Fall 2011: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
Show that there is no commutative ring with the identity whose additive group is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Problem 2:
Let $p \neq 2$ be prime and F_p be the field of p elements.
(a) How many elements of F_p have square roots in F_p?
(b) How many have cube roots in F_p?

Problem 3:
Prove that every finite group is isomorphic to a certain group of permutations (a subgroup of S_n for some n).

Problem 4:
Let G be the subgroup of S_{12} generated by $a = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)$ and $b = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8)$. Find the order of G, the number of conjugacy classes of G, and the character table of G.

Problem 5:
Prove or disprove: If the group G of order 55 acts on a set X of 39 elements then there is a fixed point.

Problem 6:
Prove or disprove: $(\mathbb{Z}/35\mathbb{Z})^* \cong (\mathbb{Z}/39\mathbb{Z})^* \cong (\mathbb{Z}/45\mathbb{Z})^* \cong (\mathbb{Z}/70\mathbb{Z})^* \cong (\mathbb{Z}/78\mathbb{Z})^* \cong (\mathbb{Z}/90\mathbb{Z})^*$. Here $(\mathbb{Z}/n\mathbb{Z})^*$ is the group of units in $\mathbb{Z}/n\mathbb{Z}$.