Spring 2010: MA Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
1. Let R be a commutative ring with identity. Recall that an ideal I of R is said to be *radical* if for every $x \in R$ such that $x^n \in I$ for some n, we have $x \in I$. Prove that I is radical if and only if I is equal to the intersection of the prime ideals containing it. Hint for one direction: if I is radical and $x \notin I$ (equivalently, no power of x is in I), by Zorn's lemma there is a largest ideal J such that no power of x is in J (this means that no ideal with the same property strictly contains J, *not* that J contains every ideal with this property). Show that J is a prime ideal.

Problem 2:
Recall the definition of a projective module M over a ring R: Whenever A and B are two other R-modules, and whenever $f : M \to A$ and $g : B \to A$ where g is surjective, are module homomorphisms, then f factors as $f = g \circ h$. For instance, $M = \mathbb{R}$ is a module over the polynomial ring $\mathbb{R}[x]$, where x acts by multiplication by 0. Is this a projective module?

Problem 3:
Prove that the group $\langle x, y : x^2 = y^3 \rangle$ is not trivial.

Problem 4: Prove that every finite group of order greater than 2 has a non-trivial automorphism.

Problem 5: Prove that if R is an integral domain with a finite group of units $R^\times$, then the group of units is cyclic.

Problem 6: Give an example of an irreducible polynomial of degree n (for some n) over $\mathbb{Q}$ whose Galois group does not have $n!$ elements.