Spring 2011: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let $\Omega = (0, 1)$, the open unit interval in $\mathbb{R}$, and consider the sequence of functions $f_n(x) = ne^{-nx}$. Prove that $f_n \not\rightharpoonup f$ weakly in $L^1(\Omega)$, i.e., the sequence f_n does not converge in the weak topology of $L^1(\Omega)$.
(**Hint**: Prove by contradiction.)

Problem 2:
Let $\Omega = (0, 1)$, and consider the linear operator $A = -\frac{d^2}{dx^2}$ acting on the Sobolev space of functions X where

$$X = \{u \in H^2(\Omega) \mid u(0) = 0\,, u(1) = 0\}\,,$$

and where

$$H^2(\Omega) = \{u \in L^2(\Omega) \mid \frac{du}{dx} \in L^2(\Omega)\,, \frac{d^2u}{dx^2} \in L^2(\Omega)\}\,.$$

Find all of the eigenfunctions of A belonging to the linear span of

$$\{\cos(\alpha x), \sin(\alpha x) \mid \alpha \in \mathbb{R}\}\,,$$

as well as their corresponding eigenvalues.

Problem 3:
Let $\Omega = (0, 1)$, the open unit interval in $\mathbb{R}$, and set

$$v(x) = (1 + |\log x|)^{-1}.$$

Show that $v \in W^{1,1}(\Omega)$ and that $v(0) = 0$, but that $\frac{v}{x} \notin L^1(\Omega)$. (This shows the failure of Hardy's inequality in L^1.) Note, that $W^{1,1}(\Omega) = \{u \in L^1(\Omega) \mid \frac{du}{dx} \in L^1(\Omega)\}$, where $\frac{du}{dx}$ denotes the weak derivative.

Problem 4:
Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Show that

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} b_n \tau_n \delta \text{ in } \mathcal{D}' \,, \tag{1}$$

that is, that equality in equation (1) holds in the sense of distributions, and relate b_n to the coefficients of the Fourier series. Note that δ denotes the Dirac distribution and τ_y is the translation operator, given by $\tau_y f(x) = f(x+y)$.
(**Hint:** Write $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with convergence in $L^2(0, 2\pi)$ and where the coefficients $c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx$.)

Problem 5:
Let $f(x)$ be a periodic continuous function on $\mathbb{R}$ with period 2π. Given $\epsilon > 0$, prove that for $N < \infty$ there is a finite Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)] \tag{2}$$

such that

$$|\phi(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R} \,.$$

This shows that the space of real-valued trigonometric polynomials on $\mathbb{R}$ (functions which can be expressed as in (2)) are *uniformly* dense in the space of periodic continuous function on $\mathbb{R}$ with period 2π.
(**Hint:** The Stone-Weierstrass theorem states that if X is compact in $\mathbb{R}^d$, $d \in \mathbb{N}$, then the algebra of all real-valued polynomials on X (with coordinates $(x_1, x_2, ..., x_d)$) is dense in $C(X)$.)

Problem 6:
For $\alpha \in (0, 1]$, the space of Hölder continuous functions on the interval $[0, 1]$ is defined as

$$C^{0,\alpha}([0,1]) = \{u \in C([0,1]) \ : \ |u(x) - u(y)| \leq C|x-y|^{\alpha} \,, x, y \in [0,1]\} \,,$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x\in[0,1]} |u(x)| + \sup_{x,y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \,.$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) \ : \ \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$ is a compact set in $C([0,1])$.
(**Hint:** The Arzela-Ascoli theorem states that if a family of continuous functions U is equicontinuous and uniformly bounded on $[0,1]$, then each sequence u_n in U has a uniformly convergent subsequence. Recall that U is uniformly bounded on $[0,1]$ if there exists $M > 0$ such that $|u(x)| < M$ for all $x \in [0,1]$ and all $u \in U$. Further, recall that U is equicontinuous at $x \in [0,1]$ if given any $\epsilon > 0$, there exists $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ for all $|x - y| < \delta$ and every $u \in U$.)