PRELIMINARY EXAM IN ALGEBRA, SPRING 2019

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $G = C_4$ be the cyclic group of order 4. Give an example of a real representation M of G that is irreducible but such that $M \otimes_{\mathbb{R}} M$ is a reducible representation of G. (In other words, M is a simple $\mathbb{R}[G]$-module with $M \otimes_{\mathbb{R}} M$ not simple.)

2. Suppose that M is a $\mathbb{Q}[x]$-module such that $\dim_{\mathbb{Q}}(M) < \infty$. Show M is not projective .

3. Let R be a unital ring, M a left R-module. Is it true $\mathrm{Hom}_R(R, M) \simeq M$ as left R-modules? If so, prove it; if not explain why or give a counter example.

4. For which values of n is the ring $\mathrm{Mat}_{2,2}(\mathbb{Z}/n\mathbb{Z})$ isomorphic to a product of matrix rings over fields? Prove your answer.

5. Consider the additive group $\mathbb{Q}$ and its quotient group $G = \mathbb{Q}/\mathbb{Z}$. Prove that for every natural number n the group G contains a unique subgroup H of order n.

6. Determine if the group G given by the presentation

$$\langle x, y | xy = y^{2020} x \rangle$$

is nilpotent.