PRELIMINARY EXAM IN ALGEBRA
FALL 2017

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Suppose that $F \subseteq K$ is an inclusion of fields and let $\alpha, \beta \in K$ be two elements which are algebraic over F. Show that $\alpha + \beta$ is also algebraic over F.

2. Let $f \in \mathbb{Q}[x]$ be the minimal polynomial of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$, and let K be the splitting field for f over $\mathbb{Q}$. What is $[K : \mathbb{Q}]$ and what is $\mathrm{Gal}(K/\mathbb{Q})$? (Note that you are not required to find f.)

3. Let $M_n(\mathbb{R})$ denote the ring of $n \times n$ matrices over $\mathbb{R}$, and consider a (possibly non-unital) ring homomorphism $f : M_{n+1}(\mathbb{R}) \to M_n(\mathbb{R})$. Can f be non-zero?

4. Find all maximal ideals of the ring $\mathbb{F}_7[x]/(x^2+1)$ and the ring $\mathbb{F}_7[x]/(x^3+1)$.

5. Let F be a field and let $p, q \in F[x]$ be polynomials over F. Show that

$$F[x]/(p) \otimes_{F[x]} F[x]/(q) \cong F[x]/(\gcd(p,q))$$

as $F[x]$-modules.

6. Prove that if p is a prime number, then every group G with p^2 elements is abelian.