Winter 2003 Mathematics Graduate Program Preliminary Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Analysis

Problem 1. Prove or disprove: Any linear bounded operator in a complex Hilbert space can be written as a linear combination of two self-adjoint operators. (Hint: Consider first the finite-dimensional case.)

Problem 2. Consider the Hilbert space $L^2[-1,1]$.
(i) Find the orthogonal complement of the space of all polynomials. (Hint: Use the Stone-Weierstrass theorem.)
(ii) Find the orthogonal complement of the space of polynomials in x^2.

Problem 3. Consider the space of all polynomials on $[0,1]$ vanishing at the origin, with the sup norm. Prove that the space is not complete and find its completion.

Problem 4. Prove that $\mathbb{R}^1$ with the metrics
(i) $\rho(x,y) = |\arctan(x) - \arctan(y)|$
or
(ii) $\rho(x,y) = |\exp(x) - \exp(y)|$
is incomplete, and find the completion in each case.

Problem 5. Consider a continuous mapping of the closed unit square $[0,1] \times [0,1]$ into some metric space X. Prove that the image of the square under such a mapping is compact.

Problem 6. Prove or disprove:
$C[0,1]$ with the usual sup norm is a Hilbert space. (Hint: Consider two continuous functions with disjoint supports and calculate the norm of their sum.)

2. Algebra and Linear Algebra

Problem 7. Suppose that A and B are complementary subgroups in a group G, meaning that $G = AB$ and that A and B intersect trivially (but perhaps neither A nor B is normal). Show that each right coset of A intersects each left coset of B in exactly one element.

Problem 8. Find all automorphisms of $\mathbb{Z}[x]$, the ring of polynomials over $\mathbb{Z}$.

Problem 9. Let R be a commutative ring with identity and prime characteristic p. Show that the map

$$\begin{aligned} \varphi : R &\to R \\ r &\mapsto r^p \end{aligned}$$

is a homomorphism of rings (it's called the Frobenius homomorphism).

Problem 10. Find a subgroup of the unit quaternions Q which is a circle. Argue a corollary: The 3-sphere is the union of disjoint circles.

Problem 11. Let G be a group and let H be a subgroup of G with finite index $n > 1$.
a. Show that the map $G \times G/H \to G/H$ defined by $(g, aH) \mapsto gaH$ gives an action of G on the space G/H of left cosets of H in G.
b. Show that if, in addition, G is finite and the order of G does not divide $n!$, then G is not simple.
c. Can a group of order $2^2 \cdot 3 \cdot 19^2$ be simple?

Problem 12. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$. Find a matrix B so that BAB^{-1} is diagonal.