Fall 2008: September 23

Preliminary Examination in Algebra for the PhilosophiæDoctor degree from the University of California at Davis

Instructions:

1. *Each problem is worth 10 points.*
2. *Explain your answers clearly to receive credit.*
3. *Use a seperate sheet for each problem.*

Problems:

1. (a) Show that if $f(x) \in \mathbf{Q}[x]$ is an irreducible (nonconstant) polynomial then $\mathbf{Q}[x,y]/(f(x))$ is a principal ideal domain.
   (b) Find a generator for the ideal (x, y).
   (c) Show that $x^2 - y^3 \in \mathbf{Q}[x,y]$ is irreducible and $(x,y) \subseteq \mathbf{Q}[x,y]/(f(x))$ is not principal.

2. Assume that p is prime, D and P are subgroups of a finite group F with D normal and having index ($[F : D]$) relatively prime to p and P a p-group. Show that $P \subseteq D$.

3. Let M be a 3 by 3 matrix of complex numbers with characteristic polynomial $x^3 + 5x^2 + 3x + (9 - i)$.
   (a) Find the determinant of M^2.
   (b) Find the trace of M^2.
   (c) Find the characteristic polynomial of M^2.

4. Assume that R is an integral domain (a commutative ring with no zero divisors) and J is a nonzero ideal of R viewed as an R-module. Is J always, sometimes, or never a direct sum of two nontrivial R-submodules?

5. If H is a subgroup of a group G, then a subgroup $K \subseteq G$ is called a *complement* of H if K has exactly one element in every left coset of H.
   (a) Show that if H is normal, then all complements of H are isomorphic to each other.
   (b) Show that the inclusion of symmetric groups $S_3 \subset S_4$ has two complements which are not isomorphic.

6. Show that every sequence of finite abelian groups $\ldots, A_2, A_1, A_0$ is the homology of some chain complex

$$\cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

   of free abelian groups (that is if $d_i : C_i \to C_{i-1}$ are the maps above then $d_{i+1}d_i = 0$ and A_i is isomorphic to $\ker(d_i)/\mathrm{im}(d_{i+1})$).

Fall 2008: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Prove that the dual space of c_o is ℓ^1, where

$$c_o = \{x = (x_n)\text{such that} \lim x_n = 0\}.$$

Problem 2: Let $\{f_n\}$ be a sequence of differentiable functions on a finite interval $[a, b]$ such that the functions themselves and their derivatives are uniformly bounded on $[a, b]$. Prove that $\{f_n\}$ has a uniformly converging subsequence.

Problem 3: Let $f \in L^1(R)$ and V_f be the closed subspace generated by the translates of f: $\{f(\cdot - y) | \forall y \in R\}$. Suppose $\hat{f}(\xi_0) = 0$ for some ξ_0. Show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(R)$, then $\hat{f}$ never vanishes.

Problem 4: (a) State the Stone-Weierstrass theorem for a compact Hausdorff space X.

(b) Prove that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$ where $g, h \in C(X)$ is dense in $C(X \times X)$.

Problem 5: For $r > 0$, define the dilation $d_r f : \mathbb{R} \to \mathbb{R}$ of a function $f : \mathbb{R} \to \mathbb{R}$ by $d_r f(x) = f(rx)$, and the dilation $d_r T$ of a distribution $T \in \mathcal{D}'(\mathbb{R})$ by

$$\langle d_r T, \phi \rangle = \frac{1}{r} \langle T, d_{1/r} \phi \rangle \qquad \text{for all test functions } \phi \in \mathcal{D}(\mathbb{R}).$$

(a) Show that the dilation of a regular distribution T_f, given by

$$\langle T_f, \phi \rangle = \int f(x)\phi(x)\, dx,$$

agrees with the dilation of the corresponding function f.

(b) A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the δ-distribution is homogeneous of degree -1.

(c) If T is a homogeneous distribution of degree n, prove that the derivative T' is a homogeneous distribution of degree $n-1$.

Problem 6: Let $\ell^2(\mathbb{N})$ be the space of square-summable, real sequences $x = (x_1, x_2, x_3, \dots)$ with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}.$$

Define $F : \ell^2(\mathbb{N}) \to \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} x_n^2 - x_n^4 \right\}$$

(a) Prove that F is differentiable at $x = 0$, with derivative $F'(0) : \ell^2(\mathbb{N}) \to \mathbb{R}$ equal to zero.

(b) Show that the second derivative of F at $x = 0$,

$$F''(0) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R},$$

is positive-definite, meaning that

$$F''(0)(h, h) > 0$$

for every nonzero $h \in \ell^2(\mathbb{N})$.

(c) Show that F does not attain a local minimum at $x = 0$.