# Fall 2002 Mathematics Graduate Program Preliminary Exam

Instructions: Explain your answers clearly. Unclear answers will not receive credit. State results and theorems that you are using.

## 1. Algebra

*Problem* 1. **a.** Let $F$ be a field. Show that every ideal in the ring of polynomials $F[x]$ is principal.
Is $F[x]$ a (UFD) Unique Factorization Domain?
**b.** Are all UFDs principal ideal domains? Why or why not?
**c.** Let $g(x) = x^3 + x + 1 \in Z_2[x]$ and let $I =< g(x) >$. Prove that $Z_2[x]/I$ is a field and determine the multiplicative inverse of $x^2 + 1 + I$.
*Problem* 2. Let $G$ be a finite group and $S$ a subgroup of $G$. Recall that the normalizer of $S$ in $G$ is $N_G(S) = \{g | g \in G \text{ and } gSg^{-1} = S\}$. **a.** Prove that $N_G(S)$ is a subgroup of $G$. Let $K$ be a subgroup of $G$ which contains $S$. Prove that $S$ is normal in $K$ if and only if $K \subset N_G(S)$.
**b.** Prove that the number of distinct conjugate subgroups of $S$ is equal to the index $[G : N_G(S)]$.
**c.** If $G = S_4$ and $S =< (1,2,3,4) >$ what is $N_G(S)$?
*Problem* 3. 3) Prove or disprove:
**a.** Let $F$ be a field. Every short exact sequence of $F$-modules splits.
**b.** Let $R$ be the ring of Gaussian integers, i.e. $R = Z[i]$. If $I$ is the ideal generated by $3 + i$ then $R/I$ is a field.
**c.** Every integral domain is a principal ideal domain.
**d.** No group of order 48 is simple.
**e.** Every group of order 209 is cyclic.

## 2. Linear Algebra and other areas

*Problem* 4. **a.** Give an example of a real $n \times n$ matrix none of whose eigenvalues are real numbers.
**b.** Show that there is no such example which is $3 \times 3$.
**c.** Show that every eigenvalue of a symmetric real matrix is real.
*Problem* 5. Suppose that $A$ is a square $n \times n$ matrix with integer entries.
**a.** If an integer $k$ is an eigenvalue of $A$ show that $k$ divides the determinant of $A$.
**b.** if the sum of the entries in each row is $k$, show that $k$ divides the determinant of $A$.

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## 3. Analysis

*Problem* 6. Let $f_n : \mathbf{R} \to \mathbf{R}$ be a differentiable mapping for each $n$, with $|f_n'(x)| \leq 1$ for all $n, x$. Show that if $g(x)$ is a function such that

$$\lim_{x\to\infty} f_n(x) = g(x)$$

then $g(x)$ is a continuous function.

*Problem* 7. **a.** State the Stone-Weierstrass theorem in the context of $C(X, \mathbb{R})$, where $X$ is a compact Hausdorff space.

**b.** State the Radon-Nikodym theorem, as it applies to a pair of $\sigma$-finite measures $\mu$ and $\nu$ defined on a measurable space $(X, \mathcal{M})$.

c State the definitions of the terms *normal* topological space and *absolutely continuous function.*

*Problem* 8. Let $f : X \to Y$ be a mapping between topological spaces $X$ and $Y$. Let $\mathcal{E}$ be a base for the topology of $Y$. Show that if $f^{-1}(E)$ is open in $X$ for each $E \in \mathcal{E}$, then $f$ is continuous.

*Problem* 9. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function, and let $\mu$ be the associated measure, so that $\mu(a, b] = F(b) - F(a)$ for all $a < b$. Prove that $\mu(\{a\}) = F(a) - F(a^-)$ and $\mu([a, b]) = F(b) - F(a^-)$ for all $a < b$.// (Notation: $F(a^-) := \lim_{x\to a, x<a} F(x)$.)

*Problem* 10. Let $f$ be a $\mathcal{B}_{[0,1]^2}$-measurable real-valued function such that the partial derivative $\frac{\partial f}{\partial t}(x, t)$ exists for each $(x, t) \in [0, 1]^2$ and $M := \sup\{|(\partial f/\partial t)(x, t)| : (x, t) \in [0, 1]^2\} < \infty$. Prove that $\frac{\partial f}{\partial t}$ is measurable, and that for all $t \in [0, 1]$,

$$\frac{d}{dt}\int_0^1 f(x, t)dx = \int_0^1 \frac{\partial f}{\partial t}(x, t)dx.$$

*Problem* 11. Let $m$ denote Lebesgue measure on $\mathbb{R}$, fix $f \in L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$, and define a function $G : \mathbb{R} \to \mathbb{R}$ by the formula

$$G(t) = \int f(x + t)dm(x), \ t \in \mathbb{R}.$$

Prove that $G$ is a continuous function.