Winter 2004 Mathematics Graduate Program Preliminary Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using. (If you find a problem ambiguous or unclear, explain why and state what assumptions you are making.)

1. Analysis

Problem 1. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a C^∞-function if f has continuous partial derivatives of all orders.

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \exp[1/(x^2 - 1)]$ if $|x| < 1$ and $f(x) = 0$ if $|x| \geq 1$. Show that f is a C^∞-function such that $\text{supp}(f) = [-1, 1]$. (Induction and L'Hospital's rule are needed here.)

(b) For $\epsilon 0$ and $a \in \mathbb{R}$, show that the function $g(x) = f[(x-a)/\epsilon]$ is also a C^∞-function with $\text{supp}(g) = [a - \epsilon, a + \epsilon]$.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable with respect to the Lebesgue measure. Show that the function $g : [0, \infty) \to \mathbb{R}$ defined by

$$g(t) = \sup\left\{\int |f(x+y) - f(x)|dx \ : \ |y| \leq t\right\}$$

for $t \geq 0$ is continuous at $t = 0$.

Problem 3. Consider the following theorem:

> Let $1 \leq p < \infty$ and $f \in L^p$, and let $\{f_n\}$ be a sequence in L^p such that $f_n \to f$ a.e. If $\lim_{n\to\infty} \|f_n\|_{L^p} = \|f\|_{L^p}$, then $\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$.

Show by an example that this theorem is false when $p = \infty$.

Problem 4. On $C^0([0,1])$ consider the two norms

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \qquad \|f\|_1 = \int_0^1 |f(x)|dx.$$

Show that the identity operator $I : (C^0[0,1], \|\cdot\|_\infty) \to (C^0[0,1], \|\cdot\|_1)$ is continuous and onto, but not open. Why does this not contradict the open mapping theorem?

Problem 5. Let $\mathcal{H}$ be a Hilbert space. For a subset A of $\mathcal{H}$, let $A^\perp$ denote the orthogonal complement of A.

(a) Prove that for any subset A, $(A^\perp)^\perp$ is the closed linear span of A.

(b) Prove that if A is a closed convex subset of $\mathcal{H}$, then A contains a unique element of minimal norm.

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $X = X^* \in \mathcal{B}(\mathcal{H})$ be compact and such that

$$\frac{1}{3}X^3 - X^2 + \frac{2}{3}X = 0 \quad .$$

($\mathcal{B}(\mathcal{H})$ is the bounded linear operators on $\mathcal{H}$)

(a) Prove that X can be written as the sum of two orthogonal projections, i.e., there exist orthogonal projections P and Q, such that $X = P + Q$.

(b) Explain why any two orthogonal projections P and Q such that $X = P+Q$, are necessarily of finite rank?

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2. Algebra and Linear Algebra

Problem 7. For each of the following, give an example or prove that no such example is possible.

(1) A nonabelian group of order five.
(2) A nonabelian group of order four.
(3) An infinite group with a subgroup of order three
(4) Two finite groups of the same order that are not isomorphic
(5) A group G with a normal subgroup H such that the factor group G/H is not isomorphic to any subgroup of G.
(6) A group G with a subgroup of index two that is not a normal subgroup.

Problem 8. Prove of disprove: $\mathbb{C}[x,y]$ is a PID.

Problem 9. Let F be a field, n, m positive integers and A an $n \times n$ matrix with coefficents in F. Suppose that $A^m = 0$. Show that $A^n = 0$.

Problem 10. Consider the natural homomorphism from the ring of polynomials with coefficients in $\mathbb{Z}/5\mathbb{Z}$ into the ring of $\mathbb{Z}/5\mathbb{Z}$-valued functions on $\mathbb{Z}/5\mathbb{Z}$ (the evaluation homomorpism).
a) Prove that the kernel of this homomorphism is infinite.
b) Find at least one element belonging to the kernel.
c) Describe all the elements of the kernel.

Problem 11. Prove that $\mathbb{Q}$ is not a nontrivial direct sum $A \oplus B$ of two subgroups.

Problem 12. A complex matrix A has a characteristic polynomial $(x-3)^4 \times (x+4)^3$.
a)Calculate the trace of A and A^2
b)Describe all possible Jordan normal forms of A if it is known that A has two linearly independent eigenvectors with eigenvalue 3 and one eigenvector with eigenvalue -4.