Winter 2009: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Let $1 < p < 2$.
(a) Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^p(\mathbb{R})$ and a function $g \in L^2(\mathbb{R})$ such that $g \notin L^p(\mathbb{R})$.
(b) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, prove that $f \in L^p(\mathbb{R})$

Problem 2:
(a) State the Weierstrass approximation theorem.
(b) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 x^n f(x)\, dx = 0$$

for all non-negative integers n. Prove that $f = 0$.

Problem 3:
(a) Define strong convergence, $x_n \to x$, and weak convergence, $x_n \rightharpoonup x$, of a sequence (x_n) in a Hilbert space $\mathcal{H}$.
(b) If $x_n \rightharpoonup x$ weakly in $\mathcal{H}$ and $\|x_n\| \to \|x\|$, prove that $x_n \to x$ strongly.
(c) Give an example of a Hilbert space $\mathcal{H}$ and sequence (x_n) in $\mathcal{H}$ such that $x_n \rightharpoonup x$ weakly and

$$\|x\| < \liminf_{n\to\infty} \|x_n\|.$$

Problem 4: Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a complex Hilbert space $\mathcal{H}$ such that

$$T^* = -T, \qquad T^2 = -I$$

and $T \neq \pm iI$. Define

$$P = \frac{1}{2}(I + iT), \qquad Q = \frac{1}{2}(I - iT).$$

(a) Prove that P, Q are orthogonal projections on $\mathcal{H}$.

(b) Determine the spectrum of T, and classify it.

Problem 5: Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R} \to \mathbb{C}$. Define an operator $H : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\widehat{(Hf)}(\xi) = i\mathrm{sgn}(\xi)\hat{f}(\xi) = \begin{cases} i\hat{f}(\xi) & \text{if } \xi > 0, \\ -i\hat{f}(\xi) & \text{if } \xi < 0, \end{cases}$$

where $\hat{f}$ denotes the Fourier transform of f.

(a) Why is $Hf \in L^2(\mathbb{R})$ for any $f \in \mathcal{S}(\mathbb{R})$?

(b) If $f \in \mathcal{S}(\mathbb{R})$ and $Hf \in L^1(\mathbb{R})$, show that

$$\int_{\mathbb{R}} f(x)\, dx = 0.$$

(Hint: you may want to use the Riemann-Lebesgue Lemma)

Problem 6: Let Δ denote the Laplace operator in $\mathbb{R}^3$.

(a) Prove that

$$\lim_{\epsilon \to 0} \int_{B_\epsilon^c} \frac{1}{|\mathbf{x}|} \Delta f(\mathbf{x}) d\mathbf{x} = 4\pi f(0), \quad \forall f \in \mathcal{S}(\mathbb{R}^3)$$

where B_ϵ^c is the complement of the ball of radius ϵ centered at the origin.

(b) Find the solution u of the Poisson problem

$$\Delta u = 4\pi f(\mathbf{x}), \quad \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0$$

for $f \in \mathcal{S}(\mathbb{R}^3)$.