Name ________________________________

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21C FINAL EXAMINATION

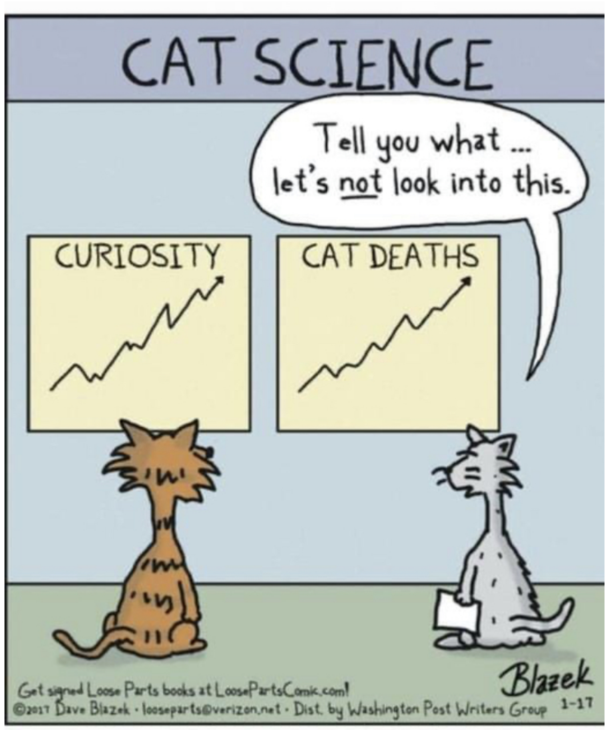
Friday December 9, 2022

Communicating or sharing work with other students is NOT allowed.

Declaration of honesty: I, the undersigned, do hereby swear that this work was unaided by any electronic device, notes or collaboration of any kind. I am aware that cheating on this exam may lead to a failing grade and referral to Student Judicial Affairs.

Signature ____________________ **Date** _______________

Take your time and do not panic. Set your work out carefully. Relevant pictures may improve your grade. You can score partial credit for a well-explained attempt to solve problems. If you are guessing, admit this and justify why you made that guess. Also, do try to enjoy yourself. This is your chance to showcase the math you learnt this quarter! If all else fails, stare at this cartoon:



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Question 1

Is it possible for the number $|a-b|$ to be larger than $|a-c|+|b-c|$? *Explain!*

What do the symbols

$$\lim_{n\to\infty} a_n = L$$

mean? *Include a picture and precise definition in your answer.*

Suppose $\varepsilon > 0$. *Find* a number N such that

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

for all integers $n > N$.

Assuming you solved the previous question correctly, what result did you just prove?

Suppose the sequence $\{a_n\}$ obeys $\lim_{n\to\infty} a_n = L$ and that $\varepsilon > 0$. Explain why there exists a number N such that for all *pairs* of integers $n, m > N$,

$$|a_n - L| < \varepsilon/2 \text{ and } |a_m - L| < \varepsilon/2\,.$$

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If $|a_n - L| < \varepsilon/2$ and $|a_m - L| < \varepsilon/2$, what can you say about $|a_n - a_m|$?

A sequence $\{a_n\}$ is said to be *Cauchy* if for all $\varepsilon > 0$ there exists a number N such that

$$n, m > N \Rightarrow |a_n - a_m| < \varepsilon\,.$$

If you got this far, you just proved that every convergent sequence is Cauchy (well done!). This gives us a new way to test if a sequence diverges—check that it is not Cauchy. Consider the sequence

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\,.$$

This is the partial sum of a harmonic series, does this converge or diverge? (No explanation needed.)

To show your previous answer was correct using the Cauchy test, we must demonstrate that no matter which positive integer N we choose, there are numbers n and m such that $|a_n - a_m| \geq \varepsilon$. Let $m > n$ and now *show* that

$$a_m - a_n \geq 1 - \frac{n}{m}\,.$$

Since ε is arbitrary, lets show something goes wrong at $\varepsilon = \frac{1}{2}$. Can you find two numbers $m, n > N$ such that $|a_m - a_n| > \frac{1}{2}$? (This time do include a short explanation.)

If you think you answered everything above correctly, you proved $\lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{m} = \infty$. In that case write the letters QED in the box below. If you are unsure, leave this box empty.

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Question 2

Suppose the vectors $\vec{u}$ and $\vec{v}$ are both non-zero and non-parallel. Show how to compute the compute the area of a triangle whose side lengths are defined by $\vec{u}$ and $\vec{v}$ using the cross product. A good answer will include an explanation of why your answer works. Include pictures as well as an example where the triangle area is $\frac{1}{2}|\vec{u}|\,|\vec{v}|$.

Suppose the three points $P, Q, R \in \mathbb{R}^3$ are distinct and do not all lie on a common line. Moreover, assume that a fourth point O is not in the plane defined by P, Q and R. Sketch this situation.

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The volume of a tetrahedron (if you forgot what a tetrahedron is, consider the pyramid shaped region defined by the four points O, P, Q and R in your sketch above) is given by $\frac{1}{3}Bh$ where B is the area of any one of the triangular faces and h is the distance of the remaining corner from that triangle. Use both the cross and dot product to find a vector formula for the volume of the tetrahedron defined by the points O, P, Q and R in your sketch. Include pictures and explanations in your answer. Also, show how your formula works in a simple example.

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Question 3

Consider the function

$$f(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2 \, .$$

Compute the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yx}$ and f_{yy} (recall $f_x := \frac{\partial f}{\partial x}$).

Sketch the graph $z = f(x, y)$ in the three cases

$$(\alpha = 1, \beta = 0, \gamma = 1) \, , \qquad (\alpha = 1, \beta = 0, \gamma = -1) \, , \qquad (\alpha = -1, \beta = 0, \gamma = -1) \, .$$

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From now on, let us assume (α, β, γ) are one of the three above cases. For which points (x, y) does

$$f_x = 0 = f_y\,?$$

(Your solutions are known as critical points.)

The Hessian of a function of two variables $f(x, y)$ is defined by

$$\mathrm{Hess}(f) := f_{xx}f_{yy} - f_{xy}^2\,.$$

Compute $\mathrm{Hess}(\alpha x^2 + 2\beta xy + \gamma y^2)$.

The Hessian is useful because

- $\mathrm{Hess}(f) > 0$ and $f_{xx} < 0$ at a critical point $\Rightarrow$ the critical point is a local maximum (for example the top of a hill).
- $\mathrm{Hess}(f) > 0$ and $f_{xx} > 0$ at a critical point $\Rightarrow$ the critical point is a local minimum.
- $\mathrm{Hess}(f) < 0$ at a critical point $\Rightarrow$ the critical point is a saddle point (like at the center of a horse's back).
- $\mathrm{Hess}(f) = 0$ is inconclusive.

Label your five graphs above by the value of their Hessians at a critical point. Discuss whether your graphs and Hessian results fit with the above classification.