Winter 2002 Mathematics Graduate Program Preliminary Exam (post F2001 system)

·Instructions: There are 11 problems on two pages. Do as many as you can. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

1. Analysis

Problem 1. Show that the l^2 norm is indeed a norm.

Problem 2. Prove that $C[0,1]$, the space of continuous functions on $[0,1]$, is not complete in the L^1 metric: $\rho(f,g) = \int |f(x) - g(x)| dx$.

Problem 3. Let $C([0,1])$ be the space of continuous functions on the unit interval with the uniform norm and let $R[x]$ be the subspace of polynomials. Give an example of an unbounded linear transformation $T : R[x] \to R$.

Problem 4. Let X be a metric space. Prove or find a counterexample :
a) If X is compact, then it is complete
b) If X is complete, then it is compact

Problem 5. Let H be a Hilbert space and let V be a vector subspace of H. Show that $(V^\perp)^\perp$ is the closure of V in H.

Problem 6. State Jensen's inequality. Show that the function $x \to \log(1/x)$ is convex on $(0, +\infty)$. Suppose that $a_1, a_2, \ldots, a_n$ and $p_1, p_2, \ldots; p_n$ are nonnegative real numbers such that

$$p_1 + p_2 + \cdots + p_n = 1.$$

Prove that

$$a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n} \leq p_1 a_1 + p_2 a_2 + \cdots + p_n a_n.$$

2. Algebra and Linear Algebra

Problem 7. 1. Let G be a group acting on a set S. Let $x \in S$, $G_x = \{g \in G \mid gx = x\}$ and $\bar{x} = \{gx \mid g \in G\}$. Prove that $|\bar{x}| = [G : G_x]$.

2. Recall that G can act on itself by conjugation $G \times G \to G$ where $(g, x) \mapsto gxg^{-1}$. From this action one obtains the class equations. State and prove the class equations explaining all terms.

Problem 8. Let $(\mathbb{Z}, +)$ be the additive group of integers.

1. Prove that if H is a subgroup of $\mathbb{Z}$, then $H = \{0\}$ or $H = m\mathbb{Z}$ for some positive integer m.
2. Let R be a commutative ring. Give the definition of prime ideals and maximal ideals in R.
3. $\mathbb{Z}$ also has a ring structure. What are the prime and maximal ideals in $\mathbb{Z}$? Justify your answer.

Problem 9. Let H and K be subgroups of a group G. It is known that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G if K is normal in G. Assume that H, K are both normal in G and that $H \cap K = 1$.

1. Show that $hk = kh$ for all $h \in H$ and $k \in K$.
2. Prove that each element in HK can be written uniquely as hk where $h \in H$ and $k \in K$.
3. Prove that $HK \cong H \times K$.

Problem 10. Let R be a commutative ring with identity and prime characteristic p. Show that the map

$$\begin{aligned} \varphi : R &\to R \\ r &\mapsto r^p \end{aligned}$$

is a homomorphism of rings (it's called the Frobenius homomorphism).

Problem 11. Let $\mathbb{F}$ be a finite field with q elements and let W be a k-dimensional vector space over $\mathbb{F}$. Show that the number of distinct bases of W is

$$(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1}).$$