Preliminary exam in Algebra
Fall 2021

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a di erent sheet for the solution to each problem.

1. Choose a monomial order on $R = \mathbb{R}[x\ ,\quad ,x\ ]$, and let $I \subset R$ be an ideal. Show that if $R/LT(I)$ is a finite-dimensional vector space over $\mathbb{R}$, then $\dim(R/LT(I)) = \dim(R/I)$, where $LT(I)$ is the ideal of leading terms of elements in I.

2. Either find a field F, an irreducible polynomial $f(x) \in F[x]$ over F and an element of the splitting field K of f which is not in F butis fixed by every element of $G = Aut(K/F)$ (the group of automorphisms of K that preserve F) or show that no such choices are possible.

3. If G is a finite group and H is a normal subgroup of G, then there is a composition series of G, in which H is one of the groups in the series.

4. Determine $Z(\mathbb{M}_5\quad)$, that is, determine the center of the ring of 5×5 matrices over , and justify your answer.

5. Let R be a unique factorization domain (UFD) and $r \in R$ Show r is irreducible if and only if r is prime.

6. How many idempotents are there in the ring $\mathbb{Q}[x]\quad(x^7\quad x^3)$? Justify your answer.