

Spring 2010: PhD Algebra Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1:

1. Let R be a commutative ring with identity. Recall that an ideal I of R is said to be *radical* if for every $x \in R$ such that $x^n \in I$ for some n , we have $x \in I$. Prove that I is radical if and only if I is equal to the intersection of the prime ideals containing it. Hint for one direction: if I is radical and $x \notin I$ (equivalently, no power of x is in I), by Zorn's lemma there is a largest ideal J such that no power of x is in J (this means that no ideal with the same property strictly contains J , *not* that J contains every ideal with this property). Show that J is a prime ideal.

Problem 2:

Recall the definition of a projective module M over a ring R : Whenever A and B are two other R -modules, and whenever $f : M \rightarrow A$ and $g : B \rightarrow A$ where g is surjective, are module homomorphisms, then f factors as $f = g \circ h$. For instance, $M = \mathbb{R}$ is a module over the polynomial ring $\mathbb{R}[x]$, where x acts by multiplication by 0. Is this a projective module?

Problem 3:

Prove that the group $\langle x, y : x^2 = y^3 \rangle$ is not trivial.

Problem 4: Prove that every finite group of order greater than 2 has a non-trivial automorphism.

Problem 5: Prove that if R is an integral domain with a finite group of units R^\times , then the group of units is cyclic.

Problem 6: Give an example of an irreducible polynomial of degree n (for some n) over \mathbb{Q} whose Galois group does not have $n!$ elements.