PhD Algebra Preliminary Exam for 2005-06

Instructions: *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let field E be a finite extension of a field F, and let R be a subring of E that contains F. Prove that R is a field.

Problem 2. Let R be a commutative ring with a unit. Prove that the following two properties of R are equivalent:

(a) If $a, b \in R$ and $a + b$ is invertible, then either a or b is invertible.

(b) R is local, that is, R has a unique maximal ideal.

Problem 3. Describe all possible Jordan forms of an $n \times n$ matrix X obeying $X^n = 0$.

Problem 4. Show that $\mathbb{Q}$ (the additive group of rational numbers) is not finitely generated.

Problem 5. Determine all finitely generated abelian groups which have finite group of automorphisms.

Problem 6. Suppose that $H \subset G$ is a subgroup which is contained in every nontrivial subgroup of G. Show that H is contained in the center of G.

Analysis Preliminary Exam for 2005-06

Instructions: *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. (a) Prove that there is no continuous map from the closed interval $[0, 1]$ onto the open interval $(0, 1)$.

(b) Construct a continuous map from the interval $(0, 1)$ onto the interval $[0, 1]$.

Problem 2. Define the Fibonacci sequence (x_n) of integers by $x_1 = 1$, $x_2 = 1$ and

$$x_{n+1} = x_n + x_{n-1}, \qquad n = 2, 3, \ldots.$$

Let $r_n = x_{n+1}/x_n$ be the ratio of successive terms. Prove that r_n converges to ϕ as $n \to \infty$, where ϕ is the "golden ratio"

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

Problem 3. Suppose that X is a complete metric space with metric d. Let $(F_n)_{n=1}^{\infty}$ be a decreasing (i.e. $F_{n+1} \subset F_n$ for all n) sequence of nonempty, closed subsets of X such that $\operatorname{diam} F_n \to 0$ as $n \to \infty$. Here,

$$\operatorname{diam} F = \sup\{d(x, y) \mid x, y \in F\}$$

denotes the diameter of F. Prove that the intersection $\cap_{n=1}^{\infty} F_n$ consists of a single point.

Problem 4. Let $f, g \in L^2(\mathbb{T})$, where $\mathbb{T}$ is the circle, identified with the quotient of $\mathbb{R}$ by the subgroup $2\pi\mathbb{Z}$. Let $*$ denote the convolution on $L^2(\mathbb{T})$. Show that the identity

$$f * g = \frac{1}{2}(f * f + g * g)$$

holds if and only if $f = g$.

Problem 5. Let $\{u_k \mid k \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. Find (i.e. characterize) all sequences of scalars $\{a_k \mid k \in \mathbb{N}\}$ such that the set $\{a_k u_k \mid k \in \mathbb{N}\}$ is compact in $\mathcal{H}$.

Problem 6. Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a compact linear operator on a complex Hilbert space $\mathcal{H}$. If $\lambda \in \mathbb{C}$ is nonzero, prove that the range of $\lambda I - T$ is closed.