Fall 2007: MA Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Suppose that ρ is a complex matrix representation of a finite group G. Show that every matrix $\rho(g)$ is diagonalizable.

Problem 2. Consider the ring $\mathbb{R}[[x]]$ of formal power series in x with real coefficients. Namely, this is the set of all infinite series $a_0 + a_1x + a_2x^2 + \ldots$ with no conditions on convergence. What are the units (invertible elements) in this ring? What are the ideals?

Problem 3. If H is a subgroup of a group G, then the normalizer $N(H)$ of H is defined as the set of g in G such that $gHg^{-1} = H$. It is the largest subgroup of G that contains H as a normal subgroup. Let G be the symmetric group S_7; let $H \subset G$ be the cyclic subgroup generated by a 7-cycle.

Find the number of elements of the normalizer $N(H)$ of H in G.

Problem 4. Compute the number of groups of order ≤ 1029 each of which contains exactly three elements of order 3.

Problem 5. Show that the group $\mathbb{Q}$ of rational numbers (with respect to the addition operation) is not finitely generated.

Problem 6. Show that

$$\det(\exp(A)) = e^{tr(A)}$$

for every complex $n \times n$ matrix A, where $\exp(A)$ is defined as

$$\exp(A) = 1 + A + \frac{A^2}{2} + \ldots + \frac{A^k}{k!} + \ldots$$