Algebra Prelim Problems

Spring, 2015

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1.

Show that if M is a nondiagonalizable complex matrix and M^n is diagonalizable then $\det(M) = 0$.

Problem 2.

Find the degree of the field extension $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}$.

Problem 3.

Show that if G is an infinite simple group then every proper subgroup has infinitely many conjugates. Use this to conclude that G has an infinite automorphism group.

Problem 4.

Find a quotient ring of $Z[x]$ which is a principal ideal domain but not a field.

Problem 5.

Let $R = \mathbb{Q}[X]/(X^3 - 2)$.

(a) Is R a field? Explain.

(b) Run the extended Euclidean algorithm on $X^3 - 2$ and $X^2 - X + 1$ to find polynomials $A(x)$ and $B(x)$ with

$$A(X)(X^3 - 2) + B(X)(X^2 - X + 1) = gcd(X^3 - 2, X^2 - X + 1).$$

(c) Does $[X^2 - X + 1]$ have a multiplicative inverse in R? If yes, find it.

Problem 6.

Let G be a finite group and $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ a representation.

(a) Show: $\delta\colon G \to \mathbb{C}$, $g \mapsto det(\rho(g))$ is a linear character of G (i.e. a group homomorphism to the multiplicative group).

(b) Show: If $\delta(g) = -1$ for some $g \in G$, then G has a normal subgroup of index 2.

(c) Show: If G has order $2k$, k odd, then G has a normal subgroup of index 2.

(d) Let $\chi(g) = tr(\rho(r))$ and $g \in G$ an involution. Show: (i) $\chi(g)$ is an integer; (ii) $\chi(g) \equiv \chi(1) \bmod 2$; (iii) if G has no normal subgroup of index 2, then $\chi(g) \equiv \chi(1) \bmod 4$.