PRELIMINARY EXAM IN ANALYSIS
FALL, 2016

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let A and B be two bounded, self-adjoint operators on a Hilbert space $\mathcal{H}$. Prove that

$$||Af||\,||Bf|| \geq \frac{|\langle [A,B]f, f\rangle|}{2},$$

where $[A,B] = AB - BA$ is the commutator of A and B. In addition, prove that equality holds if and only if $Af = cBf$ for some $c \in \mathbb{R}$.

2. Let $f : \mathbb{T} \to \mathbb{C}$ be a C^1 function such that $\int\limits_{-\pi}^{\pi} f(x)dx = 0$. Show that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

3. Let $y = \{a_n\}_{n=1}^{\infty}$ be a sequence of real-valued scalars and assume that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$. Show that $y \in \ell^2(\mathbb{N})$.

4. Let $f \in L^1(\mathbb{R})$ and assume that $\hat{f}$, the Fourier transform of f, is supported on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Let $\operatorname{sinc}(x) = \frac{\sin x}{x}$. Prove that

$$f(x) = \sum_{n \in \mathbb{Z}} f(n)\operatorname{sinc}(x - n),$$

where the equality holds in the L^2-sense. (Here, the Fourier transform of a function g is given by $\hat{g}(\omega) = \int\limits_{-\infty}^{\infty} g(x)e^{-2\pi i x\omega}dx$.)

5. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and fix $1 < p < \infty$. Given $f \in L^p([0,1])$, define $Tf : [0,1] \to \mathbb{R}$ by

$$Tf(x) = \int_0^1 K(x,y)f(y)dy.$$

(a) Prove that Tf is a continuous function.

(b) Prove that the image under T of the unit ball in $L^p([0,1])$ is precompact in $C([0,1])$.

6. Let D denote the closed unit disk in $\mathbb{C}$, and consider the complex Hilbert space

$$\mathcal{H} \stackrel{\text{def}}{=} \left\{ f : D \to \mathbb{C} \,\middle|\, f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } ||f||_{\mathcal{H}}^2 \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (1 + k^2)|a_k|^2 < \infty \right\}.$$

Prove that the linear functional $L : \mathcal{H} \to \mathbb{C}$ defined by $L(f) = f(1)$ is bounded, and find an element $g \in \mathcal{H}$ such that $L(f) = \langle g, f\rangle$. (In other words, so that g represents L as in the Riesz representation theorem.)