## Sample Graduate Preliminary Exam (post F2001 system)

(The Ph.D. Preliminary Examination is a written exam, covering graduate material in analysis and algebra, as in 201ABC and 250AB.)

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

### 1. Analysis

*Problem* 1. a. Find a sequence of continuous functions on $[0,1]$ converging pointwise but not uniformly.
b. Prove that $C[0,1]$, the space of continuous functions on $[0,1]$, is not complete in the $L^1$ metric $\rho(f,g) = \int |f(x) - g(x)| dx$.

*Problem* 2. Let $T$ be the union of the graph of $\sin(x^{-1})$ on $(0,1)$ and $\{(0,0)\}$ with the topology induced from $R^2$. Prove that $T$ is connected but not arcwise connected.

*Problem* 3. a. Prove that in any Hilbert space the parallelogram identity takes place: $||x-y||^2 + ||x+y||^2 = 2||x||^2 + 2||y||^2$.
b. Prove that if $X$ is a Banach space over complex numbers such that the parallelogram identity takes place then one can make $X$ into a Hilbert space by defining a scalar product $(x,y)$ such that $(x,x) = ||x||^2$.

*Problem* 4. Suppose that $f \in L^1(\mu)$. Prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_A |f| \, d\mu < \epsilon \qquad \text{whenever} \quad \mu(A) < \delta.$$

*Problem* 5. State Hölder's inequality. Suppose that $1 \le p < q < r \le \infty$. Prove that if $u \in L^p \cap L^r$, then $u \in L^q$ and $\|u\|_{L^q} \le \|u\|_{L^p}^{\theta} \|u\|_{L^r}^{1-\theta}$, where $\theta = \dfrac{1/q - 1/r}{1/p - 1/r}$.

*Problem* 6. Let $C^k([0,1])$, $k \ge 1$, denote the set of all functions $[0,1] \to R^1$ with a continuous $k^{th}$ order derivative. Prove that $C^k([0,1])$ is dense in $C([0,1])$ with the supremum norm for all $k \ge 1$.

### 2. Algebra

*Problem* 7. Suppose a group $G$ acts on a set $X$. Show that if $x, y \in X$ belong to the same $G$-orbit, then $|G_x| = |G_y|$ where $G_x = \{g \in G : gx = x\}$ denotes the stabilizer of $x \in X$.

*Problem* 8. Prove or give a counter example: If $0 \to K \to G \to H \to 0$ is an exact sequence of groups with both $K$ and $H$ abelian, then $G$ is abelian.

*Problem* 9. Prove or disprove: $Z[x]$ is a Principle Ideal Domain.

*Problem* 10. Argue that the commutator subgroup of a group $G$ is characteristic, and so is the center.

*Problem* 11. a. Give an example of a finite field of order 3 and a field of order 9.
b. Let $F$ be a finite field. Show that the order of $F$ is equal to $p^n$ for some prime number $p$ and a positive integer $n$.
c. Show that the multiplicative group $F^\times$ consisting of the non-zero elements of a finite field $F$ is a cyclic group.