Preliminary Exam in Analysis

Spring, 2016

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1. Let $f(x)$ be a continuous function on $\mathbb{R}$ such that for any polynomial $P(x)$ we have

$$\int_{\mathbb{R}} f(x)P(x)dx = 0.$$

Show that $f(x)$ is identically zero.

Problem 2. Let M be a multiplication on $L^2(\mathbb{R})$ defined by

$$Mf(x) = m(x)f(x),$$

where $m(x)$ is continuous and bounded. Prove that M is a bounded operator on $L^2(\mathbb{R})$ and that its spectrum is given by

$$\sigma(M) = \{m(x) : x \in \mathbb{R}\}^{cl},$$

where A^{cl} denotes the closure of A. Can M have eigenvalues?

Problem 3. Show that the closed unit ball of a Hilbert space H is compact if and only if dim H is finite.

Problem 4. Suppose f is a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ which satisfies the normalizing condition $\int_{-\infty}^{+\infty} |f(x)|^2 dx = 1$. Let $\hat{f}$ denote the Fourier transform of f. Show that

$$\left(\int_{-\infty}^{+\infty} x^2 |f(x)|^2 dx \right) \left(\int_{-\infty}^{+\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega \right) \geq \frac{1}{16\pi^2}.$$

Problem 5. Let $f(x) \in W^{1,1}([0,1])$. Let $\bar{f} = \int_0^1 f(x)dx$. Show that

$$\|f - \bar{f}\|_{L^1([0,1])} \leq 2\|f(x)x(1-x)\|_{L^1([0,1])}.$$

Problem 6. Let H be a Hilbert space and let U be a unitary operator, that is surjective and isometric, on H. Let $I = \{v \in H : Uv = v\}$ be the subspace of invariant vectors with respect to U.

a) Show that $\{Uw - w : w \in H\}$ is dense in $I^\perp$ and that I is closed.

b) Let P be the orthogonal projection onto I. Show that

$$\frac{1}{N}\sum_{n=1}^{N} U^n v \to Pv.$$