Spring 2013: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Consider the Hilbert space $\mathcal{H} = L^2([0,1])$ with inner product $\langle f, h\rangle = \int\limits_0^1 f(x)h(x)\,dx$. Let

$$V = \{f \in L^2([0,1]) : \int_0^1 xf(x)dx = 0\} \subset \mathcal{H}$$

and $g(x) \equiv 1$. Find the closest element to g in V. Justify your answer.

Problem 2: Let $(B, \|\|)$ be a Banach space. Recall that the spectrum of a bounded linear operator $A \in \mathcal{L}(B)$ is defined as

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}.$$

Consider a sequence of bounded linear operators $A_n \in \mathcal{L}(B)$ which converges in norm to a bounded linear operator $A \in \mathcal{L}(B)$. Assume that all spectra are the same, i.e. $\sigma_0 := \sigma(A_1) = \sigma(A_2) = \dots$. Show that $\sigma_0 \subset \sigma(A)$.

Problem 3: Consider the function

$$f(x) = \begin{cases} 2\sin(x) + 3, & x > 0 \\ -2\sin(x) + c, & x \leq 0 \end{cases}.$$

Find its distributional derivative $\frac{df}{dx}$ and $\frac{d^2 f}{dx^2}$. For which values of c will $f \in W^{1,p}_{loc}(\mathbb{R})$? Justify your answer.

Problem 4: Prove that

$$\lim_{\epsilon \to 0^+} \int_0^\infty \frac{\epsilon}{\epsilon^2 + x} \sin(1/x)dx = 0\,.$$

Problem 5: Prove that the image of the space $C^k(\mathbb{T})$ of k times continuously differentiable functions on the unit circle under the Fourier transform is contained in the set of sequences satisfying $|c_n| = o(|n|^{-k})$ and contains the set of sequences satisfying $|c_n| = o(|n|^{-k-1-\epsilon})$, $\epsilon > 0$.
(Recall: $f(n) = o(h(n))$ as $n \to \infty$ means that for every $\delta > 0$ there exists an N such that $|f(n) \leq \delta|g(n)|$ for all $n > N$).

Problem 6: Let I be the interval $(0, 1)$ and $q \geq p \geq 1$. Show that there exists a constant $C = C(p, q, I)$ such that

$$\|u\|_{L^q(I)} \leq C \|u\|_{W^{1,p}(I)}$$

for all $u \in W_0^{1,p}(I)$. (Hint: First show that $\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)}$ for all $u \in W_0^{1,p}(I)$.)