(4, 5) = 15$, *and* $M = 2$, *where* $r(2, n) = n$. *(Non-constructive.)*

Theorem (Carlini–Catalisano–Geramita, 2011)

Minimal, constructive solution for monomials $\mathbf{x}^{\mathbf{M}}$, $M_1 \leq \cdots \leq M_n$ *with* $\prod_{i=2}^{n}(M_i + 1)$, *involving roots of unity.*

Effective (constructive) version?

First numerical procedure given by J. Brachat, P. Comon, B. Mourrain, E. Tsigaridas (Lin. Alg. Appl., 2010)

Simple (suboptimal) rational constructions

$$\mathbf{x}^{\mathbf{M}} = \frac{1}{|\mathbf{M}|!} \sum_{0 \leq p_i \leq M_i} \alpha_{\mathbf{p}} (p_1 x_1 + \cdots + p_n x_n)^{|\mathbf{M}|}$$

with $\alpha_p = (-1)^{|\mathbf{M}| - (p_1 + \cdots + p_n)} \binom{M_1}{p_1} \cdots \binom{M_n}{p_n}$

Theorem (Alexander–Hirschowitz, 1995)

A generic homogeneous polynomial of degree M in n variables is expressible as the sum of

$$r(M,n) = \left\lceil \frac{\binom{n+M-1}{M}}{n} \right\rceil$$

M-th powers of linear forms, with the exception of the cases $r(3,5)=8$, $r(4,3)=6$, $r(4,4)=10$, $r(4,5)=15$, and $M=2$, where $r(2,n)=n$. (Non-constructive.)

Theorem (Carlini–Catalisano–Geramita, 2011)

Minimal, constructive solution for monomials $\mathbf{x}^{\mathbf{M}}$, $M_1 \le \cdots \le M_n$ with $\prod_{i=2}^{n}(M_i+1)$, involving roots of unity.

Effective (constructive) version?

First numerical procedure given by J. Brachat, P. Comon, B. Mourrain, E. Tsigaridas (Lin. Alg. Appl., 2010)

Simple (suboptimal) rational constructions

$$\mathbf{x}^{\mathbf{M}} = \frac{1}{|\mathbf{M}|!} \sum_{0 \le p_i \le M_i} \alpha_{\mathbf{p}} (p_1 x_1 + \cdots + p_n x_n)^{|\mathbf{M}|}$$

with $\alpha_p = (-1)^{|\mathbf{M}|-(p_1+\cdots+p_n)} \binom{M_1}{p_1} \cdots \binom{M_n}{p_n}$

Powers of linear forms are enough: The polynomial Waring problem

J. Alexander and A. Hirschowitz, J. Algebraic Geom. **4** (1995), 201–222.

Theorem (Alexander–Hirschowitz, 1995)

A generic homogeneous polynomial of degree M in n variables is expressible as the sum of

$$r(M,n) = \left\lceil \frac{\binom{n+M-1}{M}}{n} \right\rceil$$

M-th powers of linear forms, with the exception of the cases $r(3,5)=8$, $r(4,3)=6$, $r(4,4)=10$, $r(4,5)=15$, and $M=2$, where $r(2,n)=n$. (Non-constructive.)

Theorem (Carlini–Catalisano–Geramita, 2011)

Minimal, constructive solution for monomials $\mathbf{x}^{\mathbf{M}}$, $M_1 \le \cdots \le M_n$ with $\prod_{i=2}^{n}(M_i+1)$, involving roots of unity.

Effective (constructive) version?

First numerical procedure given by J. Brachat, P. Comon, B. Mourrain, E. Tsigaridas (Lin. Alg. Appl., 2010)

Simple (suboptimal) rational constructions

$$\mathbf{x}^{\mathbf{M}} = \frac{1}{|\mathbf{M}|!} \sum_{0 \le p_i \le M_i} \alpha_{\mathbf{p}} (p_1 x_1 + \cdots + p_n x_n)^{|\mathbf{M}|}$$

with $\alpha_p = (-1)^{|\mathbf{M}|-(p_1+\cdots+p_n)} \binom{M_1}{p_1} \cdots \binom{M_n}{p_n}$

Powers of linear forms are enough: The polynomial Waring problem

J. Alexander and A. Hirschowitz, J. Algebraic Geom. **4** (1995), 201–222.

Theorem (Alexander–Hirschowitz, 1995)

A generic homogeneous polynomial of degree M in n variables is expressible as the sum of

$$r(M,n) = \left\lceil \frac{\binom{n+M-1}{M}}{n} \right\rceil$$

M-th powers of linear forms, with the exception of the cases $r(3,5)=8$, $r(4,3)=6$, $r(4,4)=10$, $r(4,5)=15$, and $M=2$, where $r(2,n)=n$. (Non-constructive.)

Theorem (Carlini–Catalisano–Geramita, 2011)

Minimal, constructive solution for monomials $\mathbf{x}^{\mathbf{M}}$, $M_1 \le \cdots \le M_n$ with $\prod_{i=2}^{n}(M_i+1)$, involving roots of unity.

Effective (constructive) version?

First numerical procedure given by J. Brachat, P. Comon, B. Mourrain, E. Tsigaridas (Lin. Alg. Appl., 2010)

Simple (suboptimal) rational constructions

$$\mathbf{x}^{\mathbf{M}} = \frac{1}{|\mathbf{M}|!} \sum_{0 \le p_i \le M_i} \alpha_{\mathbf{p}} (p_1 x_1 + \cdots + p_n x_n)^{|\mathbf{M}|}$$

with $\alpha_p = (-1)^{|\mathbf{M}|-(p_1+\cdots+p_n)} \binom{M_1}{p_1} \cdots \binom{M_n}{p_n}$

Computational results with LattE integrale

Average and standard deviation of integration time in seconds of a random monomial over a d-simplex (average over 50 random monomials)

	Degree										
Dimension	1	2	5	10	20	30	40	50	100	200	300
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.0	3.8
	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.4	1.7
3	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	2.3	38.7	162.0
	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.4	24.2	130.7
4	0.0	0.0	0.0	0.0	0.0	0.1	0.4	0.7	22.1	–	–
	0.0	0.0	0.0	0.0	0.0	0.1	0.3	0.7	16.7	–	–
5	0.0	0.0	0.0	0.0	0.1	0.3	1.6	4.4	–	–	–
	0.0	0.0	0.0	0.0	0.0	0.2	1.3	3.5	–	–	–
7	0.0	0.0	0.0	0.0	0.2	2.2	12.3	63.2	–	–	–
	0.0	0.0	0.0	0.0	0.2	1.7	12.6	66.9	–	–	–
8	0.0	0.0	0.0	0.0	0.4	4.2	30.6	141.4	–	–	–
	0.0	0.0	0.0	0.0	0.3	3.0	31.8	127.6	–	–	–
10	0.0	0.0	0.0	0.0	1.3	19.6	–	–	–	–	–
	0.0	0.0	0.0	0.0	1.4	19.4	–	–	–	–	–
15	0.0	0.0	0.0	0.1	5.7	–	–	–	–	–	–
	0.0	0.0	0.0	0.0	3.6	–	–	–	–	–	–
20	0.0	0.0	0.0	0.2	23.3	–	–	–	–	–	–
	0.0	0.0	0.0	1.3	164.8	–	–	–	–	–	–
30	0.0	0.0	0.0	0.6	110.2	–	–	–	–	–	–
	0.0	0.0	0.1	4.0	779.1	–	–	–	–	–	–
40	0.0	0.0	0.0	1.0	–	–	–	–	–	–	–
	0.0	0.0	0.3	7.0	–	–	–	–	–	–	–
50	0.0	0.0	0.1	1.8	–	–	–	–	–	–	–
	0.0	0.1	0.5	12.9	–	–	–	–	–	–	–

A change of variables to exponential sums

Set $\mathbf{z} = \mathrm{e}^{\mathbf{y}} = (\mathrm{e}^{y_1}, \dots, \mathrm{e}^{y_d})$ with complex variables $y_1, \dots, y_d$.

The generating function

$$g(P; \mathbf{z}) = \sum_{\mathbf{x} \in P \cap \mathbf{Z}^d} \mathbf{z}^{\mathbf{x}} = \sum_i \epsilon_i \frac{\mathbf{z}^{\mathbf{u}^i}}{\prod_{j=1}^d (1 - \mathbf{z}^{\mathbf{v}^{i,j}})}$$

changes to the exponential sum

$$S(P; \mathbf{y}) = \sum_{\mathbf{x} \in P \cap \mathbf{Z}^d} \exp\{\langle \mathbf{y}, \mathbf{x} \rangle\}$$

(discrete all-sided Laplace transform of the indicator function of P)

Intermediate sums

The idea to use intermediate sums appeared first in Barvinok (2006), for the computation of the top k Ehrhart coefficients of a rational simplex in varying dimension. We take them to the generating-function (Laplace-transform) level and use them for mixed-integer optimization.

Theorem (S^L version of the Khovanskii–Pukhlikov theorem)

Let $L \subseteq V$ be a rational subspace. There exists a unique valuation S^L which to every rational polyhedron $P \subset V$ associates a meromorphic function with rational coefficients $S^L(P) \in \mathcal{M}(V^*)$ so that the following properties hold:

1. *If P contains a line, then $S^L(P) = 0$.*
2. $$S^L(P)(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{P \cap (y+L)} e^{\langle \xi, x \rangle} \, dm_L(x),$$ *for every $\xi \in V^*$ such that the above sum converges.*
3. *For every point $s \in \Lambda + L$, we have* $$S^L(s+P)(\xi) = e^{\langle \xi, s \rangle} S^L(P)(\xi).$$

Intermediate sums for "parallel" cones

V. Baldoni, N. Berline, J. De Loera, **Kö.**, M. Vergne: Computation of the highest coefficients of weighted Ehrhart quasi-polynomials of rational polyhedra.

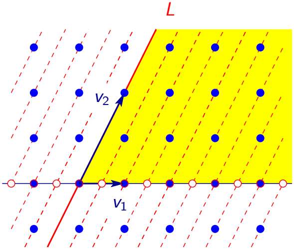
Let

$$C = \operatorname{cone}\{v_1, \dots, v_d\}$$

be a simplicial cone with one face parallel to the subspace

$$L = L_{I^c} = \operatorname{lin}\{v_i : i \in I^c\}$$

.



Theorem

The intermediate sum for the full cone $s + C$ breaks up into the product

$$S^{L_{I^c}}(s + C)(\xi) = S(s_I + C_I, \Lambda_I)(\xi)\, I(s_{I^c} + C_{I^c}, L_{I^c} \cap \Lambda)(\xi),$$

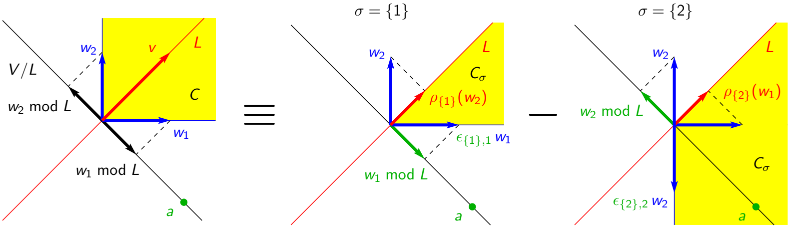
where

$$I(s_{I^c} + C_{I^c}, L_{I^c} \cap \Lambda)(\xi) = e^{\langle \xi, s_{I^c} \rangle} \operatorname{vol}_{L_{I^c} \cap \Lambda}(\mathbf{B}_{I^c})(-1)^{|I^c|} \prod_{j \in I^c} \frac{1}{\langle \xi, v_j \rangle}$$

is the integral over the slice $(s + C) \cap L_{I^c}$.

Arbitrary cones and subspaces: Use Brion–Vergne decomposition

M. Brion and M. Vergne, Residue formulae, vector partition functions and lattice points in rational polytopes, J. Amer. Math. Soc. 10 (1997), 797–833



Theorem

Let L be a linear subspace of $V = \mathbf{R}^d$. Let C be a full dimensional simplicial cone in V with generators $w_1, \dots, w_d$. You can't read this: Let $a \in V/L$ be generic, belong to the projection of C on V/L. For $\sigma \in \mathcal{B}(C, L)$, let $a = \sum_{j\in\sigma} a_{\sigma,j}(w_j \bmod L)$. Let $\epsilon_{\sigma,j}$ be the sign of $a_{\sigma,j}$ and $\epsilon(\sigma) = \prod_{j\in\sigma} \epsilon_{\sigma,j}$. Denote by $C_\sigma \subset V$ the cone with edge generators $\epsilon_{\sigma,j} w_j$ for $j \in \sigma$, and $\rho_\sigma(w_k)$ for $k \notin \sigma$, Then we have the following relation between indicator functions of cones.

$$[C] \equiv \sum_{\sigma\in\mathcal{B}(C,L)} \epsilon(\sigma)\,[C_\sigma] \;\; mod\ \mathcal{L}(V). \qquad (1)$$

If codim L is fixed, can compute in polynomial time.

Short formula for intermediate valuations

V. Baldoni, N. Berline, J. De Loera, **Kö.**, M. Vergne: Computation of the highest coefficients of weighted Ehrhart quasi-polynomials of rational polyhedra.
V. Baldoni, N. Berline, **Kö.**, M. Vergne: Intermediate Sums on Polyhedra: Computation and Real Ehrhart Theory.

Theorem (Short formula for $S^L(P)(\xi)$)

Fix a non-negative integer k_0. There exists a polynomial time algorithm for the following problem. Given the following input:

- (I$_1$) *a number d in unary encoding,*
- (I$_2$) *a simple polytope $P \subset \mathbf{R}^d$, represented by its vertices, rational vectors $s_1, \dots, s_{d+1} \in \mathbf{Q}^d$ in binary encoding,*
- (I$_3$) *a subspace $L \subseteq \mathbf{Q}^d$ of codimension k_0, represented by $d - k_0$ linearly independent vectors $b_1, \dots, b_{d-k_0} \in \mathbf{Q}^d$ in binary encoding,*

compute the rational data such that we have the following equality of meromorphic functions:

$$S^L(P)(\xi) = \sum_{n \in N} \alpha^{(n)} \left(e^{\langle \xi, s^{(n)} \rangle} \prod_{i=1}^{k_0} T(z_i^{(n)}, \langle \xi, w_i^{(n)} \rangle) \right) \frac{1}{\prod_{i=1}^{d} \langle \xi, w_i^{(n)} \rangle}.$$

From this, we can extract intermediate sums of polynomial functions using series expansions.

Ehrhart polynomials from generating functions

If vertices are lattice points and dilation factors n are integers: When P is replaced with nP, the vertex s is replaced with ns but the tangent cone C_s does not change. We replace ξ by $t\xi$ with $t \in \mathbf{C}$. We obtain

$$\sum_{x \in nP \cap \Lambda} \mathrm{e}^{\langle t\xi, x \rangle} = \sum_{s \in \mathcal{V}(P)} S(ns + C_s)(t\xi) = \sum_{s \in \mathcal{V}(P)} \mathrm{e}^{nt\langle \xi, s \rangle} S(C_s)(t\xi). \qquad (*)$$

The decomposition into homogeneous components (of equal ξ-degree) gives

$$S(C_s)(t\xi) = t^{-d} I(C_s)(\xi) + t^{-d+1} S(C_s)_{[-d+1]}(\xi) + \cdots + t^k S(C_s)_{[k]}(\xi) + \cdots .$$

Expanding the exponential, we find that the t^M-term in the right-hand side of $(*)$ is equal to

$$\sum_{k=0}^{M+d} (nt)^{M+d-k}\, t^{-d+k} \frac{\langle \xi, s \rangle^{M+d-k}}{(M+d-k)!} S(C_s)_{[-d+k]}(\xi).$$

Thus:

$$\sum_{x \in nP \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{s \in \mathcal{V}(P)} n^{M+d} \frac{\langle \xi, s \rangle^{M+d}}{(M+d)!} I(C_s)(\xi) + n^{M+d-1} \frac{\langle \xi, s \rangle^{M+d-1}}{(M+d-1)!} S(C_s)_{[-d+1]}(\xi) + \cdots + S(C_s)_{[M]}(\xi).$$

Approximation Theorem

Let $\mathcal{J}^d_{\geq d_0}$ be the poset of subsets of $\{1,\ldots,d\}$ of cardinality $\geq d_0$.

Patching functions λ

For $1 \leq i \leq d$, let $F_i(z) \in \mathbf{C}[[z]]$ be any formal power series (in one variable) with constant term equal to 1. Then

$$\prod_{1\leq i\leq d} F_i(z_i) \equiv \sum_{I\in\mathcal{J}^d_{\geq d_0}} \lambda(I) \prod_{i\in I^c} F_i(z_i) \quad \text{mod terms of } z\text{-degree } \geq d-d_0+1.$$

Theorem (Approximation by a patched generating function)

Let $C \subset V$ be a rational simplicial cone with edge generators $v_1,\ldots,v_d$. Let $s \in V_{\mathbf{Q}}$. Let $I \mapsto \lambda(I)$ be a patching function on the poset $\mathcal{J}^d_{\geq d_0}$.
For $I \in \mathcal{J}^d_{\geq d_0}$ let L_I be the linear span of $\{v_i\}_{i\in I}$. Then we have

$$S(s+C,\Lambda)(\xi) \equiv A^\lambda(s+C,\Lambda)(\xi) := \sum_{I\in\mathcal{J}^d_{\geq d_0}} \lambda(I)\, S^{L_I}(s+C,\Lambda)(\xi)$$

$$\textit{mod terms of } \xi\textit{-degree } \geq -d_0+1.$$

Approximation Theorem: Example

Let C be the first quadrant in $\mathbf{R}^2$, and $d_0 = 1$. Thus $\mathcal{J}^2_{\geq 1}$ consists of three subsets, $\{1\}, \{2\}$ and $\{1, 2\}$. A patching function is given by $\lambda(\{i\}) = 1$ and $\lambda(\{1, 2\}) = -1$. We consider the affine cone $s + C$ with $s = (-\frac{1}{2}, -\frac{1}{2})$. Let $\xi = (\xi_1, \xi_2)$. We have

$$I(s_i + C_{\{i\}})(\xi) = \frac{-\mathrm{e}^{-\xi_i/2}}{\xi_i}, \qquad I(s + C)(\xi) = \frac{\mathrm{e}^{-\xi_1/2-\xi_2/2}}{\xi_1\xi_2},$$

$$S(s_i + C_{\{i\}})(\xi) = \frac{1}{1-\mathrm{e}^{\xi_i}}, \qquad S(s + C)(\xi) = \frac{1}{(1-\mathrm{e}^{\xi_1})(1-\mathrm{e}^{\xi_2})}.$$

The approximation theorem claims that

$$\frac{1}{(1-\mathrm{e}^{\xi_1})(1-\mathrm{e}^{\xi_2})} \equiv \frac{1}{1-\mathrm{e}^{\xi_2}} \cdot \frac{-\mathrm{e}^{-\xi_1/2}}{\xi_1} + \frac{1}{1-\mathrm{e}^{\xi_1}} \cdot \frac{-\mathrm{e}^{-\xi_2/2}}{\xi_2} - \frac{\mathrm{e}^{-\xi_1/2-\xi_2/2}}{\xi_1\xi_2}$$

mod terms of ξ-degree ≥ 0.

Indeed, the difference between the two sides is equal to

$$\Big(\frac{1}{1-\mathrm{e}^{\xi_1}} + \frac{\mathrm{e}^{-\xi_1/2}}{\xi_1}\Big)\Big(\frac{1}{1-\mathrm{e}^{\xi_2}} + \frac{\mathrm{e}^{-\xi_2/2}}{\xi_2}\Big)$$

which is analytic near 0.

"Top Ehrhart" theorem

For every fixed number $k_0 \in \mathbf{N}$, there exists a polynomial-time algorithm for the following problem.

Input:

(I$_1$) a simple polytope P, given by its vertices, rational vectors $\mathbf{s}_j \in \mathbf{Q}^d$ for $j \in \mathcal{V}$ (a finite index set) in binary encoding,

(I$_2$) a rational vector $\ell \in \mathbf{Q}^d$ in binary, a number $M \in \mathbf{N}$ in unary encoding.

Output, in binary encoding,

(O$_1$) polynomials $f^{\gamma,m} \in \mathbf{Q}[r_1, \dots, r_{k_0}]$ and integer numbers $\zeta_i^{\gamma,m} \in \mathbf{Z}$, $q_i^{\gamma,m} \in \mathbf{N}$ for $\gamma \in \Gamma$ (a finite index set) and $m = M + d - k_0, \dots, M + d$ and $i = 1, \dots, k_0$,

such that the Ehrhart quasi-polynomial

$$E(P, \ell, M; n) = \sum_{x \in nP \cap \Lambda} \frac{\langle \ell, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(P, \ell, M; \{n\}_q)\, n^m$$

agrees in n-degree $\geq M + d - k_0$ with the quasi-polynomial

$$\sum_{\gamma \in \Gamma} \sum_{m=M+d-k_0}^{M+d} f^{\gamma,m}\Big(\{\zeta_1^{\gamma,m} n\}_{q_1^{\gamma,m}}, \dots, \{\zeta_{k_0}^{\gamma,m} n\}_{q_{k_0}^{\gamma,m}}\Big)\, n^m.$$

$E_m(P, \ell, M, \{n\}_q)$, when P is the simplex in $\mathbf{R}^5$ with vertices:

$$(0,0,0,0,0),\ (\tfrac{1}{2},0,0,0,0),\ (0,\tfrac{1}{2},0,0,0),\ (0,0,\tfrac{1}{2},0,0),\ (0,0,0,\tfrac{1}{6},0),\ (0,0,0,0,\tfrac{1}{6}).$$

We consider the linear form ℓ on $\mathbf{R}^5$ given by the scalar product with $(1,1,1,1,1)$. If $M = 0$, the coefficients of $E_m(P, \ell, M = 0; \{n\}_q)$ are just the coefficients of the unweighted Ehrhart quasi-polynomial $S(nP, 1)$. We obtain

$$S(nP,1) = \frac{1}{34560}n^5 + \Big(\frac{5}{3456} - \frac{1}{6912}\{n\}_2\Big)n^4 + \Big(\frac{139}{5184} - \frac{5}{864}\{n\}_2 + \frac{1}{3456}(\{n\}_2)^2\Big)n^3 + \cdots .$$

Now if $M = 1$, all integral points $(x_1, x_2, x_3, x_4, x_5)$ are weighted with the function $h(x) = x_1 + x_2 + x_3 + x_4 + x_5$, and we obtain

$$S(nP,h) = \frac{11}{1244160}n^6 + \Big(\frac{19}{41472} - \frac{11}{207360}\{n\}_2\Big)n^5 + \Big(\frac{553}{62208} - \frac{95}{41472}\{n\}_2 + \frac{11}{82944}(\{n\}_2)^2\Big)n^4 + \cdots .$$

Note period collapse: Although $q = 6$ is the smallest integer such that qP is a lattice polytope, only periodic functions of n mod 2 enter in the top three Ehrhart coefficients.

Random lattice simplices.

	Average runtime (CPU seconds)			
	Full Ehrhart polynomial			Top 3 coefficients
Dimension	Dual	Primal	Primal_{1000}	
3	0.16	0.10	0.04	1.12
4	28.00	4.68	0.28	4.31
5		317.5	5.8	13.4
6			198.0	37.4
7				103
8				294
9				393
10				1179
11				1681

LattE command line options for Ehrhart computations

Dual method (default):

```
count --ehrhart-polynomial
```

Primal "irrational" method:

```
count --ehrhart-polynomial --irrational-primal
```

Primal "irrational" method with stopped decomposition:

```
count --ehrhart-polynomial --irrational-primal --maxdet=1000
```

Ehrhart quasi-polynomial, incremental computation of coefficients:

```
integrate --valuation=top-ehrhart
```

Same, but output formulas valid for arbitrary real dilations:

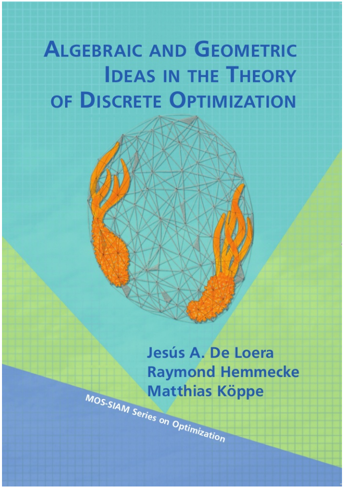
```
integrate --valuation=top-ehrhart --real-dilations
```

V. Baldoni, N. Berline, J. A. De Loera, M. Köppe, and M. Vergne.
How to integrate a polynomial over a simplex.
Mathematics of Computation, 80(273):297–325, 2011.

V. Baldoni, N. Berline, J. A. De Loera, M. Köppe, and M. Vergne.
Computation of the highest coefficients of weighted Ehrhart quasi-polynomials of rational polyhedra.
Foundations of Computational Mathematics, 12:435–469, 2012.

V. Baldoni, N. Berline, M. Köppe, and M. Vergne.
Intermediate sums on polyhedra: Computation and real Ehrhart theory.
Mathematika, 59(1):1–22, September 2013.

J. A. De Loera, B. Dutra, M. Köppe, S. Moreinis, G. Pinto, and J. Wu.
Software for exact integration of polynomials over polyhedra.
Computational Geometry: Theory and Applications, 46(3):232–252, 2013.



available now