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Algebra Prelim Exam for 2004-05

Instructions: *explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let V be a nonzero finite-dimensional complex vector space, and let $f, g \colon V \to V$ be two linear maps. Prove that there exists a non-zero vector $v \in V$ such that the vectors $f(v), g(v)$ are collinear (that is, $\dim(\mathrm{Span}(f(v), g(v))) \leq 1$).

Warning: neither of f, g is assumed to be non-singular.

Problem 2. Prove that an infinite simple (not having proper normal subgroups) group does not have proper subgroups of finite index.

Problem 3. Let G be a finitely generated abelian group. Prove that there are no non-zero homomorphisms $\mathbb{Q} \to G$ (here $\mathbb{Q}$ is the additive group of rational numbers).

Problem 4. Prove of disprove: $\mathbb{C}[x, y]$ is a PID (Principal Ideal Domain).

Problem 5. Give examples of each of the following

a) a finite nonabelian group

b) an infinite nonabelian group

c) a group that is not finitely generated

d) a group that is not solvable

Problem 6.

a) Construct infinitely many non-isomorphic quadratic extensions of $\mathbb{Q}$.

b) Use (a) to show that the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ does not have finitely generated abelianization.

Here $\mathbb{Q}$ is the field of rational numbers.

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Analysis Prelim Exam for 2004-05

Instructions: *explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let $f : (-1, 1) \to \mathbb{R}$ be a differentiable function such that there exists a limit

$$\lim_{x\to 0} \frac{f(x)}{x^2} = L \in \mathbb{R}.$$

Does it follow that the second derivative $f''(0)$ exist and equals L? Give a proof or a counter-example.

Problem 2. For functions from $[0, 1]$ to $\mathbb{R}$ do the following:

a) Define what it means for a sequence of functions to converge uniformly.

b) Explain what it means for a sequence of functions to be equicontinuous.

c) Does every equicontinuous sequence of functions converge uniformly to a continuous function? Is the converse true? Give examples or prove.

Problem 3. Define two sequences of functions, (f_n) and (g_n), on the interval $[0, 1]$ as follows:

$$\begin{aligned} f_n(x) &= (1 + \cos 2\pi x)^{1/n}, \quad n \geq 1 \\ g_n(x) &= (1 + \frac{1}{2}\cos 2\pi x)^{1/n}, \quad n \geq 1 \end{aligned}$$

a) What are the pointwise limits, f and g, of the sequences (f_n) and (g_n) respectively?

b) For each sequence, determine whether the convergence is uniform. Explain your answer.

Problem 4. Let X and Y be a topological spaces. Prove that if $f : X \to Y$ is continuous and X is compact, then $f(X)$ is also compact.

Problem 5. Let X be a normed linear space and let X^* be its topological dual. Suppose that for $x, y \in X$ are such that for all $\varphi \in X^*$, $\varphi(x) = \varphi(y)$. Prove that $x = y$.

Problem 6. Consider the following equation for an unknown function $f : [0, 1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x - y)^2 f(y)\, dy + \frac{1}{2}\sin(f(x)) \tag{1}$$

Prove that there exists a number $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all continuous functions g on $[0, 1]$, the equation (1) has a unique continuous solution.